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# STRONG $n$-SHAPE THEORY

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## INTRODUCTION

Let $\mu^{n+1}$ be the $(n+1)$-dimensional universal Menger compactum. In [Chi$_1$], A. Chigogidze introduced the concept of $n$-shape and established the $(n+1)$-dimensional analogue of Chapman's complement theorem [Cha, Theorem 2], that is, two $Z$-sets $X$ and $Y$ in $\mu^{n+1}$ have the same $n$-shape type if and only if their complements $\mu^{n+1}\setminus X$ and $\mu^{n+1}\setminus Y$ are homeomorphic ($\approx$), where $X\subset M$ is a *$Z$-set* in $M$ if there are maps $f\colon M\to M\setminus X$ arbitrarily close to $\mathrm{id}_M$. The $n$-shape category of compacta was discussed in [Chi$_2$] (cf. [Chi$_3$]). Later, corresponding to [Cha, Theorem 1], Y. Akaike [Aka] defined the weak proper $n$-homotopy category of complements of $Z$-sets in $\mu^{n+1}$ which is isomorphic to the $n$-shape category of $Z$-sets in $\mu^{n+1}$. Then, as Strong Shape Theory ([EH], [DS], [KO], etc.), it is a natural attempt to define the strong $n$-shape category which corresponds to the proper $n$-homotopy category of complements of $Z$-sets in $\mu^{n+1}$. Properly, one require this category to factorize the natural functor (called the *$n$-shape functor*) from the $n$-homotopy category to the $n$-shape category into two functors through it. In this paper, we introduce the $(n+1)$-skeletal conic telescope to define the strong $n$-shape category of compacta.

Throughout the paper, spaces are separable metrizable and maps are continuous. It is said that two (proper) maps $f,g\colon X\to Y$ are *(properly) $n$-homotopic relative to $A\subset X$* and denoted by $f\simeq^n g$ rel. $A$ ($f\simeq^n_p g$ rel. $A$) if, for any (proper) map $\varphi\colon Z\to X$, there is a (proper) homotopy $h\colon Z\times\mathbf{I}\to Y$ such that $h_0=f\varphi$, $h_1=g\varphi$ $h_t|\varphi^{-1}(A)=f\varphi|\varphi^{-1}(A)$ for each $t\in\mathbf{I}$. When $A=\emptyset$, we say that $f$ and $g$ are *(properly) $n$-homotopic* and denote $f\simeq g$ ($f\simeq_p g$).

A map $\varphi\colon M\to X$ is said to be *$n$-invertible* if any map $\psi\colon Z\to X$ of a space $Z$ with $\dim Z\leqslant n$ lifts to $M$, that is, there exists a map $\tilde\psi\colon Z\to M$ such that $\varphi\tilde\psi=\psi$. In case $\varphi$ is a proper map, if $\psi$ is proper then $\tilde\psi$ is also proper. For an $n$-invertible map $\varphi\colon M\to X$ and $A\subset X$, $\varphi|\varphi^{-1}(A)\colon\varphi^{-1}(A)\to A$ is also $n$-invertible. By the result of Dranishnikov [Dra, Theorem 1], for any compactum $X$, there exists an $n$-invertible map $\varphi\colon M\to X$ of a compactum $M$ with $\dim M\leqslant n$. Then, for two (proper) maps $f,g\colon X\to Y$, $f\simeq^n g$ rel. $A$ ($f\simeq^n_p g$ rel. $A$) if and only if $f\varphi\simeq g\varphi$ rel. $\varphi^{-1}(A)$ ($f\varphi\simeq_p g\varphi$ rel. $\varphi^{-1}(A)$) for an invertible (proper) map $\varphi\colon M\to X$.

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## 1. The polyhedral telescope

The $n$-skeleton of a simplicial complex $K$ is denoted by $K^{(n)}$, whence $K^{(0)}$ is the set of vertices of $K$. The polyhedron of $K$ is denoted by $|K|$ (i.e., $|K| = \bigcup_{\sigma\in K}\sigma$). By $\langle v_1,\dots,v_n\rangle$, we denote the simplex with vertices $v_1,\dots,v_n$. A subdivision $\delta K$ of $K$ induces the subdivision $\delta K^{(n)}$ of $K^{(n)}$. It should be remarked that $\delta K^{(n)} \subset (\delta K)^{(n)}$ but $\delta K^{(n)} \neq (\delta K)^{(n)}$ in general. The following is well known:

**Fact 1.** *Let $L$ be a subcomplex of $K$ and $Z$ a space with* $\dim Z \leqslant n$. *Then, for any map $\varphi\colon Z \to |K|$, there is a map $\psi\colon Z \to |K^{(n)} \cup L|$ such that $\varphi \simeq \psi$* rel. $\varphi^{-1}(|K^{(n)} \cup L|)$.

An ordered simplicial complex is a simplicial complex with an order of vertices such that the set of vertices of each simplex is totally ordered. The barycentric subdivision $\operatorname{Sd} K$ of a simplicial complex $K$ is an ordered simplicial complex with the following order:

$$\hat\sigma \leqslant \hat\tau \quad \underset{\text{def}}{\iff} \quad \sigma \text{ is a face of } \tau,$$

where $\hat\sigma$ is the barycenter of $\sigma$.

Let $I = \{0, 1, \mathbf{I}\}$ be the natural triangulation of the unit interval $\mathbf{I} = [0,1]$. Then, $I$ is an ordered simplicial complex with the natural order $0 < 1$. For an ordered simplicial complex $K$, the product simplicial complex $K \times I$ is defined as follows:

$$\begin{aligned} K\times I = \big\{\sigma\times\{0\},\ \sigma\times\{1\} \bigm| \sigma\in K\big\} \\ \cup \big\{\langle (v_1,0),\dots,(v_i,0),(v_j,1),\dots,(v_k,1)\rangle \bigm| \langle v_1,\dots,v_k\rangle \in K \\ v_1<\dots<v_k \in K^{(0)},\ 1\leqslant i\leqslant j\leqslant k\big\}. \end{aligned}$$

Then $K\times I$ is an ordered simplicial complex with the following order on $(K\times I)^{(0)} = K^{(0)} \times \{0,1\}$:

$$(v,i) \leqslant (v',i') \quad \underset{\text{def}}{\iff} \quad v\leqslant v' \text{ and } i \leqslant i'.$$

Let $K$ and $L$ be ordered simplicial complexes and $f\colon K \to L$ a simplicial map. The simplicial mapping cylinder $M(f)$ is defined as follows:

$$\begin{aligned} M(f) = K\cup L\cup \big\{\langle f(v_1),\dots,f(v_i),v_j,\dots,v_k\rangle \bigm| \\ \langle v_1,\dots,v_k\rangle \in K,\ v_1<\dots<v_k,\ 1\leqslant i\leqslant j\leqslant k\big\}. \end{aligned}$$

When $L$ is degenerate (i.e., a singleton), $M(f)$ is the simplicial cone $C(K)$ over $K$. We have the natural simplicial map $q_f\colon K\times I \to M(f)$ which is naturally defined by $q_f(v,0) = f(v)$ and $q_f(v,1) = v$ for $v \in K^{(0)}$. The simplicial collapsing map $c_f\colon M(f) \to L$ is defined by $c_f(v) = f(v)$ for $v\in K^{(0)}$ and $c_f(u) = u$ for $u \in L^{(0)}$. Then $c_f q_f = f\operatorname{pr}_X$ and $c_f \simeq \mathrm{id}$ rel. $|L|$ in $|M(f)|$. Extending the orders on $K^{(0)}$ and $L^{(0)}$ to $M(f)^{(0)} = K^{(0)} \cup L^{(0)}$ so that $u < v$ for each $u \in L^{(0)}$ and $v \in K^{(0)}$, $M(f)$ is an ordered simplicial complex. Let $f^{(n)} = f|K^{(n)}\colon K^{(n)} \to L^{(n)}$ be the restriction of $f$. Observe that

$$M(f)^{(n)} \subset M(f^{(n)}) \subset M(f)^{(n+1)} \subset M(f^{(n)}) \cup K \cup L$$

and $c_f|M(f^{(n)}) = c_{f^{(n)}} \simeq \mathrm{id}$ rel. $|L^{(n)}|$ in $|M(f^{(n)})|$.

**Fact 2.** *For a simplicial map $f\colon K \to L$, $c_f||M(f)^{(n+1)} \cup K \cup L| \simeq^n \mathrm{id}$ rel. $|L|$ in $|M(f)^{(n+1)} \cup K \cup L|$, hence $f = c_f|K \simeq^n \mathrm{id}_K$ in $|M(f)^{(n+1)} \cup K \cup L|$.*

Since $K \times I$ can be regarded as $M(\mathrm{id}_K)$, we have the following:

**Fact 3.** *Let $p\colon |(K^{(n)} \times I) \cup (K \times \{0,1\})| \to |K \times \{0\}|$ be the retraction defined by $p(x,t) = (x,0)$. Then, $p \simeq^n \mathrm{id}$ rel. $|K \times \{0\}|$ in $|(K^{(n)} \times I) \cup (K \times \{0,1\})|$, where we identify $K = K \times \{0\}$.*

Let $\mathbf{K} = (|K_i|, q_{i,i+1})_{i\in\mathbb{N}}$ be an inverse sequence of ordered simplicial complexes such that each $q_{i,i+1}\colon K_{i+1} \to \delta K_i$ is simplicial, where $\delta K_i$ is some subdivision of $K_i$. Let $q_i\colon \varprojlim \mathbf{K} \to |K_i|$ be the projection of the inverse limit of $\mathbf{K}$ to $|K_i|$ and denote

$$q_{i,j} = q_{i,i+1}\circ\cdots\circ q_{j-1,j}\colon |K_j| \to |K_i|,\ i < j.$$

We define

$$\mathrm{Tel}_{[j,\infty)}(\mathbf{K}) = \bigcup_{i=j}^{\infty} |M(q_{i,i+1})| \quad \text{and} \quad \mathrm{Tel}_{[j,k]}(\mathbf{K}) = \bigcup_{i=j}^{k-1} |M(q_{i,i+1})|,\ j < k,$$

where $|M(q_{i,i+1})| \cap |M(q_{i+1,i+2})| = |K_{i+1}|$ and $|M(q_{i,i+1})| \cap |M(q_{j,j+1})| = \emptyset$ for $|i-j| > 1$. The polyhedron $\mathrm{Tel}_{[1,\infty)}(\mathbf{K})$ is called the *polyhedral telescope* for $\mathbf{K}$. One should note that $\bigcup_{i=1}^{\infty} M(q_i)$ is not a simplicial complex unless $\delta K_i = K_i$ for every $i \in \mathbb{N}$. Let

$$\mathrm{Tel}_{[0,\infty)}(\mathbf{K}) = |C(K_1)| \cup \mathrm{Tel}_{[1,\infty)}(\mathbf{K}) \quad \text{and} \quad \mathrm{Tel}_{[0,k]}(\mathbf{K}) = |C(K_1)| \cup \mathrm{Tel}_{[1,k]}(\mathbf{K}),$$

where $|C(K_1)| \cap \mathrm{Tel}_{[1,\infty)}(\mathbf{K}) = |K_1|$. We call $\mathrm{Tel}_{[0,\infty)}(\mathbf{K})$ the *polyhedral conic telescope*.

The simplicial collapsing map $c_{q_{i,i+1}}\colon M(q_{i,i+1}) \to \delta K_i$ extends to the deformation retraction

$$c^{\mathbf{K}}_{i,i+1}\colon \mathrm{Tel}_{[0,i+1]}(\mathbf{K}) = \mathrm{Tel}_{[0,i]}(\mathbf{K}) \cup |M(q_{i,i+1})| \to T_{[0,i]}(\mathbf{K}).$$

The following diagram is commutative:

$$\begin{array}{ccccccc}
\mathrm{Tel}_{[0,1]}(\mathbf{K}) & \xleftarrow[\subset]{c^{\mathbf{K}}_{1,2}} & \mathrm{Tel}_{[0,2]}(\mathbf{K}) & \xleftarrow[\subset]{c^{\mathbf{K}}_{2,3}} & \mathrm{Tel}_{[0,3]}(\mathbf{K}) & \xleftarrow[\subset]{c^{\mathbf{K}}_{3,4}} & \cdots \\
\cup & & \cup & & \cup & & \cdots \\
|K_1| & \xleftarrow[q_{1,2}]{} & |K_2| & \xleftarrow[q_{2,3}]{} & |K_3| & \xleftarrow[q_{3,4}]{} & \cdots.
\end{array}$$

The inverse limit of the upper sequence is denoted by $\mathrm{Tel}_{[0,\infty]}(\mathbf{K})$ with the projection $c^{\mathbf{K}}_i\colon \mathrm{Tel}_{[0,\infty]}(\mathbf{K}) \to \mathrm{Tel}_{[0,i]}(\mathbf{K})$. We denote

$$c^{\mathbf{K}}_{i,j} = c^{\mathbf{K}}_{i,i+1}\circ\cdots\circ c^{\mathbf{K}}_{j-1,j}\colon \mathrm{Tel}_{[0,j]}(\mathbf{K}) \to \mathrm{Tel}_{[0,i]}(\mathbf{K}),\ i < j.$$

Regarding $\mathrm{Tel}_{[0,\infty)}(\mathbf{K})$ as an open subspace of $\mathrm{Tel}_{[0,\infty]}(\mathbf{K})$, we have

$$\mathrm{Tel}_{[0,\infty]}(\mathbf{K}) \setminus \mathrm{Tel}_{[0,\infty)}(\mathbf{K}) = \varprojlim \mathbf{K} \quad \text{and} \quad c^{\mathbf{K}}_i | \varprojlim \mathbf{K} = q_i,\ i \in \mathbb{N}.$$

It is easy to see that each $c_i^{\mathbf{K}}$ is a strong deformation retraction. Hence, it follows that $\mathrm{Tel}_{[0,\infty)}(\mathbf{K})$ is homotopy dense in $\mathrm{Tel}_{[0,\infty]}(\mathbf{K})$, that is, there is a homotopy $h\colon \mathrm{Tel}_{[0,\infty]}(\mathbf{K}) \times \mathbf{I} \to \mathrm{Tel}_{[0,\infty]}(\mathbf{K})$ such that $h_0 = \mathrm{id}$ and $h_t(\mathrm{Tel}_{[0,\infty]}(\mathbf{K})) \subset \mathrm{Tel}_{[0,\infty)}(\mathbf{K})$ for $t > 0$. Since $\mathrm{Tel}_{[0,\infty)}(\mathbf{K})$ is a polyhedron, $\mathrm{Tel}_{[0,\infty]}(\mathbf{K})$ is an ANR by Hanner's characterization of ANR's (cf. [Hu]). Since $\mathrm{Tel}_{[0,\infty]}(\mathbf{K})$ is contractible, it is an AR. The above construction was founded in [Ko, Theorem 1 and Corollary 1]. For each $j \in \mathbb{N}$, we can similarly define $\mathrm{Tel}_{[j,\infty]}(\mathbf{K})$, which is an ANR and a closed subspace of $\mathrm{Tel}_{[0,\infty]}(\mathbf{K})$. Clearly,

$$\mathrm{Tel}_{[j,\infty]}(\mathbf{K}) \setminus \mathrm{Tel}_{[j,\infty)}(\mathbf{K}) = \mathrm{Tel}_{[0,\infty]}(\mathbf{K}) \setminus \mathrm{Tel}_{[0,\infty)}(\mathbf{K}) = \varprojlim \mathbf{K}.$$

Each $d_j^{\mathbf{K}} = c_j^{\mathbf{K}} | \mathrm{Tel}_{[j,\infty]}(\mathbf{K})\colon \mathrm{Tel}_{[j,\infty]}(\mathbf{K}) \to |K_j|$ is a strong deformation retraction and $q_{i,j} d_j^{\mathbf{K}} = d_i^{\mathbf{K}} | \mathrm{Tel}_{[j,\infty]}(\mathbf{K})$.

Now, we define

$$\mathrm{Tel}^{n+1}_{[j,\infty)}(\mathbf{K}) = \bigcup_{i=j}^{\infty} |K_i| \cup \bigcup_{i=j}^{\infty} \left| M(q_{i,i+1})^{(n+1)} \right| \quad \text{and}$$
$$\mathrm{Tel}^{n+1}_{[j,k]}(\mathbf{K}) = \bigcup_{i=j}^{k} |K_i| \cup \bigcup_{i=j}^{k-1} \left| M(q_{i,i+1})^{(n+1)} \right|, \ j < k.$$

These are subpolyhedra of $\mathrm{Tel}_{[1,\infty)}(\mathbf{K})$. Recall that $\bigcup_{i=1}^{\infty} M(q_i)$ is not a simplicial complex in general. We call $\mathrm{Tel}^{n+1}_{[1,\infty)}(\mathbf{K})$ the $(n+1)$-*skeletal telescope* for $\mathbf{K}$. Let

$$\mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) = \left| C(K_1)^{(n+1)} \right| \cup \mathrm{Tel}^{n+1}_{[1,\infty)}(\mathbf{K}) \quad \text{and}$$
$$\mathrm{Tel}^{n+1}_{[0,k]}(\mathbf{K}) = \left| C(K_1)^{(n+1)} \right| \cup \mathrm{Tel}^{n+1}_{[1,k]}(\mathbf{K}).$$

These are $n$-connected. The polyhedron $\mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K})$ is called the $(n+1)$-*skeletal conic telescope* for $\mathbf{K}$.

Observe that $c_i^{\mathbf{K}}(\mathrm{Tel}^{n+1}_{[0,i+1]}(\mathbf{K})) = \mathrm{Tel}^{n+1}_{[0,i]}(\mathbf{K})$. The following diagram is commutative:

$$\begin{array}{ccccccc}
\mathrm{Tel}^{n+1}_{[0,1]}(\mathbf{K}) & \xleftarrow[\subset]{c^{\mathbf{K}}_{1,2}|} & \mathrm{Tel}^{n+1}_{[0,2]}(\mathbf{K}) & \xleftarrow[\subset]{c^{\mathbf{K}}_{2,3}|} & \mathrm{Tel}^{n+1}_{[0,3]}(\mathbf{K}) & \xleftarrow[\subset]{c^{\mathbf{K}}_{3,4}|} & \cdots \\
\cup & & \cup & & \cup & & \cdots \\
|K_1| & \xleftarrow[q_{1,2}]{} & |K_2| & \xleftarrow[q_{2,3}]{} & |K_3| & \xleftarrow[q_{3,4}]{} & \cdots.
\end{array}$$

Then the inverse limit of the upper sequence is the closed subspace

$$\mathrm{Tel}^{n+1}_{[0,\infty]}(\mathbf{K}) = \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) \cup \varprojlim \mathbf{K} \subset \mathrm{Tel}_{[0,\infty]}(\mathbf{K}).$$

For each $j \in \mathbb{N}$, let $\mathrm{Tel}^{n+1}_{[j,\infty]}(\mathbf{K}) = \mathrm{Tel}^{n+1}_{[j,\infty)}(\mathbf{K}) \cup \varprojlim \mathbf{K}$.

**Fact 4.** *For each* $j \in \mathbb{N} \cup \{0\}$, $\mathrm{Tel}^{n+1}_{[j,\infty]}(\mathbf{K}) \setminus \mathrm{Tel}^{n+1}_{[j,\infty)}(\mathbf{K}) = \varprojlim \mathbf{K}$ *is a* $Z$*-set in* $\mathrm{Tel}^{n+1}_{[j,\infty]}(\mathbf{K})$.

Let $\psi\colon Z \to \mathrm{Tel}^{n+1}_{[j,\infty]}(\mathbf{K})$ be a map of a space $Z$ with $\dim Z \leqslant n$. Then it is easy to construct a homotopy $h\colon Z \times \mathbf{I} \to \mathrm{Tel}^{n+1}_{[j,\infty]}(\mathbf{K})$ such that $h_0 = \psi$ and $h_t(Z) \subset \mathrm{Tel}^{n+1}_{[j,\infty)}(\mathbf{K})$ for $t > 0$. In general, $\mathrm{Tel}^{n+1}_{[0,\infty]}(\mathbf{K})$ is not an ANR, but we have the following:

**Fact 5.** *Each* $\mathrm{Tel}^{n+1}_{[j,\infty]}(\mathbf{K})$ *is* $LC^n$*, hence it is an ANE*$(n+1)$*. Moreover, the space* $\mathrm{Tel}^{n+1}_{[0,\infty]}(\mathbf{K})$ *is* $n$*-connected, so it is an AE*$(n+1)$*.*[1]

The following follows from Fact 2:

**Fact 6.** *For* $i < j \in \mathbb{N} \cup \{0\}$, $d^{\mathbf{K}}_{i,j} | \mathrm{Tel}^{n+1}_{[i,j]}(\mathbf{K}) \simeq^n \mathrm{id}$ *in* $\mathrm{Tel}^{n+1}_{[i,j]}(\mathbf{K})$*, hence* $q_{i,j} \simeq^n \mathrm{id}_{K_j}$ *in* $\mathrm{Tel}^{n+1}_{[i,j]}(\mathbf{K})$*. Moreover,* $d^{\mathbf{K}}_i | \mathrm{Tel}^{n+1}_{[i,\infty]}(\mathbf{K}) \simeq^n \mathrm{id}$ *in* $\mathrm{Tel}^{n+1}_{[i,\infty]}(\mathbf{K})$*, so* $q_i \simeq^n \mathrm{id}_{K_j}$ *in* $\mathrm{Tel}^{n+1}_{[i,j]}(\mathbf{K})$.

## 2. The strong $n$-shape category $\mathrm{Sh}^n_S$

Let $\mathcal{H}^n$ be the $n$-homotopy category of compacta and $\mathrm{Sh}^n$ the $n$-shape category of compacta. In this section, we define the strong $n$-shape category $\mathrm{Sh}^n_S$ of compacta and show that the $n$-shape functor from $\mathcal{H}^n$ to $\mathrm{Sh}^n$ is factorized into two functors through the category $\mathrm{Sh}^n_S$.

Every compactum $X$ is the limit of an inverse sequence $\mathbf{K} = (K_i, q_i)_{i\in\mathbb{N}}$ of finite simplicial complexes such that each $q_{i,i+1}\colon K_{i+1} \to \mathrm{Sd}\, K_i$ is simplicial for the barycentric subdivision $\mathrm{Sd}\, K_i$ of $K_i$ and $\dim K_i \leqslant \dim X$ for all $i \in \mathbb{N}$ [Isb, Lemma 33] (cf. Proof of [Ko$_2$, Theorem 1]). We call $\mathbf{K}$ a *barycentric sequence associated with* $X$. It should be noted that $q_{i,i+1}\colon K_{i+1} \to K_i$ is not simplicial in general. In fact, there exists a 1-dimensional compact AR which is not the limit of any inverse sequence of simplicial complexes and *simplicial* maps [Ko$_1$, Theorem 1(2)] (cf. [Ko$_2$, p.536]). It should be also noted that a barycentric sequence associated with $X$ is an $LC^n(n+1)$-sequence associated with $X$ (cf. [Chi$_2$]).

**Theorem 1.** *Let* $X$ *and* $Y$ *be compacta and* $\mathbf{K}$, $\mathbf{L}$ *be barycentric sequences associated with* $X$ *and* $Y$*, respectively.*

(1) *Every map* $f\colon X \to Y$ *extends to a map* $\bar{f}\colon \mathrm{Tel}_{[0,\infty]}(\mathbf{K}) \to \mathrm{Tel}_{[0,\infty]}(\mathbf{L})$ *such that* $\bar{f}(\mathrm{Tel}^k_{[0,\infty)}(\mathbf{K})) \subset \mathrm{Tel}^k_{[0,\infty)}(\mathbf{L})$ *for each* $k \in \mathbb{N}$.

(2) *For two maps* $f, g\colon \mathrm{Tel}^{n+1}_{[0,\infty]}(\mathbf{K}) \to \mathrm{Tel}^{n+1}_{[0,\infty]}(\mathbf{L})$ *with* $f^{-1}(Y) = g^{-1}(Y) = X$*, if* $f|X \simeq^n g|X$ *in* $Y$ *then* $f|\,\mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) \simeq^n_p g|\,\mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K})$ *in* $\mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L})$.

In Theorem 1(1) above, a proper map $\bar{f}|\,\mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K})\colon \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) \to \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L})$ is said to be *induced* by $f$. By Theorem 1(2), the proper homotopy class of such a map is unique. The following is a direct consequence of Theorem 1.

[1] A space $Y$ is an AE$(n+1)$ (or an ANE$(n+1)$) if every map of any closed set $A$ in an arbitrary metrizable space $X$ with $\dim X \leqslant n+1$ extends over $X$ (or a neighborhood of $A$). A space $Y$ is an AE$(n+1)$ if and only if $Y$ is an $n$-connected ANE$(n)$, and $Y$ is an ANE$(n+1)$ if and only if $Y$ is $LC^n$.

**Corollary 1.** *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be barycentric sequences associated with the same compactum* $X$. *Then a proper map* $h\colon \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) \to \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L})$ *induced by* $\mathrm{id}_X$ *is a proper* $n$*-homotopy equivalence.*

**Definition of** $\mathrm{Sh}^n_S$**.** Let $X$ and $Y$ be compacta. Let $\mathbf{K}$, $\mathbf{K}'$ be barycentric sequences associated with $X$ and $\mathbf{L}$, $\mathbf{L}'$ barycentric sequences associated with $Y$. Two proper maps $F\colon \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) \to \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L})$ and $F'\colon \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}') \to \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L}')$ are *$n$-fundamentally equivalent* (written by $F \simeq^n_f F'$) if $h'F \simeq^n_p F'h$ for some proper $n$-homotopy equivalences $h\colon \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) \to \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}')$ and $h'\colon \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L}') \to \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L})$ induced by $\mathrm{id}_X$ and $\mathrm{id}_Y$, respectively. A *strong $n$-shape morphism* from $X$ to $Y$ is the $n$-fundamentally equivalence class of a proper map $F\colon \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}) \to \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{L})$, where $\mathbf{K}$ and $\mathbf{L}$ are barycentric sequences associated with $X$ and $Y$ respectively. Thus, the strong $n$-shape category $\mathrm{Sh}^n_S$ of compacta can be defined.

The following follows immediately from Theorem 1 and the definition above.

**Corollary 2.** *There exists a functor* $\Xi\colon \mathcal{H}^n \to \mathrm{Sh}^n_S$ *which maps objects identically.*

For simplicity, let us assign each compactum $X$ to a barycentric sequence $\mathbf{K}^X = (K^X_i, q^X_{i,i+1})_{i\in\mathbb{N}}$ associated with $X$ and denote as follows:

$$\mathrm{Tel}^{n+1}_{[0,\infty)}(X) = \mathrm{Tel}^{n+1}_{[0,\infty)}(\mathbf{K}^X),\ \mathrm{Tel}^{n+1}_{[j,k]}(X) = \mathrm{Tel}^{n+1}_{[j,k]}(\mathbf{K}^X),$$
$$c^X_{i,i+1} = c^{\mathbf{K}^X}_{i,i+1} | \mathrm{Tel}^{n+1}_{[0,i+1]}(\mathbf{K}^X),\ c^X_i = c^{\mathbf{K}^X}_i | \mathrm{Tel}^{n+1}_{[0,\infty]}(\mathbf{K}^X),$$
$$d^X_i = d^{\mathbf{K}^X}_i | \mathrm{Tel}^{n+1}_{[i,\infty]}(\mathbf{K}^X),\ \text{etc.}$$

Thus, $X$ is assigned to the following commutative diagram of inverse sequences:

$$\begin{array}{ccccccc}
\mathrm{Tel}^{n+1}_{[0,1]}(X) & \xleftarrow[\subset]{c^X_{1,2}} & \mathrm{Tel}^{n+1}_{[0,2]}(X) & \xleftarrow[\subset]{c^X_{2,3}} & \mathrm{Tel}^{n+1}_{[0,3]}(X) & \xleftarrow[\subset]{c^X_{3,4}} & \cdots \\
\cup & & \cup & & \cup & & \cdots \\
|K^X_1| & \xleftarrow[q^X_{1,2}]{} & |K^X_2| & \xleftarrow[q^X_{2,3}]{} & |K^X_3| & \xleftarrow[q^X_{3,4}]{} & \cdots .
\end{array}$$

Now, we prove the following:

**Theorem 2.** *There exists a full*[2] *functor* $\Theta\colon \mathrm{Sh}^n_S \to \mathrm{Sh}^n$ *such that* $\Theta\circ\Xi\colon \mathcal{H}^n \to \mathrm{Sh}^n$ *is the* $n$*-shape functor.*

*Remarks.* The following proposition can be proved similarly to Theorem 1(1).

**Proposition.** *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be barycentric sequences associated with compacta* $X$ *and* $Y$*, respectively. Every proper map* $f\colon \mathrm{Tel}_{[0,\infty)}(\mathbf{K}) \to \mathrm{Tel}_{[0,\infty)}(\mathbf{L})$ *is properly homotopic to a proper map* $\bar{f}\colon \mathrm{Tel}_{[0,\infty)}(\mathbf{K}) \to \mathrm{Tel}_{[0,\infty)}(\mathbf{L})$ *such that* $\bar{f}(\mathrm{Tel}^k_{[0,\infty)}(\mathbf{K})) \subset \mathrm{Tel}^k_{[0,\infty)}(\mathbf{L})$ *for each* $k \in \mathbb{N}$.

By the same proof, Theorem 1(2) is valid even if $\mathrm{Tel}^{n+1}_{[0,\infty]}$ is replaced with $\mathrm{Tel}_{[0,\infty]}$. Then, in the definition of $\mathrm{Sh}^n_S$, replacing $\mathrm{Tel}^{n+1}_{[0,\infty)}$ by $\mathrm{Tel}_{[0,\infty)}$, we can define the

[2]The functor is *full* if the induced maps of the sets of morphisms are surjective.

category $\overline{\mathrm{Sh}}^n_S$ which factorizes the $n$-shape functor into two functors through $\overline{\mathrm{Sh}}^n_S$. In fact, the functor $\Xi$ in Corollary 2 is factorized into two natural functors through $\overline{\mathrm{Sh}}^n_S$, where the natural functor from $\overline{\mathrm{Sh}}^n_S$ to $\mathrm{Sh}^n_S$ can be obtained by the proposition above. As is easily observed, the functor from $\overline{\mathrm{Sh}}^n_S$ to $\mathrm{Sh}^n_S$ is injective, but it is a problem whether it is surjective or not.

$$\begin{array}{ccc}
\mathcal{H}^n & \longrightarrow & \mathrm{Sh}^n \\
\big\downarrow & & \big\uparrow \\
\overline{\mathrm{Sh}}^n_S & \longrightarrow & \mathrm{Sh}^n_S
\end{array}$$

In the definition of $\mathrm{Sh}^n_S$, replacing $\mathrm{Tel}^{n+1}_{[0,\infty)}$ and $\simeq^n_p$ by $\mathrm{Tel}_{[0,\infty)}$ and $\simeq_p$, we can obtain the strong shape category $\mathrm{Sh}_S$ (cf. [DS]). Then, we can easily obtain the natural functor from $\mathrm{Sh}_S$ to $\overline{\mathrm{Sh}}^n_S$. Let $\mathcal{H}$ be the homotopy category of compacta. We have the following diagram of categories and functors:

$$\begin{array}{ccccccc}
\mathcal{H} & \longrightarrow & \mathrm{Sh}_S & = & \mathrm{Sh}_S & \longrightarrow & \mathrm{Sh} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{H}^n & \longrightarrow & \overline{\mathrm{Sh}}^n_S & \longrightarrow & \mathrm{Sh}^n_S & \longrightarrow & \mathrm{Sh}^n
\end{array}$$

Restricting the objects to compacta with $\dim \leqslant k$, we have the subcategories $\mathrm{Sh}(k)$, $\mathrm{Sh}^n(k)$, $\mathrm{Sh}_S(k)$, $\mathrm{Sh}^n_S(k)$ and $\overline{\mathrm{Sh}}^n_S(k)$ of $\mathrm{Sh}$, $\mathrm{Sh}^n$, $\mathrm{Sh}_S$, $\mathrm{Sh}^n_S$ and $\overline{\mathrm{Sh}}^n_S$, respectively. Then, $\mathrm{Sh}^n_S(n) = \overline{\mathrm{Sh}}^n_S(n)$ because $\mathrm{Tel}^{n+1}_{[0,\infty)}(X) = \mathrm{Tel}_{[0,\infty)}(X)$ if $\dim X \leqslant n$. Moreover, $\mathrm{Sh}^n_S(n-1) = \overline{\mathrm{Sh}}^n_S(n-1) = \mathrm{Sh}_S(n-1)$ because $\dim \mathrm{Tel}_{[0,\infty)}(X) \leqslant n$ if $\dim X \leqslant n-1$. Although $\mathrm{Sh}^n(n) = \mathrm{Sh}(n)$, it is not known whether $\mathrm{Sh}^n_S(n) = \mathrm{Sh}_S(n)$ or not.

## 3. An isomorphism between $\mathrm{Sh}^n_S(\mathcal{Z}(\mu^{n+1}))$ and $\mathcal{H}^n_P(\mathcal{M}_{n+1})$

Let $\mathcal{Z}(\mu^{n+1})$ be the class of $Z$-sets in $\mu^{n+1}$ and $\mathcal{M}_{n+1}$ the class of $\mu^{n+1}$-manifolds $\mu^{n+1} \setminus X$, $X \in \mathcal{Z}(\mu^{n+1})$. In this section, we prove that the strong $n$-shape category $\mathrm{Sh}^n_S(\mathcal{Z}(\mu^{n+1}))$ of $\mathcal{Z}(\mu^{n+1})$ is categorically isomorphic to the proper $n$-homotopy category $\mathcal{H}^n_P(\mathcal{M}_{n+1})$ of $\mathcal{M}_{n+1}$.

**Lemma 1.** *Let $f\colon X \to Y$ be a map from a locally compact separable metrizable space $X$ with* $\dim X \leqslant n+1$ *to a completely metrizable* $\mathrm{ANE}(n+1)$ $Y$. *For any closed set $A \subset X$ and a $Z$-set $B \subset Y$, $f$ is approximated by maps $g\colon X \to Y$ such that $g|A = f|A$ and $g(X \setminus A) \subset Y \setminus B$.*

As in §2, we assign each $X \in \mathcal{Z}(\mu^{n+1})$ to the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Tel}^{n+1}_{[0,1]}(X) & \xleftarrow[\subset]{c^X_{1,2}} & \mathrm{Tel}^{n+1}_{[0,2]}(X) & \xleftarrow[\subset]{c^X_{2,3}} & \mathrm{Tel}^{n+1}_{[0,3]}(X) & \xleftarrow[\subset]{c^X_{3,4}} & \cdots \\
\cup & & \cup & & \cup & & \cdots \\
|K^X_1| & \xleftarrow[q^X_{1,2}]{} & |K^X_2| & \xleftarrow[q^X_{2,3}]{} & |K^X_3| & \xleftarrow[q^X_{3,4}]{} & \cdots,
\end{array}$$

where the lower sequence is a barycentric sequence associated with $X$. To prove Theorem 3, we apply the construction in [Sa] to this diagram.

Let $M_1^X = C(K_1^X)^{(n+1)}$. Then $|M_1^X| = \mathrm{Tel}^{n+1}_{[0,1]}(X)$. We inductively define a simplicial complex

$$M_{i+1}^X = \left(\operatorname{Sd} M_i^X \times I\right)^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)},$$

where we identify $\operatorname{Sd} M_i^X = \operatorname{Sd} M_i^X \times \{0\}$. So we have

$$M(q_{i,i+1}^X)^{(n+1)} \cap \left(\operatorname{Sd} M_i^X \times I\right) = M(q_{i,i+1}^X)^{(n+1)} \cap \operatorname{Sd} M_i^X = \operatorname{Sd} K_i.$$

Observe that $\mathrm{Tel}^{n+1}_{[0,i+1]}(X) = \mathrm{Tel}^{n+1}_{[0,i]}(X) \cup |M(q_{i,i+1}^X)^{(n+1)}| \subset |M_{i+1}^X|$. The simplicial collapsing map $c_{q_{i,i+1}^X} : M(q_{i,i+1}^X) \to \operatorname{Sd} K_i^X$ extends to the simplicial retraction

$$\tilde{c}_{i,i+1} \colon M_i^X = \left(\operatorname{Sd} M_{i-1}^X \times I\right)^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)} \to \left(\operatorname{Sd} M_{i-1}^X \times I\right)^{(n+1)}.$$

We define $r_{i,i+1}^X = \mathrm{pr}_i\, \tilde{c}_{i,i+1} \colon M_{i+1}^X \to M_i^X$, where $\mathrm{pr}_i \colon (\operatorname{Sd} M_i^X \times I)^{(n+1)} \to M_i^X$ is the projection. Let $\pi_1^X = \mathrm{id} \colon |M_1^X| \to \mathrm{Tel}^{n+1}_{[0,1]}(X)$ $(= |M_1^X|)$ and inductively define the retraction $\pi_{i+1}^X \colon |M_{i+1}^X| \to \mathrm{Tel}^{n+1}_{[0,i+1]}(X)$ by $\pi_{i+1}^X \big| |M(q_{i,i+1}^X)^{(n+1)}| = \mathrm{id}$ and $\pi_{i+1}^X \big| |(\operatorname{Sd} M_i^X \times I)^{(n+1)}| = \pi_i^X\, \mathrm{pr}_i$. Thus, we obtain the following commutative diagram of the inverse sequences:

$$\begin{array}{ccccccc}
|M_1^X| & \xleftarrow[\supset]{r_{1,2}^X} & |M_2^X| & \xleftarrow[\supset]{r_{2,3}^X} & |M_3^X| & \xleftarrow[\supset]{r_{3,4}^X} & \cdots \\
\| & & \pi_2^X \downarrow \cup & & \pi_3^X \downarrow \cup & & \\
\mathrm{Tel}^{n+1}_{[0,1]}(X) & \xleftarrow[\supset]{c_{1,2}^X} & \mathrm{Tel}^{n+1}_{[0,2]}(X) & \xleftarrow[\supset]{c_{2,3}^X} & \mathrm{Tel}^{n+1}_{[0,3]}(X) & \xleftarrow[\supset]{c_{3,4}^X} & \cdots \\
\cup & & \cup & & \cup & & \\
|K_1^X| & \xleftarrow[q_{1,2}^X]{} & |K_2^X| & \xleftarrow[q_{2,3}^X]{} & |K_3^X| & \xleftarrow[q_{3,4}^X]{} & \cdots .
\end{array}$$

Recall that $\mathrm{Tel}^{n+1}_{[0,\infty)}(X) = \bigcup_{i \in \mathbb{N}} \mathrm{Tel}^{n+1}_{[0,i]}(X)$, $\mathrm{Tel}^{n+1}_{[0,\infty]}(X) = \mathrm{Tel}^{n+1}_{[0,\infty)}(X) \cup X$ is the inverse limit of the middle sequence and $X$ is the inverse limit of the bottom sequence. Let $M^X$ be the inverse limit of the upper sequence. Then $X \subset \mathrm{Tel}^{n+1}_{[0,\infty]}(X) \subset M^X$ but $M^X \neq X \cup \bigcup_{i \in \mathbb{N}} |M_i^X|$. Applying Bestvina's characterization of $\mu^{n+1}$ [Be], one can see that $M^X \approx \mu^{n+1}$ (cf. [Sa] and [Iwa, Proposition 2.1]). It is easily seen that $X$ is a $Z$-set in $M^X$ (it is also a $Z$-set in $\mathrm{Tel}^{n+1}_{[0,\infty]}(X)$ [Sa]). Since $(M^X, X) \approx (\mu^{n+1}, X)$ by the $Z$-set unknotting theorem [Be], we have a homeomorphism $h_X \colon M^X \setminus X \to \mu^{n+1} \setminus X$. On the other hand, we have the retraction of $\pi^X \colon M^X \to \mathrm{Tel}^{n+1}_{[0,\infty]}(X)$ induced by $\pi_i^X$. Observe that $\pi^X|X = \mathrm{id}$ and $\pi^X(M^X \setminus X) = \mathrm{Tel}^{n+1}_{[0,\infty]}(X)$.

**Lemma 2.** $\pi^X|M^X \setminus X \simeq_p^n \mathrm{id}$ *in* $M^X \setminus X$.

Now we have the following:

**Theorem 3.** *There is a categorical isomorphism* $\Phi \colon \mathrm{Sh}^n_S(\mathcal{Z}(\mu^{n+1})) \to \mathcal{H}^n_P(\mathcal{M}_{n+1})$ *such that* $\Phi(X) = \mu^{n+1} \setminus X$ *for* $X \in \mathcal{Z}(\mu^{n+1})$.

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