数理解析研究所講究録
1122巻 2000年 115-125

THE INDEX OF A LOG-CANONICAL SINGULARITY

SHIHOKO ISHII 石井志保子(東工大)

ABSTRACT. In this paper we study the index of an isolated strictly log-canonical singularity. As a result, we obtain the boundedness of indices of these singularities of dimension 3 and determine all possible indices.

1. Introduction

A log-canonical, non-log-terminal singularity is called strictly log-canonical. Let (X, x) be an isolated strictly log-canonical singularity over $\mathbb{C}$. If its dimension is 2, then the index is 1, 2, 3, 4, or 6. This is observed by checking the list of the weighted dual graphs of all strictly log-canonical singularities. This is also proved by Shokurov [14] by means of complements and by Okuma [13] by means of plurigenera. In the 3-dimensional case, the author heard that boundedness of indices of such singularities is proved by Shokurov in [15]. In this paper, we study the quotient of isolated strictly log-canonical singularities by finite group actions. First, in case that the group acts freely in codimension 1, we obtain the formula of the index of the quotient singularity (Lemma 3.3). By this it follows a different proof of above fact on indices for dimension 2. We then prove that the index of 3-dimensional strictly log-canonical singularity is less than or equal to 66. More precisely, a positive integer r is the index of such a singularity if and only if $\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function. This is related to finite automorphisms on $K3$-surfaces, Abelian surfaces and elliptic curves.

The author would like to express her gratitude to Professor Viyacheslav Shokurov for asking her the question on index, which gave the motivation for this work. She is also grateful to Professors Viyacheslav Nikulin, Shigeyuki Kondo and Keiji Oguiso for providing her with useful information.

2. Isolated strictly log-canonical singularities.

2.1. Isolated strictly log-canonical singularities are studied in [5]. In this section we summarize those results and add some basic facts on these singularities.

[1]partially supported by the Grant-in-Aid for Scientific Research(No.09640016), the Ministry of Education, Japan.

Definition 2.2. Let (X, x) be a germ of normal singularity. If there is an integer r such that $\omega_X^{[r]}$ is invertible, the singularity is called a *$\mathbb{Q}$-Gorenstein singularity*. We call the minimum positive such number r the *index* of (X, x) and denote by $\mathrm{Ind}(X, x)$.

Definition 2.3. A $\mathbb{Q}$-Gorenstein singularity (X, x) is called a *log-canonical singularity* (resp.*log-terminal singularity*) if for a good resolution $f : Y \to X$ the canonical divisor on Y has an expression in $\mathrm{Div}(Y) \otimes \mathbb{Q}$:

$$K_Y = f^*K_X + \sum_i m_i E_i$$

with $m_i \geq -1$ (resp. $m_i > -1$) for every irreducible exceptional divisor E_i with $x \in f(E_i)$. Here a good resolution means a resolution whose exceptional set is a normally crossing divisor with the non-singular irreducible components. We call m_i the *discrepancy* over X at E_i or the *discrepancy* for f at E_i for each irreducible component E_i.

2.4. In case of index 1, a strictly log-canonical singularity is equivalent to a purely elliptic singularity ([5]). In this case we define the essential divisor in the exceptional divisor of a good resolution. It actually plays an essential role in the exceptional divisor (cf. Lemma 3.7 [5]).

Definition 2.5. Let (X, x) be an isolated strictly log-canonical singularity of index 1 and $f : Y \to X$ a good resolution. Then one has a representation

$$K_Y = f^*K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} E_j,$$

with $m_i \geq 0$, $I \cap J = \emptyset$ and $J \neq \emptyset$. The divisor $E_J := \sum_{j \in J} E_j$ is called the *essential divisor* for a good resolution f.

2.6. Let (X, x) be an n-dimensional isolated strictly log-canonical singularity of index 1 and $f : Y \to X$ a good resolution with the essential divisor E_J. Since E_J is a complete variety with normal crossings,

$$H^{n-1}(E_J, \mathcal{O}_{E_J}) \simeq Gr_F^0 H^{n-1}(E_J, \mathbb{C}) = \bigoplus_{i=0}^{n-1} H_{n-1}^{0,i}(E_J),$$

where F is the Hodge filtration and $H_m^{i,j}(*)$ is the (i, j)-Hodge-component of $H^m(*, \mathbb{C})$. As the left hand side is 1-dimensional $\mathbb{C}$-vector space (Lemma 3.7 [5]), it must coincide with one of $H_{n-1}^{0,i}(E_J)$ $(i = 0, 1, 2, \ldots, n-1)$.

Definition 2.7. An n-dimensional isolated strictly log-canonical singularity (X, x) of index 1 is called of type $(0, i)$, if $H^{n-1}(E_J, \mathcal{O}_{E_J}) = H_{n-1}^{0,i}(E_J)$.

2.8. The type is independent of the choice of a good resolution (Proposition 4.2 in [5]).

Example 2.9. A 2-dimensional srictly log-canonical singularity (X, x) of index 1 is of type $(0,1)$ if and only if (X, x) is a simple elliptic singularity and of type $(0,0)$ if and only if it is a cusp singularity.

Proposition 2.10. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type $(0,2)$ and $f : Y \to X$ the canonical model, i.e. Y has at worst canonical singularities and K_Y is f-ample. Let D be the exceptional divisor of f with the reduced structure. Then Y has at worst terminal singularities and D is isomorphic to either a normal $K3$-surface or an Abelian surface. Here a normal $K3$-surface is a normal surface whose minimal resolution is a $K3$-surface.*

Proof. First note that E_J is irreducible by Lemma 6, [7]. Since the discrepancy for f at each exceptional component is negative (the proof of Lemma 3.7 [7]), D is irreducible. Let $g : Y' \to Y$ be a proper birational morphism whose composite $f \circ g : Y' \to X$ is a good resolution. One sees that Y has at worst terminal singularities. Indeed, if not, there exists an exceptional divisor E_0 which is crepant for g. Then the discrepancy at E_0 for $f \circ g$ is less than 0, so E_0 becomes another component of the essential divisor, which is a contradiction. Now one can prove that Y is non-singular away from finite points. If D has 1-dimensional singular locus, then by the blowing-up at a 1-dimensional irreducible component of the singular locus one obtains a component E_1 whose discrepancy for $f \circ g$ is $-m+1 < 0$, where m is the multiplicity of D at a general point on the curve. It implies that E_1 is another component of the essential divisor, which is a contradiction. Therefore D is non-singular away from finite points. On the other hand, since $\omega_Y \simeq \mathcal{O}_Y(-D)$ is Cohen-Macaulay, so is D. Hence by Serre's criterion D is normal. The condition $\omega_Y \simeq \mathcal{O}_Y(-D)$ yields that $\omega_D \simeq \mathcal{O}_D$. A normal surface with this condition and $H^2(E_J, \mathcal{O}_{E_J}) = \mathbb{C}$, where E_J is a resolution of D, is either a normal $K3$-surface or an Abelian surface ([16]). □

Proposition 2.11. *([6]) Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type $(0,1)$ and a finite group G act on (X, x). Then either:*

(i) for every good resolution $f : \tilde{X} \to X$, the essential divisor E_J is a cycle $E_1 + E_2 + \ldots + E_s$, $(s \geq 2)$ of elliptic ruled surfaces, where E_i and E_{i+1} intersect at a section on each component for $i = 1, \ldots, s$ $(E_{s+1} = E_1)$ or

(ii) there is a G-equivariant good resolution $f : \tilde{X} \to X$ such that the essential divisor E_J contains a G-invariant chain $E^{(0)} = E_1 + \ldots + E_s$ $(s \geq 1)$ of elliptic ruled surfaces, where E_i and E_{i+1} intersect at a section on each component for $i = 1, \ldots, s-1$. There are disjoint subdivisors $E^{(-)}$ and $E^{(+)}$ of E_J such that $E_J = E^{(-)} + E^{(0)} + E^{(+)}$, where $E^{(-)} \cap E^{(0)}$ is a section of E_1 and $E^{(+)} \cap E^{(0)}$ is a section of E_s.

3. Finite groups which act freely in codimension 1.

Definition 3.1. Let G be a group and (X,x) a germ of a singularity. We say that G acts on (X,x) if G acts on a neighbourhood of x and fixes the point x. We say that G acts on (X,x) freely in codimension 1, if there exists a closed subset S of codimension greater than or equal to 2 on a neighbourhood X such that G acts freely on $X \setminus S$.

3.2. We denote the set of non-singular points of X by X_{reg}. Let (X,x) be a $\mathbb{Q}$-Gorenstein singularity of index m and a group G act on (X,x). We denote the germ $(X/G, x')$ by $(X,x)/G$, where $x' \in X/G$ is the image of x. Denote the maximal ideal of x by $\mathfrak{m}_x$. Then it induces a canonical representation

$$\rho : G \to GL(\omega_X^{[m]}/\mathfrak{m}_x\omega_X^{[m]}) \simeq \mathbb{C}^*.$$

because G fixes the point x.

Lemma 3.3. *Let (X,x) be a $\mathbb{Q}$-Gorenstein normal singularity of index m. Let G be a finite group which acts on (X,x) freely in codimension 1 and $\rho : G \to GL(\omega_X^{[m]}/\mathfrak{m}_x\omega_X^{[m]}) \simeq \mathbb{C}^*$ the canonical representation. Then*

$$\mathrm{Ind}((X,x)/G) = m|Im\rho|.$$

In particular,

$$\mathrm{Ind}((X,x)/G) \le m|G|.$$

Proof. Denote the order of G by d, $|Im\rho|$ by r and $\mathrm{Ind}((X,x)/G)$ by I. Let g be a generator of $Im\rho$ and ϵ the primitive r-th root of 1 which corresponds to g. Let ω be a generator of $\omega_X^{[m]}$. By the pull-back of a generator of $\omega_{X/G}^{[I]}$, one has a G-invariant I-ple n-form θ which is holomorphic and does not vanish on X_{reg}. Therefore $I = mm'$ for some $m' \in \mathbb{N}$ and $\theta = h\omega^{\otimes m'}$, where h is a nowhere vanishing holomorphic function on X. Since $\theta^g = \theta$ as an element of $\omega_X^{[I]}/\mathfrak{m}_x\omega_X^{[I]}$, one obtains that $\epsilon^{m'}h(x)\omega^{\otimes m'} = h(x)\omega^{\otimes m'}$. Hence $\epsilon^{m'} = 1$. This shows $I \ge mr$. Next, to prove $I \le mr$, we construct a G-invariant mr-ple n-form which is holomorphic and does not vanish on X_{reg}. Denote an element of G which corresponds to $g \in Im\rho$ by the same symbol g. Let θ be an mr-ple n-form $\omega \otimes \omega^g \ldots \otimes \omega^{g^{r-1}}$ and $\tilde{\theta}$ be $(1/d)\sum_{\sigma \in G} \theta^\sigma$. Then $\tilde{\theta}$ is an invariant mr-ple n-form. Let $\rho(\sigma) = g^i$ for $\sigma \in G$. Then in $\omega_X^{[mr]}/\mathfrak{m}_x\omega_X^{[mr]}$, $\theta^\sigma = \epsilon^{ri+(1+2+\ldots+r-1)}\omega^{\otimes r}$ which is $\omega^{\otimes r}$ if r is odd and $-\omega^{\otimes r}$ if r is even. Therefore $\tilde{\theta} = \pm\omega^{\otimes r} + \lambda$, where $\lambda \in \mathfrak{m}_x\omega_X^{[mr]}$. Since $\tilde{\theta} \notin \mathfrak{m}_x\omega_X^{[mr]}$, $\tilde{\theta}$ does not vanish on X_{reg}, which shows that $\tilde{\theta}$ is a required form. □

Corollary 3.4. *Let (X,x) be an isolated strictly log-canonical singularity of index 1 on which a finite group G acts. Let $f : \tilde{X} \to X$ be a G-equivariant resolution of the singularities and $\rho : G \to GL(\omega_X/f_*\omega_X) \simeq \mathbb{C}$ the induced representation. Then* $\mathrm{Ind}((X,x)/G) = |Im\rho|$.

Proof. For an isolated strictly log-canonical singularity of index 1, it follows that $\mathfrak{m}_x\omega_X = f_*\omega_{\tilde{X}}$. □

Corollary 3.5. *Let (X, x) be an n-dimensional isolated strictly log-canonical singularity of index 1 on which a finite group G acts. Assume there exists the canonical model $\varphi : X' \to X$ and let E be the reduced exceptional divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E, \mathcal{O}_E))$ and* $\mathrm{Ind}(X, x)/G = |Im\rho|$.

Proof. Take a G-equivariant resolution $f : \tilde{X} \to X$. Then $\bigoplus_{m\geq 0} f_*\omega_{\tilde{X}}^{\otimes m}$ admits the action of G. So the canonical model admits the equivariant action of G, therefore the exceptional divisor E also does. Since $\omega_{X'} \simeq \mathcal{O}_{X'}(-E)$ (proof of Lemma 7 of [7]) and X' is Gorenstein in codimension 2, E is Cohen-Macaulay and $\omega_E \simeq \mathcal{O}_E$. These yield that $H^{n-1}(E, \mathcal{O}_E) = \mathbb{C}$. As $R^{n-1}\varphi_*\mathcal{O}_{X'} \simeq R^{n-1}f_*\mathcal{O}_{\tilde{X}} \simeq \mathbb{C}$, the surjection $R^{n-1}\varphi_*\mathcal{O}_{X'} \to H^{n-1}(E, \mathcal{O}_E)$ is an isomorphism. On the other hand $R^{n-1}f_*\mathcal{O}_{\tilde{X}}$ is dual to $\omega_X/f_*\omega_{\tilde{X}}$, on which one can apply Corollary 3.4. □

Corollary 3.6. *Let (X, x) be an n-dimensional isolated strictly log-canonical singularity of index 1 on which a finite group G acts. Let $f : Y \to X$ be a G-equivariant good resolution and E_J the essential divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E_J, \mathcal{O}_{E_J}))$ and* $\mathrm{Ind}(X, x)/G = |Im\rho|$.

Proof. It is clear that G acts on E_J. Since E_J is the essential divisor, $R^{n-1}f_*\mathcal{O}_{X'} \simeq H^{n-1}(E_J, \mathcal{O}_{E_J})$ by Lemma 3.7 [5]. On the other hand $R^{n-1}f_*\mathcal{O}_{\tilde{X}}$ is dual to $\omega_X/f_*\omega_{\tilde{X}}$, on which one can apply Corollary 3.4. □

4. Index of isolated strictly log-canonical singularities

4.1. In this section, one proves that the indices of isolated strictly log-canonical singularities of dimension 2 and 3 are determined. Here one should note that the boundedness of indices does not hold for log-terminal singularities and non-log-canonical singularities even for 2-dimensional case.

Example 4.2. (1) Let (Z_m, z_m) be the cyclic quotient singularity $\mathbb{C}^2/G$, where G is generated by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Here ϵ is a primitive m-th root of unity. Then the exceptional curve on the minimal resolution is $\mathbb{P}^1$ and its self-intersection number is $-m$. Therefore the index of (Z_m, z_m) is m if m is odd and $m/2$ if m is even. This shows that the indices of log-terminal singularities are not bounded.

(2) Let $(X, x) \subset (\mathbb{C}^3, 0)$ be a hypersurface singularity defined by $x^4 + y^4 + z^4 = 0$ and (Z_m, z_m) is its quotient by the cyclic group generated by

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix},$$

where ϵ is a primitive m-th root of unity. Then the index of (Z_m, z_m) is m. This shows that the indices of non-log-canonical singularities are not bounded.

4.3. Let $\pi : (X, x) \to (Z, z)$ be a finite morphism étale in codimension 1. Then (X, x) is strictly log-canonical if and only if (Z, z) is (see for example Proposition 1.7, [?]). Hence by the canonical cover, an arbitrary strictly log-canonical singularity is regarded as the quotient of such a singularity of index 1 by a finite group which acts on the singularity freely in codimension 1.

Definition 4.4. An isolated strictly log-canonical singularity is called of type $(0, i)$, if its canonical cover is of type $(0, i)$.

Theorem 4.5. *An arbitrary dimensional isolated strictly log-canonical singularity of type* $(0, 0)$ *has index either 1 or 2.*

Proof. This is proved in Theorem 3.10, [?]. One can also prove it by using 3.6. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of an n-dimensional isolated strictly log-canonical singularity (Z, z) and $G =< g >$ the associated cyclic group. Let $f : \tilde{X} \to X$ be a G-equivariant good resolution of (X, x) such that $\pi \circ f$ factors through a good resolution $g : \tilde{Z} \to Z$ of (Z, z). Denote the essential divisor for f by E_J and its dual complex by Γ. Then g induces an automorphism g^* on $H^{n-1}(\Gamma, \mathbb{Z})$. Since (X, x) is of type $(0, 0)$, $\mathbb{C} \simeq H^{0,0}_{n-1}(E_J)$ and this is isomorphic to $H^{n-1}(\Gamma, \mathbb{C})$ by 2.5, [9]. Therefore $H^{n-1}(\Gamma, \mathbb{Z})$ is of rank 1. Let λ be a free generator of $H^{n-1}(\Gamma, \mathbb{Z})$ Then $g^*(\lambda) = \pm\lambda + (torsion)$ in $H^{n-1}(\Gamma, \mathbb{Z})$. Therefore $g^*(\lambda) = \pm\lambda$ in $H^{n-1}(\Gamma, \mathbb{C})$. Hence the order of the action of G on $H^{n-1}(E_J, \mathcal{O}_{E_J})$ is 1 or 2. Now apply 3.6. □

4.6. A non-singular projective variety X is called a Calabi-Yau variety, if it satisfies that $\omega_X \simeq \mathcal{O}_X$. It is well known that a 1-dimensional Calabi-Yau variety is an elliptic curve and 2-dimensional one is either a $K3$-surface or an Abelian surface. An automorphism g on X induces a linear automorphism g^* on $\Gamma(X, \omega_X) = \mathbb{C}$ which is dual to $H^n(X, \mathcal{O}_X)$, where $n = \dim X$. Now let us introduce a conjecture on finite automorphisms on Calabi-Yau varieties, which is essential to our problem.

Conjecture 4.7. For $n \in \mathbb{N}$, there is a number B_n such that n-dimensional Calabi-Yau variety X and a finite automorphism g on X, the order of the induced automorphism g^* on $H^n(X, \mathcal{O}_X) = \mathbb{C}$ is bounded by B_n.

For $n = 1, 2$, the conjecture holds true.

Proposition 4.8. *For an arbitrary elliptic curve X, denote the order $|Im\rho|$ by r, where $\rho : Aut(X) \to GL(H^1(X, \mathcal{O}_X)) = \mathbb{C}^*$ is the induced representation. Then $\varphi(r) \leq 2$, which means $r = 1, 2, 3, 4$ or 6.*

Proof. This is a classical result and proved in various ways. For example, note that an automorphism of X is the composite of a group homomorphism and a translation. Since the translation has no effect on $H^1(X, \mathcal{O}_X) = \mathbb{C}$, $Im\rho$ is $\rho(Aut(X,0))$, where $Aut(X,0)$ is the group of automorphisms. Since $Aut(X,0)$ fixes the zero element of the group, it is a finite group of order 1, 2, 4 or 6 (see, for example, IV, 4.7, [4]). □

Proposition 4.9. *(i) (10.1.2, [11]) For an arbitrary $K3$-surface X, denote the order $|Im\rho|$ by r, where $\rho : Aut(X) \to GL(H^2(X, \mathcal{O}_X)) = \mathbb{C}^*$ is the induced representation. Then $\varphi(r) \leq 20$, in particular $r \leq 66$. Here φ is the Euler function.*

(ii) (3.2, [3]) For an arbitrary Abelian surface X, the order r of a finite automorphism on X satisfies $\varphi(r) \leq 4$, which means that $r = 1, 2, 3, 4, 5, 6, 8, 10, 12$.

Now one obtains a new proof of the following result.

Theorem 4.10. *A 2-dimensional strictly log-canonical singularity has index $1, 2, 3, 4$ or 6.*

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of the strictly log-canonical singularity (Z, z) and G be the associated cyclic group. By 4.5, it is sufficient to prove for the case that (X, x) is of type $(0, 1)$. Let $f : Y \to X$ be the minimal resolution and E the exceptional curve. Then f is a G-equivariant good resolution with the essential divisor E which is an elliptic curve. By 4.8, $|Im\rho| = 1, 2, 3, 4$, or 6, where $\rho : G \to GL(H^1(E, \mathcal{O}_E)) = \mathbb{C}^*$ is the induced representation. Now apply 3.6. □

Theorem 4.11. *An isolated 3-dimensional strictly log-canonical singularity of type $(0, 2)$ has index r, where $\varphi(r) \leq 20$.*

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of a 3-dimensional strictly log-canonical singularity (Z, z) and G the associated cyclic group. Let E be the exceptional divisor on the canonical model of X. Then by 2.10 E is either a normal $K3$-surface or an Abelian surface. Note that the action of G on E is lifted onto the minimal resolution $\tilde{E}$ of E. Since the singularities on E are at worst rational double, one obtains that $\Gamma(E, \omega_E) = \Gamma(\tilde{E}, \omega_{\tilde{E}})$. By the Serre duality, the action of G on $H^2(E, \mathcal{O}_E)$ is the same as the one on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$. Therefore by 3.5 and 4.9 $r = \mathrm{Ind}(Z, z)$ satisfies $\varphi(r) \leq 20$. □

Theorem 4.12. *An isolated 3-dimensional strictly log-canonical singularity of type $(0, 1)$ has index $1, 2, 3, 4$ or 6.*

4.13. For the proof of Theorem 4.12 one needs the discussion on the following divisor: Let E_J be a simple normal crossing divisor on a non-singular 3-fold. Assume $E_J = E_1 + E_2 + \ldots + E_s$ is a cycle of elliptic ruled surfaces E_i and every intersection curve is a section on the ruled surfaces. Decompose E_J into two connected chains $E^{(i)}$ $(i = 1, 2)$ with no common components. Let C_1 and C_2 be the irreducible curves of $E^{(1)} \cap E^{(2)}$. Let $p : E^{(1)} \to C$ and $q : E^{(2)} \to C$ be the rulings and $p_i : C_i \to C$ be the restriction of p on C_i. Then one obtains the Mayer-Vietoris exact sequence:

$$H^1(E^{(1)}, \mathbb{C}) \oplus H^1(E^{(2)}, \mathbb{C}) \to H^1(C_1, \mathbb{C}) \oplus H^1(C_2, \mathbb{C}) \to H^2(E_J, \mathbb{C}) \to 0,$$

which is an exact sequence of mixed Hodge structure. By taking Gr_F^0, where F is the Hodge filtration, one obtains the following:

$$H^1(E^{(1)}, \mathcal{O}) \oplus H^1(E^{(2)}, \mathcal{O}) \xrightarrow{\Phi} H^1(C_1, \mathcal{O}) \oplus H^1(C_2, \mathcal{O}) \xrightarrow{\Psi} H^2(E_J, \mathcal{O}) \to 0.$$

Lemma 4.14. *Assume that* $H^2(E_J, \mathcal{O}) = \mathbb{C}$. *Let* $\Phi|_{H^1(E^{(i)}, \mathcal{O})} = \varphi_i$ *and* $\Psi|_{H^1(C_i, \mathcal{O})} = \psi_i$. *Then the following hold:*

(i) $Im\varphi_1 = Im\varphi_2 = Im\Phi$;

(ii) ψ_i *is an isomorphism for* $i = 1, 2$ *and* $Ker\Psi \circ (p_1^* \oplus p_2^*) = \Delta$, *where* Δ *is the diagonal subspace of* $H^1(C, \mathcal{O}) \oplus H^1(C, \mathcal{O})$;

(iii) fix C_1, *then the isomorphism* ψ_1 *is independent of the choice of the decomposition of* E_J *as in* 4.13.

Proof. If (i) does not hold, then $Im\Phi \neq Im\varphi_1$, where $Im\varphi_1$ is of dimension 1, because φ_1 is a non-zero map from 1-dimensional vector space. Therefore Φ becomes surjective, a contradiction to $H^2(E_J, \mathcal{O}_{E_J}) \neq 0$. For (ii), consider the composite:

$$H^1(E^{(i)}, \mathcal{O}_{E^{(i)}}) \xrightarrow{\varphi_i} H^1(C_1, \mathcal{O}_{C_1}) \oplus H^1(C_2, \mathcal{O}_{C_2})$$

$$\xrightarrow{p_1^{*-1} \oplus p_2^{*-1}} H^1(C, \mathcal{O}_C) \oplus H^1(C, \mathcal{O}_C).$$

One obtains that $Im((p_1^{*-1} \oplus p_2^{*-1}) \circ \varphi_i) = \Delta$. Therefore ψ_i is not a zero map. For (iii), take another C_2' and $E^{(i)\prime}$ $(i = 1, 2)$ such that $E^{(1)\prime} \cap E^{(2)\prime} = C_1 \amalg C_2'$. One may assume that $C_2' \subset E^{(1)}$ and $E^{(1)\prime} \subset E^{(1)}$ and $E^{(2)} \subset E^{(2)\prime}$. Let $E^{(3)}$ be a subchain of E_J such that $E^{(1)} \cap E^{(2)\prime} = C_1 \amalg E^{(3)}$. Then C_2, $C_2' \subset E^{(3)}$. By these inclusions, it follows the commutative diagram :

$$\begin{array}{ccccccc}
H^1(E^{(1)}) \oplus H^1(E^{(2)}) & \to & H^1(C_1) \oplus H^1(C_2) & \xrightarrow{\Psi} & H^2(E_J) & \to 0 \\
\| \quad \uparrow\wr & & \| \quad \uparrow\wr & & \| & \\
H^1(E^{(1)}) \oplus H^1(E^{(2)\prime}) & \to & H^1(C_1) \oplus H^1(E^{(3)}) & \to & H^2(E_J) & \to 0 \\
\downarrow\wr \quad \| & & \| \quad \downarrow\wr & & \| & \\
H^1(E^{(1)\prime}) \oplus H^1(E^{(2)\prime}) & \to & H^1(C_1) \oplus H^1(C_2') & \xrightarrow{\Psi'} & H^2(E_J) & \to 0.
\end{array}$$

So the restrictions of Ψ and Ψ' on $H^1(C_1, \mathcal{O})$ are the same. □

Proof of Theorem 4.12. Let (Z, z) be an isolated strictly log-canonical singularity of type (0, 1), $\pi : (X, x) \to (Z, z)$ the canonical cover and G the associated cyclic group. Let $f : Y \to X$ be a G-equivariant good resolution and E_J the essential divisor. Then E_J is either as in (i) or (ii) of Proposition 2.11.

Case 1. The case that E_J is as in (ii) of Proposition 2.11.

Let $E_J = E^{(-)} + E^{(0)} + E^{(+)}$ be the decomposition as in (ii). Then there is a ruling $p : E^{(0)} \to C$ over an elliptic curve C. Since each fiber of p is mapped to a fiber of p by the action of G, C admits the action of G and p becomes a G-equivariant morphism. Now by Mayer-Vietoris exact sequence:

$$H^1(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^1(E^{(0)} + E^{(+)}, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O})$$

$$\to H^2(E_J, \mathcal{O}) \to H^2(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^2(E^{(0)} + E^{(+)}, \mathcal{O}) = 0,$$

one obtains a G-equivariant isomorphism $H^1(E^{(0)}, \mathcal{O}) \simeq H^2(E_J, \mathcal{O})$. On the other hand there is a G-equivariant isomorphism $p^* : H^1(C, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O})$. Since the action of G on $H^1(C, \mathcal{O})$ is induced from that on C, the order of the action on G on $H^1(C, \mathcal{O})$ is 1, 2, 3, 4, 6 by Proposition 4.8.

Case 2. The case that E_J is as in (i) of Proposition 2.11.

If the intersection curves are all fixed under the action of G, the generater g of G induces an automorphism of each intersection curve. Take C_i and $E^{(i)}$ $(i = 1, 2)$ as in 4.13. Then one obtains the commutative diagram of isomorphisms:

$$\begin{array}{ccc} H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J) \\ g|_{C_1}^* \downarrow & & \downarrow g^* \\ H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J). \end{array}$$

Since $g|_{C_1}^*$ is of order 1, 2, 3, 4, 6 by Proposition 4.8, so is g^*.

If $g(C_1) = C_2$ for $C_1 \neq C_2$, then under the notation in 4.13 let $h : C \to C$ be an automorphism $p_2 \circ g|_{C_1} \circ p_1^{-1}$. By the definition of h, it follows the commutative diagram of isomorphisms:

$$\begin{array}{ccccc} H^1(C) & \xrightarrow{p_2^*} & H^1(C_2) & \xrightarrow{\psi_2'} & H^2(E_J) \\ \downarrow h^* & & g|_{C_1}^* \downarrow & & \downarrow g^* \\ H^1(C) & \xrightarrow{p_1^*} & H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J), \end{array}$$

where ψ_2' is induced from ψ_1 through g. Here, note that $H^2(E_J, \mathcal{O}) = \mathbb{C}$ by the assumption of the singularity. So one can apply Lemma 4.14, (iii) and obtains that $\psi_2' = \psi_2$. On the other hand, as $Ker\Psi \circ (p_1^* \oplus p_2^*) = \Delta$ by Lemma 4.14, (ii), it follows that $\psi_1 \circ p_1^* = -\psi_2 \circ p_2^*$. Hence, by the diagram above, the order of g^* is 1, 2, 3, 4, 6 since that of h^* is 1, 2, 3, 4, 6 by 4.8. □

Theorem 4.15. *For a positive integer r the following are equivalent:*
(i) r is the index of a 3-dimensional strictly log-canonical singularity;
(ii) $\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function.

Proof. First assume (i), then by theorems 4.5, 4.11 and 4.12, it follows that $\varphi(r) \leq 20$. If there exists a 3-dimensional strictly log-canonical singularity (Z, z) of index 60. Then by 4.5 and 4.12, (Z, z) must be of type $(0, 2)$. Let E be the exceptional divisor on the canonical model of the canonical cover (X, x), then E is normal $K3$-surface. Let G be the corresponding group of the canonical cover, then G acts on E whose induced action on $H^2(E, \mathcal{O}_E)$ is of order 60. Since this action is lifted to the minimal resolution $\tilde{E}$ of E, one obtains a $K3$-surface $\tilde{E}$ which admits an automorphism whose action on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$ is of order 60. However, it is proved by Machida-Oguiso [10] that there is no $K3$-surface with such an automorphism.

Next assume (ii), then by [8] and [12], there is a $K3$-surface E with an automorphism $g : E \to E$ whose order and the order of induced automorphism on $H^2(E, \mathcal{O}_E)$ are both r. Let $G =< g >$, $\pi : E \to E' = E/G$ the quotient map and $\mathcal{L}$ an ample invertible sheaf on E'. Let Y' and Y be the line bundles $Spec \bigoplus_{m \geq 0} \mathcal{L}^{\otimes m}$ and $Spec \bigoplus_{m \geq 0} \pi^* \mathcal{L}^{\otimes m}$ on E' and on E, respectively. Then $Y \to E$ has the zero section E_0 whose normal bundle is $\pi^* \mathcal{L}^{-1}$, so there is a contraction $f : (Y, E_0) \to (X, x)$ of E_0. Since the exceptional divisor E_0 is $K3$-surface, the singularity (X, x) is strictly log-canonical of index 1 and of type $(0, 2)$ by [7]. One defines an action of G on (X, x) in the following way: Let σ be the action of G on E. On the other hand there is also an action τ of G on Y' which is trivial on E', because Y' admits a canonical action of $\mathbb{C}^*$ and G is considered as a subgroup of $\mathbb{C}^*$. Since Y is the fiber product $E \times_{E'} Y'$, one obtains the action of G on Y which is compatible with σ and τ. It is clear that this action is free on $Y \setminus E_0$ and E_0 is G-invariant. Therefore one can introduce the action of G on (X, x). The quotient $(Z, z) = (X, x)/G$ is strictly log-canonical of index r by Corollary 3.6. □

4.16. Boundedness of indices of higher dimensional strictly log-canonical singularities is also expected to follow from Conjecture 4.7. On the contrary, if indices of n-dimensional strictly log-canonical singularities are bounded, then Conjecture 4.7 holds for $(n-1)$-dimensional Calabi-Yau varieities. Indeed, as in the proof of Theorem 4.15, for every Calabi-Yau $(n-1)$-fold E and a finite order automorphism g, one can construct a strictly log-canonical singularity of index r, where r is the order of the induced automorphism g^* on $H^{n-1}(E, \mathcal{O}_E)$. Hence the boundedness of indices implies Conjecture 4.7.

References

1. M. Demazure, *Anneaux Gradués normaux*, Introduction a la théorie des singularités II; Methodes algebriques et géométriques, Travaux En Cours **37**, (1988) 35-68 Paris Herman.
2. A. Fujiki, *On the blowing down of analytic spaces*, Publ. RIMS, Kyoto Univ. **10**, (1975) 473-507.

3. A. Fujiki, *Finite automorphism groups of complex tori of dimension two,* Publ. RIMS, Kyoto Univ. **24**, (1988) 1-97.
4. R. Hartshorne, *Algebraic Geometry,* Graduate Text of Mathematics **52**, (1977) Springer-Verlag, New York-Heidelberg-Berlin.
5. S. Ishii, *On isolated Gorenstein singularities*, Math. Ann. **270**, (1985) 541-554.
6. S. Ishii, *The quotients of log-canonical singularities by finite groups*, to appear in Adv. Studies in Math.
7. S. Ishii & K. Watanabe, *A geometric characterization of a simple* $K3$*-singularity*, Tohoku Math. J. **44**, (1992) 19-24.
8. S. Kondo, *Automorphisms of algebraic* $K3$*-surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44**, (1992) 75-98
9. V. Kulikov, *Degenerations of* $K3$*-surfaces and Enriques surfaces*, Math. USSR Izv. **11**, (1977) 957-989.
10. N. Machida & K. Oguiso, *On* $K3$*-surfaces admitting finite non-symplectic group actions*, J. Math. Sci. Univ. Tokyo **5**, (1998) 273-297
11. V.V. Nikulin, *Factor groups of automorphisms of hyperbolic forms with respect to subgroups generated by* 3*-reflections. Algebrogeometric applications*, J. Soviet Math., **22**, (1983) 1401-1476.
12. K. Oguiso, *A remark on the global indices of* $\mathbb{Q}$*-Calabi-Yau* 3*-folds*, Math. Proc. Camb. Phil. Soc. **114**, (1993) 427-429
13. T. Okuma, *The pluri-genera of surface singularities*, Tohoku Math. J. **50**, (1998) 119-132.
14. V.V. Shokurov, 3*-fold log flips.* Russian Acad. Sci. Izv. Math. **40**, (1993) 95-202.
15. V.V. Shokurov, *Complement on surfaces*, to appear in J. of Math. Sci.
16. Y. Umezu, *On normal projective surfaces with trivial dualizing sheaf*, Tokyo J. Math. **4**, (1981) 343-354.

Department of Mathematics, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo