数理解析研究所講究録
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Visual Zariski–van Kampen Theorem

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1. Introduction

A theorem of Zariski–van Kampen tells us how to compute the fundamental group $\pi_1(\mathbb{P}^2 - C, p_0)$, where $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ is the complex projective plane, C is a plane algebraic curve and $\mathbb{P}^2 - C$ is the complement. (p_0 is a reference point.)

However, it is not easy in general to carry out the computation. This is because we can not catch in general the global behavior of the braid monodromy.

In this note, we give a simple method for the computations of fundamental groups for a special type of curves C, that is to say, for real curves with some properties. Our method can be seen more or less in some other literatures. See e.g. Kaneko [5] and Hironaka [4].

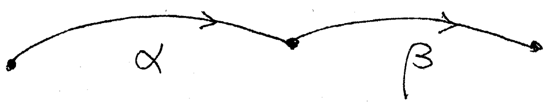
2. Notations

We use the following notations:

(i) Product of permutations:

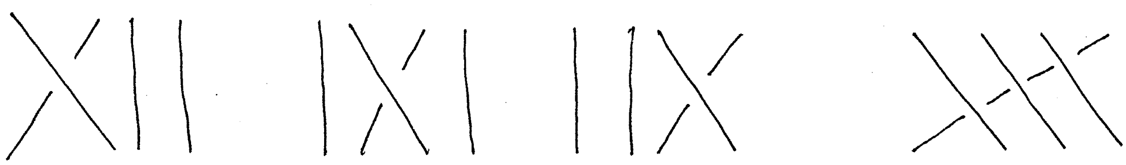
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

(ii) Product of pathes:



$= \alpha\beta$

(iii) Braids:



σ_1 σ_2 σ_3 $\sigma_3\sigma_2\sigma_1$

The Artin braid group B_d of d strings can be expressed as

$$B_d = \Big< \sigma_1, \cdots, \sigma_{d-1} \;\Big|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \quad (1 \leq i \leq d-2) \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| \geq 2) \end{array} \Big>$$

(that is, B_d is generated by $\sigma_1, \cdots, \sigma_d$ and have the generating relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, etc.)

(iv) Fundamental group of $\mathbb{C} - \{d \text{ points}\}$

Let $p_1, \cdots, p_d$ be d points in $\mathbb{C}$. Take a reference point p_0 in $\mathbb{C} - \{p_1, \cdots, p_d\}$. Then

$$\pi_1(\mathbb{C} - \{p_1, \cdots, p_d\}, p_0) = \langle \gamma_1, \cdots, \gamma_d \rangle,$$

the free group of rank d generated by $\gamma_1, \cdots, \gamma_d$, where γ_j $(1 \leq j \leq d)$ are lassos (meridians) as in Figure 1.

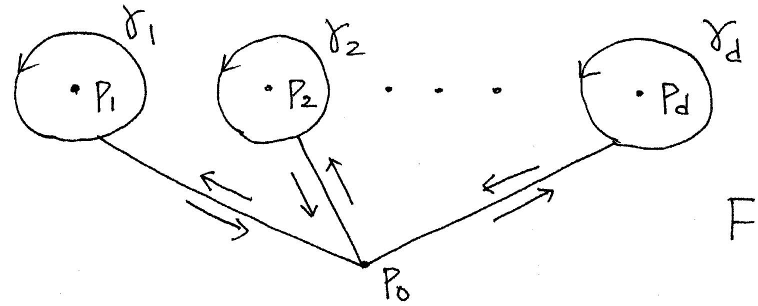


Figure 1

The Artin braid group B_d acts on the free group $F_d = \langle \gamma_1, \cdots, \gamma_d \rangle$ as follows:

$$\sigma_i(\gamma_i) = \gamma_i^{-1} \gamma_{i+1} \gamma_i, \quad \sigma_i(\gamma_{i+1}) = \gamma_i, \; (1 \leq i \leq d-2)$$

$$\sigma_i(\gamma_j) = \gamma_j \qquad (j \neq i, i+1).$$

B_d acts faithfully on F_d (see Birman [1]).

<u>Remark</u> If $p_1, \cdots, p_d$ are points in $\mathbb{P}^1$ (the complex projective line), then

$$\pi_1(\mathbb{P}^1 - \{p_1, \cdots, p_d\}, p_0) = \langle \gamma_1, \cdots, \gamma_d \mid \gamma_d \cdots \gamma_1 = 1 \rangle.$$

3. Statement of Zariski-van Kampen Theorem

Let $(X_0 : X_1 : X_2)$ be a homogeneous coordinate system in $\mathbb{P}^2$. Put $z = X_1/X_0$, $w = X_2/X_0$, the affine coordinate system. Then $L_\infty = \{X_0 = 0\}$ is the line at infinity. Let C be an algebraic curve in $\mathbb{P}^2$. Taking a suitable coordinate system, we may assume that C is defined in $\mathbb{C}^2 = \mathbb{P}^2 - L_\infty$ by the following equation:

$$C : f(z,w) = w^d + a_1(z) w^{d-1} + \cdots + a_d(z) = 0, \qquad \cdots (1)$$

where $a_j(z)$ are polynomials of z.

The discriminant $D(a_1(z), \cdots, a_d(z))$ of the equation (1) with respect to w is a polynomial of z. Let

$$\{ g_1, \cdots, g_n \} \qquad \cdots (2)$$

be the set of zeros of the discriminant. Let q_0 be a reference point in $\mathbb{C}-\{q_1,\cdots,q_n\}$. Let $\delta_1,\cdots,\delta_n$ be lassos as in Figure 2:

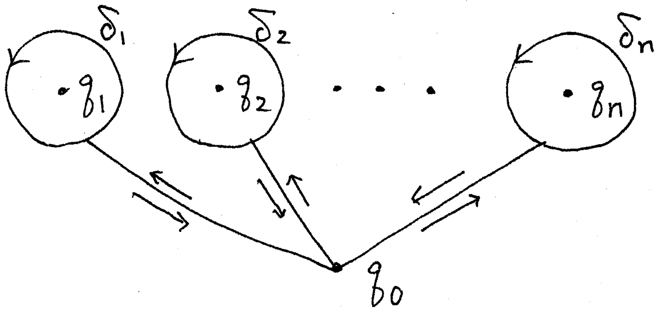


Figure 2

For a point $z\in\mathbb{C}-\{q_1,\cdots,q_n\}$, let

$$w_1(z),\ \cdots,\ w_d(z) \qquad \cdots (3)$$

be the solutions of the equation (1). We fix

$$P_1 = w_1(q_0),\ \cdots,\ P_d = w_d(q_0) \qquad \cdots (4)$$

in this order. Take a reference point $P_0\in\mathbb{C}-\{P_1,\cdots,P_d\}$. Let $\gamma_1,\cdots,\gamma_d$ be the lassos as in Figure 1.

Now, when z moves along a loop δ in $\mathbb{C}-\{q_1,\cdots,q_n\}$ starting from q_0, the points in (3) move and give a braid $\theta(\delta)$ of d strings.

$$\theta : \pi_1(\mathbb{C}-\{q_1,\cdots,q_n\},q_0) \longrightarrow B_d$$

is then a homomorphism, called the <u>braid monodromy representation</u>.

<u>Theorem 1 (Zariski – van Kampen)</u>

$$\pi_1(\mathbb{C}^2 - C, P_0) = \langle \gamma_1, \cdots, \gamma_d \mid \theta(\delta_j)\gamma_i = \gamma_i, \ (1 \le i \le d, \ 1 \le j \le n) \rangle.$$

<u>Theorem 2 (Zariski – van Kampen)</u> If C does not pass through the point $\infty = (0:0:1)$, then

$$\pi_1(\mathbb{P}^2 - C, P_0) = \langle \gamma_1, \cdots, \gamma_d \mid \theta(\delta_j)\gamma_i = \gamma_i, \ (1 \le i \le d, 1 \le j \le n), \quad \gamma_d \cdots \gamma_1 = 1 \rangle.$$

For these theorems, the reader may refer Dimca [2]. Thus if we know the braid monodromy $\theta(\delta_j)$, then the fundamental group can be calculated. But it is not easy to know it in general.

4. Strongly real curves

Let C be an algebraic curve in $\mathbb{P}^2$ given by the equation (1). Consider the following conditions:

(i) C is a real curve, that is, every polynomial $a_i(z)$ $(1 \leq i \leq d)$ has real coefficients.

(ii) Every point q_j $(1 \leq j \leq n)$ in (2) is a real point.

(iii) Every ramification point of π in every $\pi^{-1}(q_j)$ $(1 \leq j \leq n)$ is a real point, where

$$\pi : (z, w) \in C \longmapsto z \in \mathbb{C}$$

is the projection.

(iv) Every point P_i $(1 \leq i \leq d)$ in (4) is a real point.

We call a curve C with the conditions (i) $\sim$ (iv) a <u>strongly real curve</u>. For such a curve C, we will show how to carry out the computations of the fundamental groups.

We use the above notations. We may assume that

$$q_1 < \cdots < q_n \quad \text{and} \quad P_1 < \cdots < P_d .$$

Assume for example that $n=6$ and $q_2 < q_0 < q_3$. We take the lassos δ_j $(1 \leq j \leq 6)$ as in Figure 3:

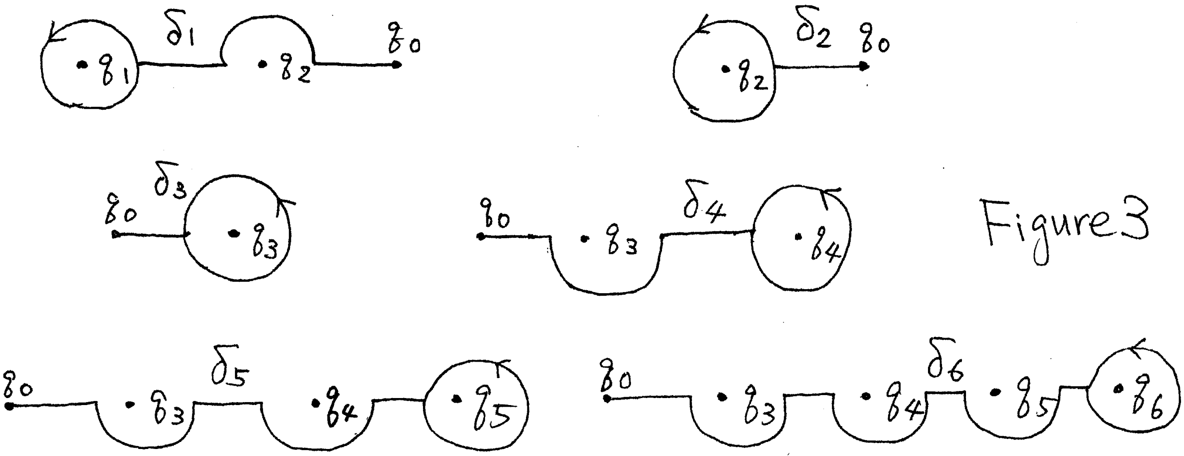


Figure 3

(The lassos are on the real line, except around q_j where they are on a circle or a hemicircle with the center q_j.)

Using these lassos, we can observe the global behavior of the braid *monodromy* $\theta(\delta_j)$ $(1 \leq j \leq n)$ and the actions on γ_i $(1 \leq i \leq d)$.

For example, if the point z on δ_4 in Figure 3 *moves* on the lower-hemicircle with the center q_3, then $w_1(z), \cdots, w_d(z)$ *move* so that they determine the <u>half</u> of the local monodromy given by the circle around q_3. Hence γ_j is changed by the action of the half monodromy, provided that every point of every $\pi^{-1}(q)$ $(q_3 - \varepsilon < q < q_3 + \varepsilon)$ is real. If some points in $\pi^{-1}(\xi)$ are not real, we must be careful and treat it case by case, as will be shown by examples.

5. Examples

We now explain our method using examples.

Example 1 Let C be the famous configuration of 6 lines as in Figure 4.

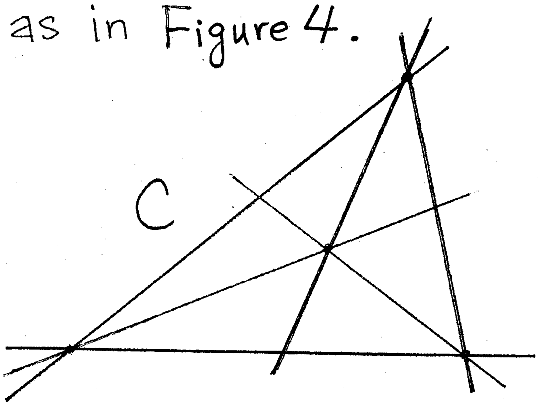


Figure 4

We take a coordinate system (z, w) as in Figure 5 so that C is strongly real. We assume that ξ_0 is the origin. We observe Figure 5 (see the next page).

The 3 lines pass through the ramification point R_3. The half monodromy of $\theta(\delta_3) = (\sigma_2\sigma_3\sigma_2)^2$ is $\sigma_2\sigma_3\sigma_2$. Thus

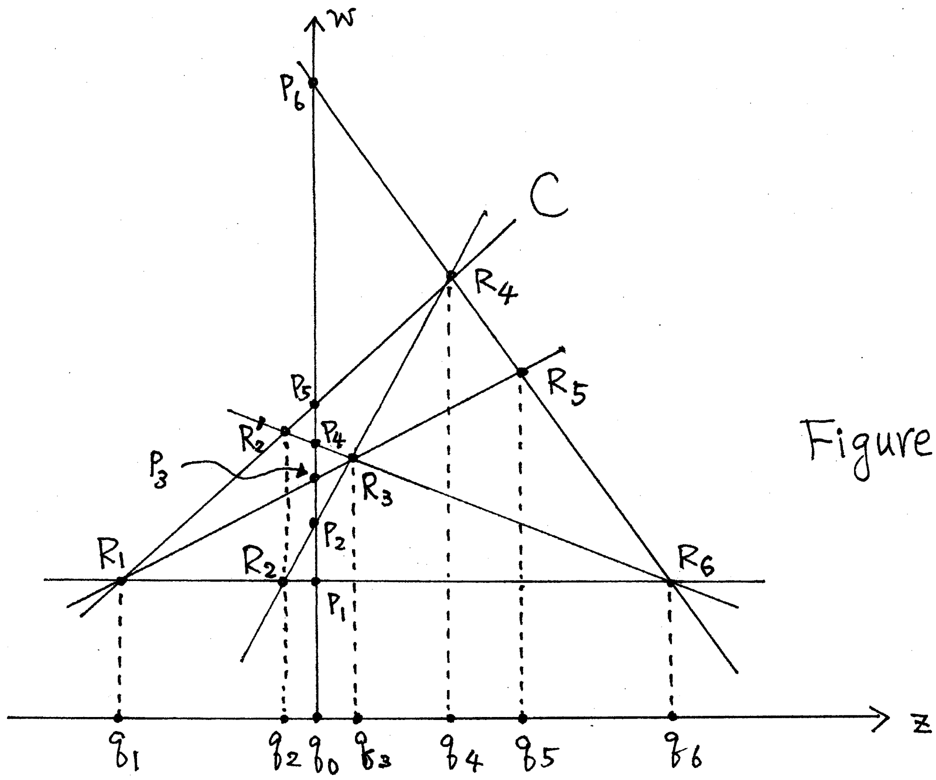


Figure 5

$$
\begin{aligned}
\gamma_1' &= (\sigma_2\sigma_3\sigma_2)(\gamma_1) = \gamma_1 , \\
\gamma_2' &= (\sigma_2\sigma_3\sigma_2)(\gamma_2) = \gamma_2^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2 , \\
\gamma_3' &= (\sigma_2\sigma_3\sigma_2)(\gamma_3) = \gamma_2^{-1}\gamma_3\gamma_2 , \\
\gamma_4' &= (\sigma_2\sigma_3\sigma_2)(\gamma_4) = \gamma_2 , \\
\gamma_5' &= (\sigma_2\sigma_3\sigma_2)(\gamma_5) = \gamma_5 , \\
\gamma_6' &= (\sigma_2\sigma_3\sigma_2)(\gamma_6) = \gamma_6 . \qquad \cdots (5)
\end{aligned}
$$

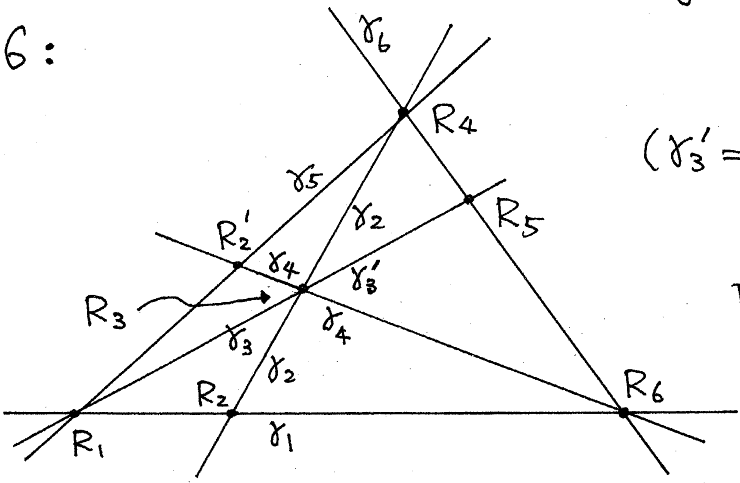
On the other hand, by Theorem 1, $\gamma_i = \theta(\delta_3)\gamma_i$ $(1 \leq i \leq 6)$. This implies that

$$\gamma_4\gamma_3\gamma_2 = \gamma_3\gamma_2\gamma_4 = \gamma_2\gamma_4\gamma_3 .$$

Hence the relations (5) can be rewritten as

$$\gamma_2' = \gamma_4 ,$$
$$\gamma_3' = \gamma_2^{-1}\gamma_3\gamma_2 ,$$
$$\gamma_4' = \gamma_2 ,$$
$$\gamma_i' = \gamma_i \qquad (i = 1, 5, 6).$$

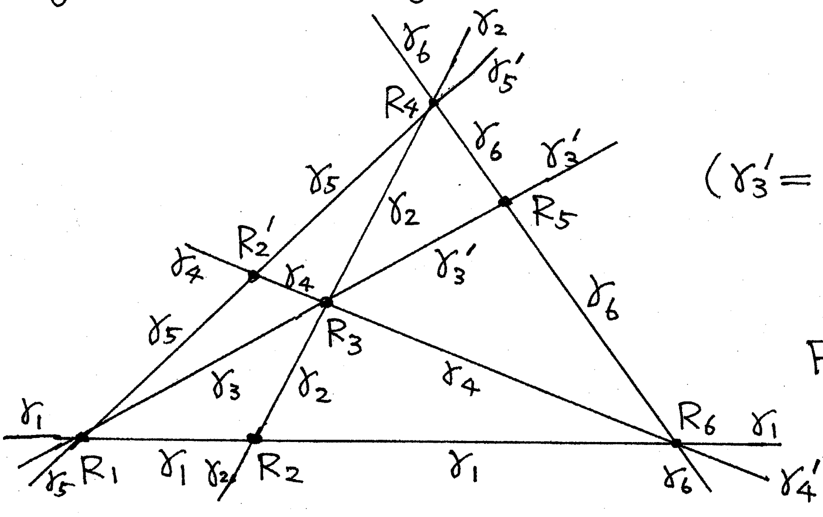
We write these data <u>on the line segments</u> as in Figure 6:



$(\gamma_3' = \gamma_2^{-1}\gamma_3\gamma_2)$

Figure 6

In a similar way, we write these data on every line segment as in Figure 7:



$(\gamma_3' = \gamma_2^{-1}\gamma_3\gamma_2)$

Figure 7

(What we write in Figure 7 are nothing but the so called <u>contiguity relations</u>.)

Observing these data in Figure 7, we can write down the fundamental groups:

$$\pi_1(\mathbb{C}^2 - C, P_0) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \mid$$
$$\gamma_5\gamma_3\gamma_1 = \gamma_3\gamma_1\gamma_5 = \gamma_1\gamma_5\gamma_3,$$
$$\gamma_2\gamma_1 = \gamma_1\gamma_2, \qquad \gamma_5\gamma_4 = \gamma_4\gamma_5,$$
$$\gamma_4\gamma_3\gamma_2 = \gamma_3\gamma_2\gamma_4 = \gamma_2\gamma_4\gamma_3,$$
$$\gamma_6\gamma_5\gamma_2 = \gamma_5\gamma_2\gamma_6 = \gamma_2\gamma_6\gamma_5,$$
$$\gamma_6 \cdot \gamma_2^{-1}\gamma_3\gamma_2 = \gamma_2^{-1}\gamma_3\gamma_2 \cdot \gamma_6,$$
$$\gamma_6\gamma_4\gamma_1 = \gamma_4\gamma_1\gamma_6 = \gamma_1\gamma_6\gamma_4 \rangle.$$

By Theorem 2, $\pi_1(\mathbb{P}^2 - C, P_0)$ is generated by $\gamma_1, \cdots, \gamma_6$ with the same relations as in $\pi_1(\mathbb{C}^2 - C, P_0)$, plus one more relation

$$\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1 = 1.$$

If we take coordinate system (z, w) such that one of the 6 lines is L_∞ (the line at infinity), then the expression of $\pi_1(\mathbb{P}^2 - C)$ becomes simpler:

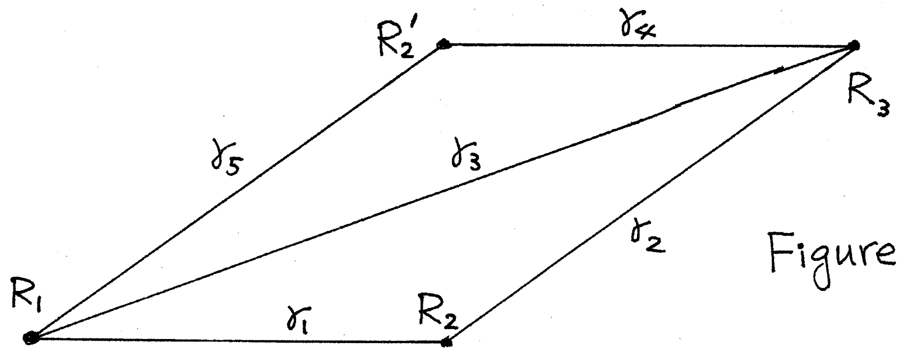


Figure 8

$$\pi_1(\mathbb{P}^2 - C, p_0) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid$$
$$\gamma_5\gamma_3\gamma_1 = \gamma_3\gamma_1\gamma_5 = \gamma_1\gamma_5\gamma_3,$$
$$\gamma_5\gamma_4 = \gamma_4\gamma_5, \quad \gamma_2\gamma_1 = \gamma_1\gamma_2,$$
$$\gamma_4\gamma_3\gamma_2 = \gamma_3\gamma_2\gamma_4 = \gamma_2\gamma_4\gamma_3 \rangle.$$

<u>Remark</u> The expression of $\pi_1(\mathbb{P}^2 - C, p_0)$ is changed if we change the position of the reference point q_0.

<u>Example 2</u> Let C be the strongly real curve consisting of an irreducible conic and 3 lines which are tangent to the conic as in Figure 9:

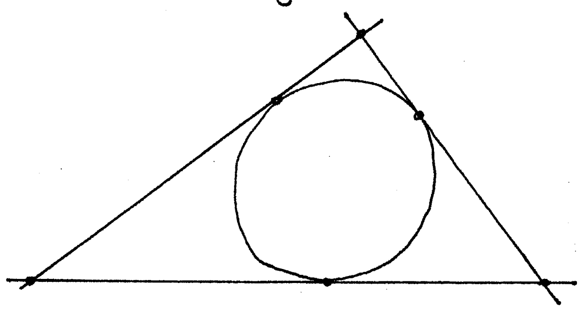


Figure 9

We take q_0 so that $q_3 < q_0 < q_4$ and assume that q_0 is the origin as in Figure 10:

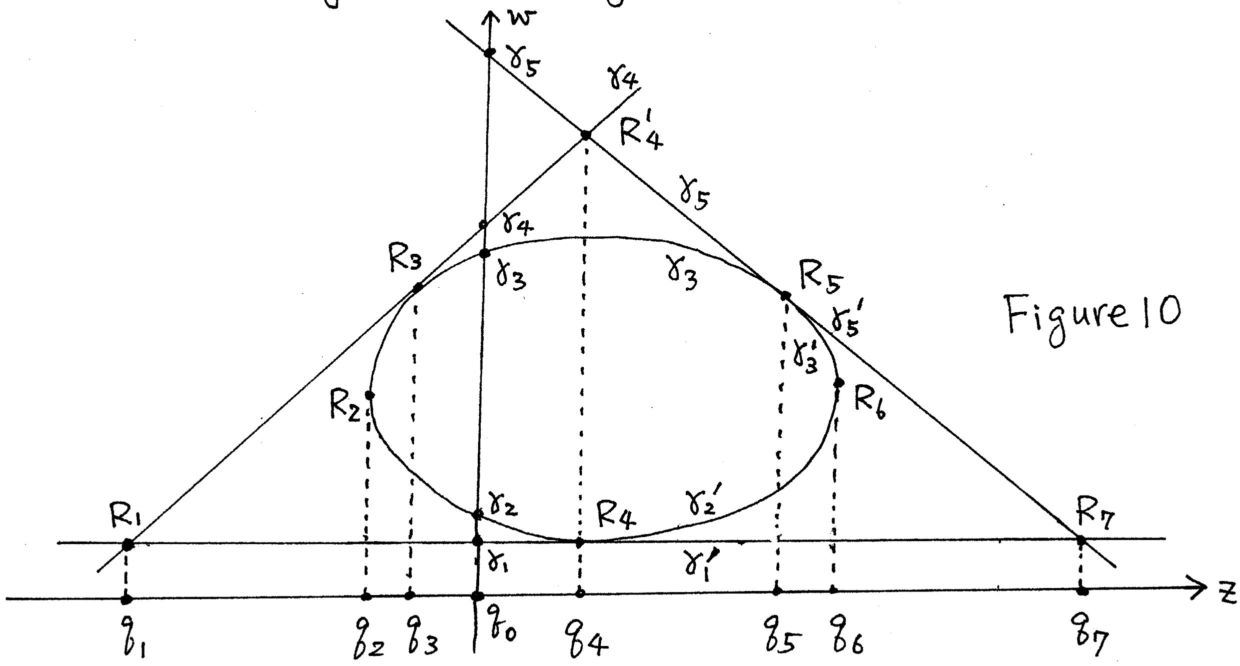


Figure 10

(We write γ_i directly instead of P_i in Figure 10).

We observe the braid monodromy $\theta(\delta_4)$, $\theta(\delta_5)$, $\theta(\delta_6)$ and $\theta(\delta_7)$.

The half monodromy of $\theta(\delta_4) = \sigma_1^4 \sigma_4^2$ is $\sigma_1^2 \sigma_4$.

Hence

$$(\gamma_2\gamma_1)^2 = (\gamma_1\gamma_2)^2, \qquad \gamma_5\gamma_4 = \gamma_4\gamma_5$$

and

$$\gamma_1' = \gamma_2\gamma_1\gamma_2^{-1},$$
$$\gamma_2' = \gamma_1^{-1}\gamma_2\gamma_1,$$
$$\gamma_4' = \gamma_5,$$
$$\gamma_5' = \gamma_4.$$

(We write γ_1' and γ_2' on the curve segments in Figure 10.)

In a similar way, we observe $\theta(\delta_5)$ and get the following relations:

$$(\gamma_5\gamma_3)^2 = (\gamma_3\gamma_5)^2,$$

$$\gamma_3' = \gamma_5\gamma_3\gamma_5^{-1},$$

$$\gamma_5' = \gamma_3^{-1}\gamma_5\gamma_3 .$$

(We write γ_3' and γ_5' on the curve segments in Figure10.)

We also observe $\theta(\delta_6)$ and get the following relation:

$$\gamma_3' = \gamma_2', \quad \text{i.e.} \quad \gamma_5\gamma_3\gamma_5^{-1} = \gamma_1^{-1}\gamma_2\gamma_1 .$$

But we must be careful about $\theta(\delta_7)$, for $w_2(z)$ and $w_3(z)$ become imaginary numbers.

When z moves to the right on the real axis with $g_5 < z < g_6 - \varepsilon$, then the points in (3) move as in Figure 11:

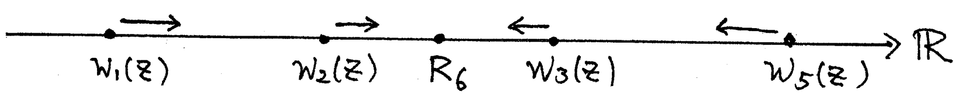
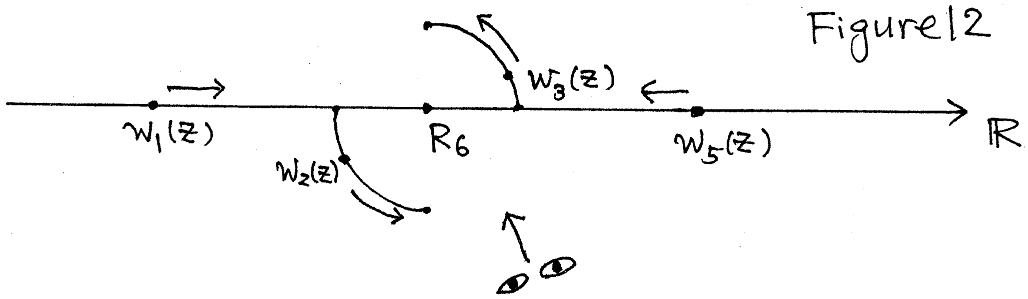


Figure11

But when z moves on the lower hemicircle around q_6, then the points in (3) move as in Figure 12 in which our eyes are at the reference point and observe the movements of points in order to determine the braid:

Figure 12



When z further moves along $\theta(\delta_7)$ with $z > q_6+\varepsilon$, then the points in (3) move as in Figure 13:

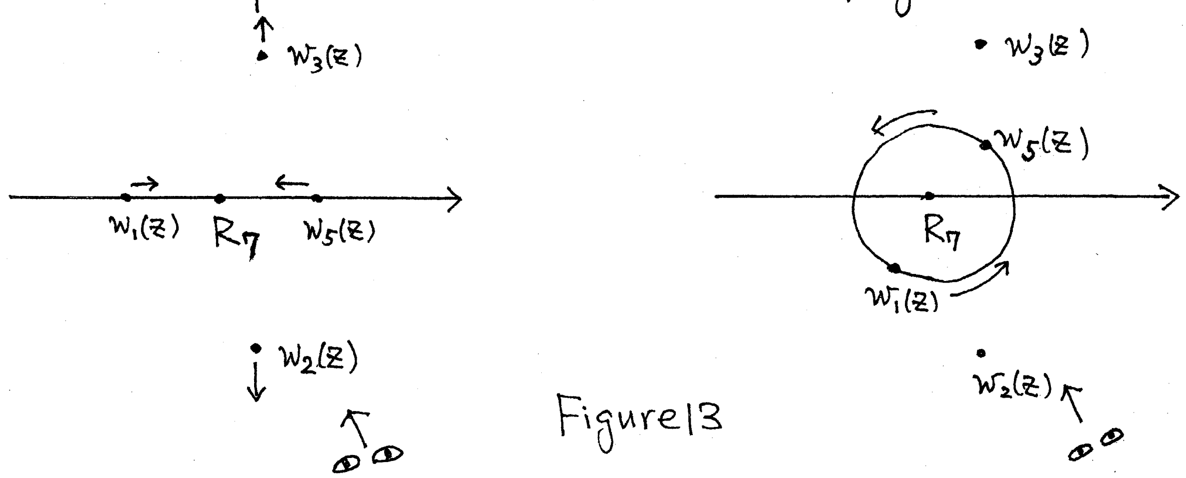


Figure 13

Hence from the point $q_6-\varepsilon$, $\theta(\delta_7)$ equals to

$$\sigma = \sigma_1^{-1}\sigma_3^{-1}\sigma_2^{2}\sigma_3\sigma_1 .$$

The relations $\sigma(\gamma_j') = \gamma_j'$ $(j=1,2,3,5)$ and the relation $\gamma_3' = \gamma_2'$ imply that

$$(\gamma_2'^{-1}\gamma_5'\gamma_2')\gamma_1' = \gamma_1'(\gamma_2'^{-1}\gamma_5'\gamma_2').$$

Similar considerations work for $\theta(\delta_1)$, $\theta(\delta_2)$ and $\theta(\delta_3)$. Thus we have

$$\begin{aligned}\pi_1(\mathbb{P}^2 - C, p_0) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid\ & (\gamma_2\gamma_1)^2 = (\gamma_1\gamma_2)^2,\ \gamma_5\gamma_4 = \gamma_4\gamma_5,\ \gamma_3' = \gamma_2', \\ & (\gamma_2'^{-1}\gamma_5'\gamma_2')\gamma_1' = \gamma_1'(\gamma_2'^{-1}\gamma_5'\gamma_2'),\ (\gamma_5\gamma_3)^2 = (\gamma_3\gamma_5)^2, \\ & (\gamma_4\gamma_3)^2 = (\gamma_3\gamma_4)^2,\ \gamma_3'' = \gamma_2, \\ & \gamma_1(\gamma_2^{-1}\gamma_4''\gamma_2) = (\gamma_2^{-1}\gamma_4''\gamma_2)\gamma_1, \\ & \gamma_5\gamma_4\gamma_3\gamma_2\gamma_1 = 1 \rangle,\end{aligned}$$

where
$$\begin{aligned}\gamma_1' &= \gamma_2\gamma_1\gamma_2^{-1}, & \gamma_2' &= \gamma_1^{-1}\gamma_2\gamma_1, \\ \gamma_3' &= \gamma_5\gamma_3\gamma_5^{-1}, & \gamma_5' &= \gamma_3^{-1}\gamma_5\gamma_3, \\ \gamma_3'' &= \gamma_4\gamma_3\gamma_4^{-1}, & \gamma_4'' &= \gamma_3^{-1}\gamma_4\gamma_3.\end{aligned}$$

We can rewrite it in a simpler form:

$$\pi_1(\mathbb{P}^2 - C, p_0) = \langle \gamma_1, \gamma_2, \gamma_4 \mid (\gamma_2\gamma_1)^2 = (\gamma_1\gamma_2)^2,\ (\gamma_4\gamma_2)^2 = (\gamma_2\gamma_4)^2,\ \gamma_4\gamma_1 = \gamma_1\gamma_4 \rangle.$$

$$(\gamma_3 = \gamma_4^{-1}\gamma_2\gamma_4,\quad \gamma_5 = (\gamma_2\gamma_4\gamma_2\gamma_1)^{-1}.)$$

Example 3 Consider the 3 irreducible conics meeting at two points A and B and 3 lines PP', QQ' and RR' meeting at a point S as in Figure 14:

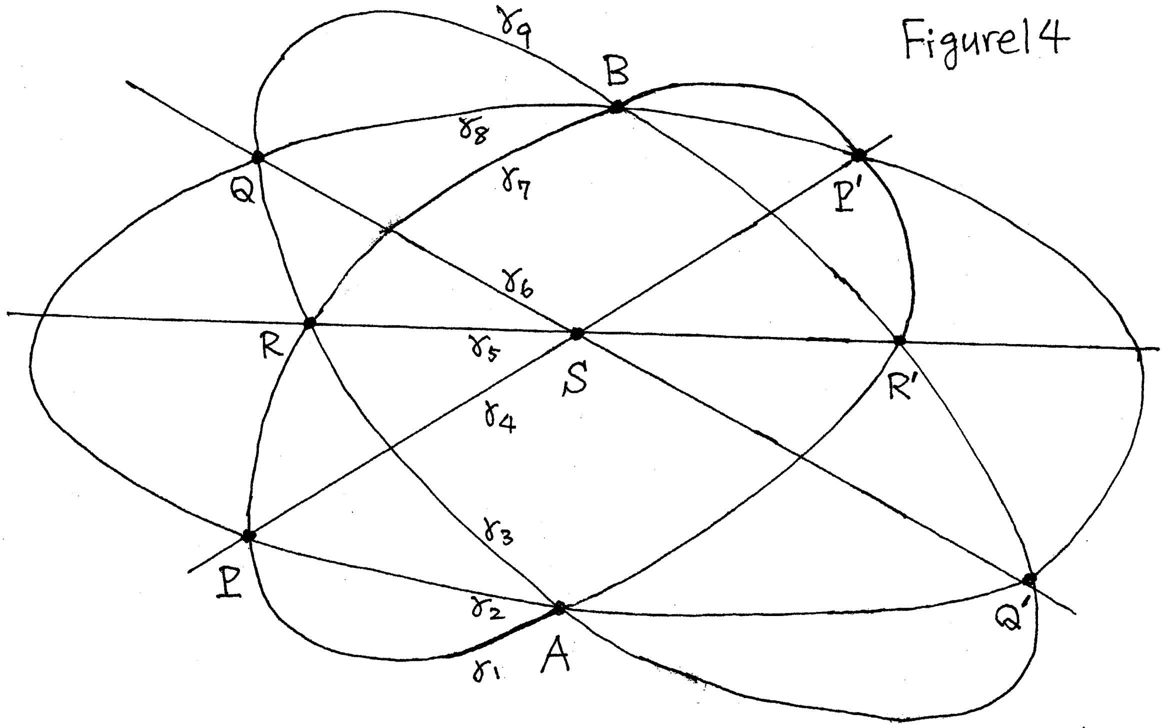


Figure 14

This configuration express a special case of Euler's theorem for cubic curves. Let C be the strongly real curve of this configuration. Then we can calculate the fundamental group as in Examples 2. The result is

$$\pi_1(\mathbb{P}^2 - C, P_0) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \mid$$
$$\gamma_i\gamma_j = \gamma_j\gamma_i \quad (i, j = 1, 2, 3),$$
$$\gamma_i\gamma_j = \gamma_j\gamma_i \quad (i = 1, 2, 3, \quad j = 4, 5, 6),$$
$$\gamma_6\gamma_5\gamma_4 = \gamma_5\gamma_4\gamma_6 = \gamma_4\gamma_6\gamma_5,$$
$$\gamma_6\gamma_5\gamma_4(\gamma_3\gamma_2\gamma_1)^2 = 1 \rangle.$$

In this way, many examples of the fundamental groups for strongly real curves C can be calculated. Interesting configurations like Puppus' theorem, Desargous' theorem, Pascal's theorem, Brianchon's theorem, etc., are expressed by strongly real curves. Thus the fundamental groups of the complements of the configurations can be calculated.

6. Braid groups as fundamental groups of strongly real curves.

If C is a strongly real curve of degree 3 and of genus 0 with a simple cusp as in Figure 15, then

$$\pi_1(\mathbb{C}^2 - C, P_0) = \langle \gamma_1, \gamma_2 \mid \gamma_2\gamma_1\gamma_2 = \gamma_1\gamma_2\gamma_1 \rangle$$
$$\simeq B_3 \quad \text{(3-rd braid group)}.$$

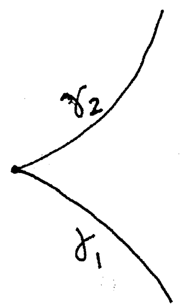


Figure 15

$$C:\ w^2 - z^3 = 0$$

is such a curve.

Next, if C is a strongly real curve of degree 4 and of genus 0 with 2 simple cusps and 1 node as in Figure 16, then

$$\pi_1(\mathbb{C}^2 - C, p_0) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_3\gamma_2\gamma_3 = \gamma_2\gamma_3\gamma_2,\ \gamma_3\gamma_1\gamma_3 = \gamma_1\gamma_3\gamma_1,\ \gamma_2\gamma_1 = \gamma_1\gamma_2 \rangle$$
$$\simeq B_4 \quad (4\text{-th braid group}).$$

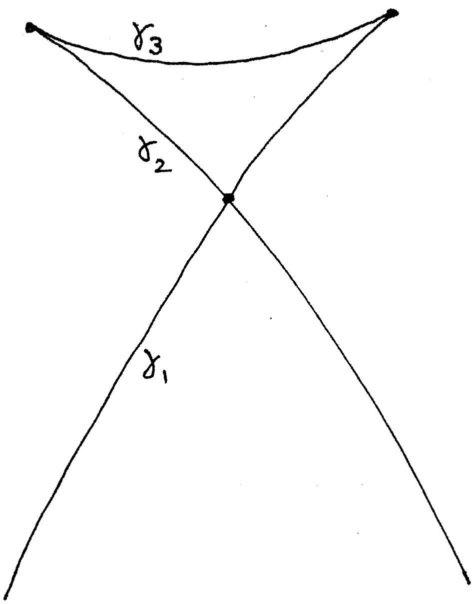


Figure 16

$$C:\ z^4 + 4z^2 - 6z^2w - 3w^2 + 4w^3 = 0$$

is such a curve. (This equation is due to M. Oka.)

In a similar way, if C are strongly real curves of degree 5 and 6, respectively, and of genus 0, with only simple cusps and nodes as in Figure 17, then $\pi_1(\mathbb{C}^2 - C, P_0)$ are isomorphic to B_5 and B_6, respectively. (But we do not know the equations for such curves.)

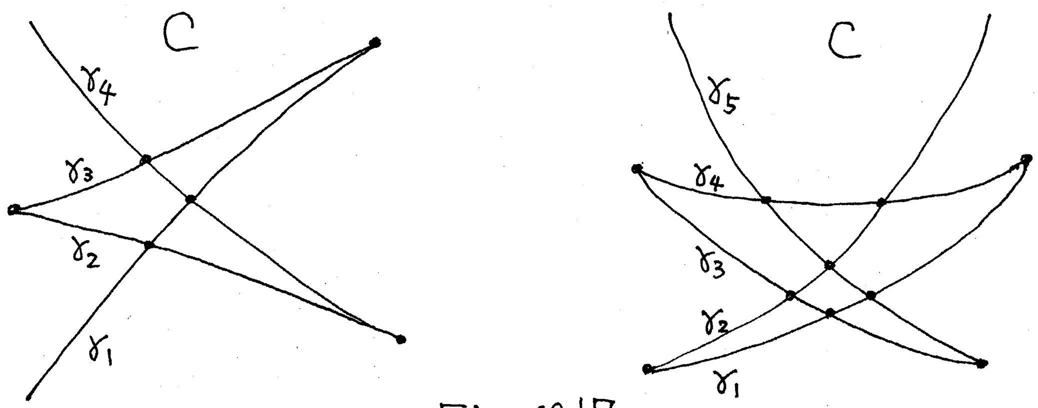


Figure 17

We can continue this process and draw a strongly real curve whose complement in $\mathbb{C}^2$ has the fundamental group isomorphic to the Artin braid group of d strings.

7. Finite branched covering of $\mathbb{P}^2$

One of the aim of Zariski [6] to study fundamental groups was to apply it to the study of algebraic functions of 2 variables, that is, of finite branched coverings of $\mathbb{P}^2$.

Here a finite branched covering

$$f : X \longrightarrow \mathbb{P}^2$$

of $\mathbb{P}^2$ is by definition, a finite proper holomorphic mapping of an irreducible normal projective surface X onto $\mathbb{P}^2$. f is determined by its permutation monodromy representation

$$\Phi_f : \pi_1(\mathbb{P}^2 - C, p_0) \longrightarrow S_d,$$

where d is the degree of f, S_d is the d-th symmetric group and C is the branch curve of f. The image of Φ_f is a transitive subgroup of S_d.

Conversely, given a homomorphism

$$\Phi : \pi_1(\mathbb{P}^2 - C, p_0) \longrightarrow S_d$$

whose image is transitive, there exist a unique (up to isomorphisms) finite branched covering

$$f : X \longrightarrow \mathbb{P}^2$$

such that (i) the branch curve is contained in

C and (ii) $\Phi_f = \Phi$.

This follows from a Theorem of Grauert–Remmert [3].

Since we know about fundamental groups $\pi_1(\mathbb{P}^2 - C, p_0)$ for many strongly real curves C, we can construct many examples of such homomorphisms Φ.

We give here only one example. Let C be the configuration of 6 lines as in Example 1. We use the same notations as in Example 1. Put

$$\Phi(\gamma_1) = \Phi(\gamma_2) = (145236),$$
$$\Phi(\gamma_4) = \Phi(\gamma_5) = (135246),$$
$$\Phi(\gamma_3) = (146235),$$
$$\Phi(\gamma_6) = (136245).$$

Then it is easy to check that

$$\Phi : \pi_1(\mathbb{P}^2 - C, p_0) \longrightarrow S_6$$

is a homomorphism, whose image is a transitive subgroup of S_6 of order 24.

References

[1] J. S. Birman: Braids, Links and Mapping Class Group, Ann. Math. Studies, 82, Princeton, (1974).

[2] A. Dimca: Singularities and Topology of Hypersurfaces, Springer-Verlag, (1992).

[3] H. Grauert, R. Remmert: Komplexe Räume, Math. Ann., 136 (1958), 245–318.

[4] E. Hironaka: Abelian coverings of the complex projective plane branched along configurations of real lines, Memoirs of AMS, 502 (1993).

[5] J. Kaneko: Monodromy group of Appell's System (F_4), Tokyo J. Math., 4 (1981), 35–54.

[6] O. Zariski, Collected papers, Vol. III, 1978.

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