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New subfactors associated with closed systems of sectors

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Abstract: A theorem is derived which (i) provides a new class of subfactors which may be interpreted as generalized asymptotic subfactors, and which (ii) ensures the existence of two-dimensional local quantum field theories associated with certain modular invariant matrices.

1 Introduction and results

We consider type III von Neumann factors throughout. $End_{\mathrm{fin}}(N)$ stands for the set of unital endomorphisms λ with finite dimension $d(\lambda)$ of a factor N.

A *closed N-system* is a set $\Delta\subset End_{\mathrm{fin}}(N)$ of mutually inequivalent irreducible endomorphisms such that (i) $\mathrm{id}_N\in\Delta$, (ii) if $\lambda\in\Delta$ then there is a conjugate endomorphism $\bar{\lambda}\in\Delta$, and (iii) if $\lambda,\mu\in\Delta$ then $\lambda\mu$ belongs to $\Sigma(\Delta)$, the set of endomorphisms which are equivalent to finite direct sums of elements from Δ.

Let $N\subset M$ be a subfactor of finite index with inclusion homomorphism $\iota\in Mor(N,M)$. An *extension of the closed N-system* Δ is a pair (ι,α), where ι is as above, and α is a map $\Delta\rightarrow End_{\mathrm{fin}}(M)$, $\lambda\mapsto\alpha_{\lambda}$, such that

(E1) $\quad \iota\circ\lambda=\alpha_{\lambda}\circ\iota$,

(E2) $\quad \iota(Hom(\nu,\lambda\mu))\subset Hom(\alpha_{\nu},\alpha_{\lambda}\alpha_{\mu})$.

Conditions (E1) and (E2) mean that (ι,α) is a monoidal functor from the full monoidal C* subcategory [3] of $End_{\mathrm{fin}}(N)$ with objects $\Pi(\Delta)$ (the set of finite products of elements from Δ) into the monoidal C* category $End_{\mathrm{fin}}(M)$. In particular, they imply that α_{λ} satisfy the same fusion rules as $\lambda\in\Delta$, and that $\alpha_{\mathrm{id}_N}=\mathrm{id}_M$ (being an idempotent within $End_{\mathrm{fin}}(M)$). It follows that if $R_{\lambda}\in Hom(\mathrm{id}_N,\bar{\lambda}\lambda)$ and $\bar{R}_{\lambda}\in Hom(\mathrm{id}_N,\lambda\bar{\lambda})$ are a pair of isometries satisfying the conjugate equations $(1_{\lambda}\times R_{\lambda}^{*})(\bar{R}_{\lambda}\times 1_{\lambda})=d(\lambda)^{-1}1_{\lambda}=(1_{\bar{\lambda}}\times\bar{R}_{\lambda}^{*})(R_{\lambda}\times 1_{\bar{\lambda}})$, and thus implementing left- and right-inverses Φ_{λ} and Ψ_{λ} for λ (i.e., linear mappings which invert the left and right monoidal products with 1_{λ}, cf. [9]), then so do $\iota(R_{\lambda})$ and $\iota(\bar{R}_{\lambda})$ for α_{λ}. (The notation $\times$ refers to the monoidal product of intertwiners [3].) In particular $\alpha_{\bar{\lambda}}$ is conjugate to α_{λ}.

While $\lambda\in\Delta$ is irreducible by definition, α_{λ} may be reducible, and its left- and right-inverses are not unique in general. But the Lemma below states that the left- and right-inverses $\Phi_{\alpha_{\lambda}}$ and $\Psi_{\alpha_{\lambda}}$ induced by $\iota(R_{\lambda})$ and $\iota(\bar{R}_{\lambda})$ are in fact the unique standard (minimal) [9] ones, provided Δ is a finite system.

We state our main result.

Theorem: *Let $N_1 \subset M$ and $N_2 \subset M$ be two subfactors of M, and (ι_1, α^1) and (ι_2, α^2) a pair of extensions of a finite closed N_1-system Δ_1 and a finite closed N_2-system Δ_2, respectively. Then there exists an irreducible subfactor*

$$A \equiv N_1 \otimes N_2^{\mathrm{opp}} \subset B$$

with dual canonical endomorphism

$$\theta \equiv \bar{\iota} \circ \iota \simeq \bigoplus_{\lambda_1 \in \Delta_1, \lambda_2 \in \Delta_2} Z_{\lambda_1, \lambda_2}\, \lambda_1 \otimes \lambda_2^{\mathrm{opp}},$$

whose "coupling matrix" Z of multiplicities is given by

$$Z_{\lambda_1, \lambda_2} = \dim Hom(\alpha^1_{\lambda_1}, \alpha^2_{\lambda_2}).$$

Here, $\iota \in Mor(A, B)$ is the inclusion homomorphism with conjugate $\bar{\iota} \in Mor(B, A)$.

The following special case when Δ_i are braided systems is of particular interest for an application in quantum field theory:

Proposition 1: *Assume in addition that the closed systems Δ_1 and Δ_2 are braided with unitary braidings ε_1 and ε_2, respectively, turning $\Pi(\Delta_1)$ and $\Pi(\Delta_2)$ into braided monoidal categories. If for any $\lambda_i, \mu_i \in \Delta_i$ and any $\phi \in Hom(\alpha^1_{\lambda_1}, \alpha^2_{\lambda_2})$, $\psi \in Hom(\alpha^1_{\mu_1}, \alpha^2_{\mu_2})$,*

(E3) $\quad (\psi \times \phi) \circ \iota_1(\varepsilon_1(\lambda_1, \mu_1)) = \iota_2(\varepsilon_2(\lambda_2, \mu_2)) \circ (\phi \times \psi)$

holds, then the canonical isometry $w_1 \in Hom(\theta, \theta^2)$ (defined below in the proof of the Theorem) and the braiding operator $\varepsilon(\theta, \theta)$ naturally induced by the braidings ε_1 and $\varepsilon_2^{\mathrm{opp}}$ satisfy

$$\varepsilon(\theta, \theta) w_1 = w_1.$$

This result answers an open question in quantum field theory, where possible matrices Z are classified which are supposed to describe the restriction of a given two-dimensional modular invariant conformal quantum field theory to its chiral subtheories, while it is actually not clear whether any given solution Z does come from a two-dimensional quantum field theory. This turns out to be true for a large class of solutions.

Namely, let $N_1 = N_2 = N$ be a local algebra of chiral observables and $\Delta_1 = \Delta_2 = \Delta$ a braided system of DHR endomorphisms. If the dual canonical endomorphism θ_M associated with $N \subset M$ belongs to $\Sigma(\Delta)$, then α-induction [8, 1] provides a pair of extensions (ι, α^+) and (ι, α^-) which satisfies (E1), (E2) as well as (E3) [1, I, Def. 3.3, Lemma 3.5 and 3.25]. The associated coupling matrix $Z_{\lambda,\mu} = \dim Hom(\alpha^+_\lambda, \alpha^-_\mu)$ is automatically a modular invariant [2]. By the characterization of extensions of local quantum field theories given in [8], the subfactor given by the Theorem induces an entire net of subfactors, indexed by the double-cones of two-dimensional Minkowski space. The statement of Proposition 1 is precisely the criterium given in [8] for the resulting two-dimensional quantum field theory to be local. Thus, every modular invariant found by the α-induction method given in [2] indeed corresponds to a local two-dimensional quantum field theory extending the given chiral nets of observables.

The case $N_1 = N_2 = M$ hence $Z = \mathbb{1}$ is known for a while [8], and was recognized [10] to yield (up to some trivial tensoring with a type III factor) the type II asymptotic subfactor [11] associated with $\sigma(N) \subset N$ where $\sigma \equiv \bigoplus_{\lambda\in\Delta} \lambda$. As the asymptotic subfactor $M \vee M^c \subset M_\infty$ associated with a fixed point inclusion $M^G \subset M$ for an outer action of a group G, provides the same category of M_∞-M_∞ bimodules as a fixed point inclusion for an outer action of the quantum double $D(G)$ on M_∞, general asymptotic subfactors in turn are considered [11, 4] as generalized quantum doubles.

Asymptotic subfactors have the properties

(A1) $M \vee M^c \simeq M \otimes M^c$ are in a tensor product position within M_∞, and every irreducible $M \vee M^c$-$M \vee M^c$ bimodule associated with the asymptotic subfactor respects the tensor product, i.e., factorizes into an M-M bimodule and an M^c-M^c bimodule [11].

(A2) M and M^c are each other's relative commutant in M_∞. We call this property of the triple (M, M^c, M_∞) *normality.*

(A3) The system of M_∞-M_∞ bimodules associated with an asymptotic subfactor has a non-degenerate braiding [11, 5].

In the type III framework, the analogous property of (A1) is that for a subfactor $A \otimes B \subset C$, the dual canonical endomorphism $\theta = \bar\iota \circ \iota$ respects the tensor product, i.e., each of its irreducible components is (equivalent to) a tensor product $\alpha \otimes \beta$ of endomorphisms of A and B, respectively. We call a subfactor with this property a *canonical tensor product subfactor (CTPS)* [12, 13].

Let (A, B, C) be a joint inclusion of von Neumann algebras, i.e., $A \vee B \subset C$. We call (A, B, C) *normal* if A and B are each other's relative commutant in C, which is equivalent to $A = A^{cc}$ (i.e., $A \subset C$ is normal in standard terminology), and $B = A^c$. For (A, B, C) normal, one has $Z(A) = (A \vee B)^c = Z(B) \supset Z(C)$, so A and likewise B are factors if and only if $A \vee B \subset C$ is irreducible, and in this case C necessarily is also a factor.

Obviously, the subfactors constructed in the Theorem are CTPS's (property (A1) of asymptotic subfactors), while we do not know at present whether they always share the property (A3) (braiding), which ought to be tested with methods as in [5]. Definitely, the joint inclusions $(N_1, N_2^{\rm opp}, B)$ in the Theorem do not share the normality property (A2) in general. The following Proposition is a characterization of normality in terms of the coupling matrix, which suggests to regard normal CTPS's as "generalized quantum doubles", beyond the class of asymptotic subfactors.

Proposition 2: *Let $A \otimes B \subset C$ be a CTPS of type III with coupling matrix Z, i.e., the dual canonical endomorphism is of the form*

$$\theta \simeq \bigoplus_{\alpha\in\Delta_A,\beta\in\Delta_B} Z_{\alpha,\beta}\, \alpha \otimes \beta,$$

where $\Delta_A \ni \mathrm{id}_A$ and $\Delta_B \ni \mathrm{id}_B$ are two sets of mutually inequivalent irreducible endomorphisms in $End_{\rm fin}(A)$ and $End_{\rm fin}(B)$. Then the following conditions are equivalent.

(N1) The joint inclusion $(A \otimes \mathbb{1}_B, \mathbb{1}_A \otimes B, C)$ is normal, i.e., $A \otimes \mathbb{1}_B$ and $\mathbb{1}_A \otimes B$ are each other's relative commutants in C.

(N2) The coupling matrix couples no non-trivial sector of A to the trivial sector of B, and vice versa, i.e.,

$$Z_{\alpha,\mathrm{id}_B} = \delta_{\alpha,\mathrm{id}_A} \quad \text{and} \quad Z_{\mathrm{id}_A,\beta} = \delta_{\beta,\mathrm{id}_B}.$$

(N3) The sets Δ_A and Δ_B are closed A- and B-systems, respectively, i.e., they are both closed under conjugation and fusion. There is a bijection $\pi : \Delta_A \to \Delta_B$ which preserves the fusion rules, i.e.,

$$\dim Hom(\alpha_1, \alpha_2\alpha_3) = \dim Hom(\pi(\alpha_1), \pi(\alpha_2)\pi(\alpha_3)).$$

The matrix Z is the permutation matrix for this bijection, i.e.,

$$Z_{\alpha,\beta} = \delta_{\pi(\alpha),\beta}.$$

2 Indication of Proofs

For complete proofs, see [12, 13].

Lemma: *Let (ι, α) be an extension of a closed N-system Δ. Let $R \in Hom(\mathrm{id}_N, \bar\lambda\lambda)$ and $\bar R \in Hom(\mathrm{id}_N, \lambda\bar\lambda)$ be a pair of isometries as before implementing the unique left- and right-inverses [9] Φ_λ and Ψ_λ for $\lambda \in \Delta$. Then $\iota(R_\lambda)$ and $\iota(\bar R_\lambda)$ implement left- and right-inverses Φ_{α_λ} and Ψ_{α_λ} for α_λ. If Δ is finite, then $d(\alpha_\lambda) = d(\lambda)$, and Φ_{α_λ} and Ψ_{α_λ} are the unique standard left- and right-inverses.*

Proof of the Lemma: The first statement is obvious, since $\iota(R_\lambda)$ and $\iota(\bar R_\lambda)$ solve the conjugate equations [9] for α_λ if R_λ and $\bar R_\lambda$ do so for λ. If Δ is finite, then the minimal dimensions $d(\alpha_\lambda)$ are uniquely determined by the fusion rules, and the latter must coincide with those of $\lambda \in \Delta$. Hence $d(\alpha_\lambda) = d(\lambda)$. Since $d(\lambda)$ are also the dimensions associated with the pair of isometries $\iota(R_\lambda)$, $\iota(\bar R_\lambda)$, the last claim follows by [9, Thm. 3.11]. □

Thus, general properties of standard left- and right-inverses [9] are applicable. We shall in the sequel repeatedly exploit the trace property

$$d(\rho)\Phi_\rho(S^*T) = d(\tau)\Phi_\tau(TS^*) \qquad \text{if} \qquad S, T \in Hom(\rho, \tau)$$

for standard left-inverses of $\rho, \tau \in End_{\mathrm{fin}}(M)$, their multiplicativity $\Phi_{\rho\tau} = \Phi_\rho\Phi_\tau$, as well as the equality of standard left- and right-inverses $\Psi_\rho = \Phi_\rho$ on $Hom(\rho, \rho)$.

Proof of the Theorem: First notice that the multiplicity of id_A in θ is $Z_{\mathrm{id}_{N_1},\mathrm{id}_{N_2}} = \dim Hom(\mathrm{id}_M, \mathrm{id}_M) = 1$, so the asserted subfactor is automatically irreducible.

In order to show that θ is the dual canonical endomorphism associated with a subfactor $A \subset B$, we make use of Longo's characterization [7] of canonical endomorphisms in terms of "canonical triples" ("Q-systems"). It says that $\theta \in End_{\mathrm{fin}}(A)$ is the dual canonical endomorphism associated with $A \subset B$ if (and only if) there is a pair of isometries $w \in Hom(\mathrm{id}_A, \theta)$ and $w_1 \in Hom(\theta, \theta^2)$ satisfying

(Q1) $w^*w_1 = \theta(w^*)w_1 = d(\theta)^{-1/2}\mathbb{1}_A$,

(Q2) $w_1w_1 = \theta(w_1)w_1$, and

(Q3) $w_1w_1^* = \theta(w_1^*)w_1$.

In order to construct the Q-system (θ, w, w_1) in the present case, we first choose a complete system of mutually inequivalent isometries $W_{(\lambda_1,\lambda_2,l)} \equiv W_l \in A \equiv N \otimes N^{\rm opp}$, where l is considered as a multi-index including $(\lambda_1 \in \Delta_1, \lambda_2 \in \Delta_2, l = 1, \dots Z_{\lambda_1,\lambda_2})$, and put

$$\theta = \sum_l W_l \, (\lambda_1 \otimes \lambda_2^{\rm opp})(\,\cdot\,) \, W_l^*.$$

The choice of these isometries is immaterial and affects the subfactor to be constructed only by inner conjugation.

Since $Hom(\mathrm{id}_A, \theta)$ is one-dimensional, the isometry w is already fixed up to an irrelevant complex phase, and we choose $w = W_0$, where 0 refers to the multi-index $l = 0 \equiv (\mathrm{id}_{N_1}, \mathrm{id}_{N_2}, 1)$. The second isometry, w_1, must be of the form

$$w_1 = \sum_{l,m,n} (W_l \times W_m) \circ \mathcal{T}_{lm}^n \circ W_n^*$$

where $\mathcal{T}_{lm}^n \in Hom(\nu_1 \otimes \nu_2^{\rm opp}, (\lambda_1 \otimes \lambda_2^{\rm opp}) \circ (\mu_1 \otimes \mu_2^{\rm opp}))$, since these operators span $Hom(\theta, \theta^2)$.

In turn, $\mathcal{T}_{lm}^n$ must be of the form

$$\mathcal{T}_{lm}^n = \sum_{e_1,e_2} \zeta_{lm,e_1e_2}^n \, T_{e_1} \otimes (T_{e_2}^*)^{\rm opp} \qquad (\zeta_{lm,e_1e_2}^n \in \mathbb{C})$$

where T_{e_i} constitute orthonormal isometric bases of the intertwiner spaces $Hom(\nu_i, \lambda_i\mu_i)$, since these operators span $Hom(\nu_1 \otimes \nu_2^{\rm opp}, (\lambda_1 \otimes \lambda_2^{\rm opp}) \circ (\mu_1 \otimes \mu_2^{\rm opp})) \equiv Hom(\nu_1, \lambda_1\mu_1) \otimes Hom(\nu_2^{\rm opp}, \lambda_2^{\rm opp}\mu_2^{\rm opp})$. Note that if $T \in Hom(\alpha, \beta)$ is isometric in N, then $(T^*)^{\rm opp} \in Hom(\beta, \alpha)^{\rm opp} \equiv Hom(\alpha^{\rm opp}, \beta^{\rm opp})$ is isometric in $N^{\rm opp}$. The labels e_i are again multi-indices of the form $(\lambda, \mu, \nu, e = 1, \dots \dim Hom(\nu, \lambda\mu))$.

It remains therefore to determine the complex coefficients $\zeta_{lm,e_1e_2}^n$, such that w_1 is an isometry satisfying Longo's relations (Q1-3) above. To specify the coefficients, we equip the spaces $Hom(\alpha_{\lambda_1}^1, \alpha_{\lambda_2}^2)$ with the non-degenerate scalar products $(\phi, \phi') := \Phi_{\lambda_1}^1(\phi^*\phi')$ (where $\Phi_{\lambda_i}^i$ stand for the induced left-inverses for $\alpha_{\lambda_i}^i$). With respect to these scalar products, we choose orthonormal bases $\{\phi_l, l = 1, \dots Z_{\lambda_1,\lambda_2}\}$ for all λ_1, λ_2, and put

$$\zeta_{lm,e_1e_2}^n = \sqrt{\frac{d(\lambda_2)d(\mu_2)}{d(\theta)d(\nu_2)}} \, \Phi_{\lambda_1}^1[\iota_1(T_{e_1}^*)(\phi_l^* \times \phi_m^*)\iota_2(T_{e_2})\phi_n].$$

Condition (Q1) is trivially satisfied, since left multiplication of w_1 by w^* singles out the term $l = 0$ due to $W_0^* W_l = \delta_{l0}$. This leaves only terms with $\lambda_i = \mathrm{id}_{N_i}$, hence $\mu_i = \nu_i$, for which T_{e_i} are trivial and $\sqrt{d(\theta)}\zeta_{0m,e_1e_2}^n = \delta_{mn}$ (up to cancelling complex phases), so $\sqrt{d(\theta)} w^* w_1 = \sum_n W_n W_n^* = \mathbb{1}_A$. For $\theta(w^*)w_1$ the argument is essentially the same.

We turn to the conditions (Q2) and (Q3). Whenever we compute either of the four products occurring, we obtain a Kronecker delta $W_s^* W_t = \delta_{st}$ for one pair of the labels $l, m, n, \dots$ involved, while the remaining operator parts are of the form

$$(W_l \times W_m \times W_k) \, [(T_{e_1} \times 1_{\kappa_1})T_{f_1} \otimes (((T_{e_2} \times 1_{\kappa_2})T_{f_2})^*)^{\rm opp}] \, W_n^*,$$

$$(W_l \times W_m \times W_k) \, [(1_{\lambda_1} \times T_{g_1})T_{h_1} \otimes (((1_{\lambda_2} \times T_{g_2})T_{h_2})^*)^{\rm opp}] \, W_n^*$$

for the left- and right-hand side of (Q2), $w_1 w_1 = \theta(w_1)w_1$, and in turn,

$$(W_l \times W_m)\left[T_{e_1}T_{f_1}^* \otimes ((T_{e_2}T_{f_2}^*)^*)^{\rm opp}\right](W_n \times W_k)^*,$$
$$(W_l \times W_m)\left[(1_{\lambda_1} \times T_{g_1}^*)(T_{h_1} \times 1_{\kappa_1}) \otimes (((1_{\lambda_2} \times T_{g_2}^*)(T_{h_2} \times 1_{\kappa_2}))^*)^{\rm opp}\right](W_n \times W_k)^*$$

for the left- and right-hand side of (Q3), $w_1 w_1^* = \theta(w_1^*)w_1$. (In these expressions, we do not specify the respective intertwiner spaces to which the various operators T belong, since these are determined by the context.)

The numerical coefficients multiplying these operators are, respectively,

$$C_{2L} = \sum_s \zeta^s_{lm,e_1e_2}\zeta^n_{sk,f_1f_2}, \quad C_{2R} = \sum_s \zeta^s_{mk,g_1g_2}\zeta^n_{ls,h_1h_2}$$

for (Q2), and

$$C_{3L} = \sum_s \zeta^s_{lm,e_1e_2}\overline{\zeta^s_{nk,f_1f_2}}, \quad C_{3R} = \sum_s \overline{\zeta^m_{sk,g_1g_2}}\zeta^n_{ls,h_1h_2}$$

for (Q3), with a summation over one common label $s = 1, \dots Z_{\sigma_1,\sigma_2}$ due to the above Kronecker δ_{st} in each case.

These summations over s can be carried out. Namely, factors $\zeta^s_{..,..}$ are in fact scalar products of the form $\Phi^1_{\sigma_1}(X\phi_s) = (X^*, \phi_s)$ within $Hom(\alpha^1_{\sigma_1}, \alpha^2_{\sigma_2})$, so summation with the operator ϕ_s^* contributing to the other factor ζ yields $\sum_s \Phi^1_{\sigma_1}(X\phi_s)\phi_s^* = X$. A factor of the form $\zeta^{\cdot}_{\cdot s,..}$ can also be rewritten with the help of the trace property for standard left inverses as a scalar product $\Phi^1_{\sigma_1}(\phi_s^* X)$ within $Hom(\alpha^1_{\sigma_1}, \alpha^2_{\sigma_2})$, and the evaluation of the sum over s is likewise possible.

After some transformations, one arrives at

$$C_{2L} \propto \Phi^1_{\nu_1}[\iota_1(T_{f_1}^*(T_{e_1}^* \times 1_{\kappa_1}))(\phi_l^* \times \phi_m^* \times \phi_k^*)\iota_2((T_{e_2} \times 1_{\kappa_2})T_{f_2})\phi_n],$$
$$C_{2R} \propto \Phi^1_{\nu_1}[\iota_1(T_{h_1}^*(1_{\lambda_1} \times T_{g_1}^*))(\phi_l^* \times \phi_m^* \times \phi_k^*)\iota_2((1_{\lambda_2} \times T_{g_2})T_{h_2})\phi_n]$$

up to a common factor $\sqrt{\frac{d(\lambda_2)d(\mu_2)d(\kappa_2)}{d(\theta)^2 d(\nu_2)}}$. Summing the operators on both sides of (Q2) as above with the coefficients C_{2L}, C_{2R}, and noting that the passage from bases $(T_e \times 1_\kappa)T_f$ to bases $(1_\lambda \times T_g)T_h$ of $Hom(\nu, \lambda\mu\kappa)$ for any fixed $\nu, \lambda, \mu, \kappa$ is described by unitary matrices, we conclude equality of both sides of (Q2).

For (Q3), similar manipulations give

$$C_{3L} \propto \frac{d(\mu_1)d(\mu_2)}{d(\sigma_2)d(\sigma_1)}\,\Phi^1_{\mu_1\lambda_1}[(\phi_l^* \times \phi_m^*)\iota_2(T_{e_2}T_{f_2}^*)(\phi_n \times \phi_k)\iota_1(T_{f_1}T_{e_1}^*)],$$
$$C_{3R} \propto \frac{d(\sigma_2)d(\sigma_1)}{d(\nu_1)d(\nu_2)} \times$$
$$\Phi^1_{\mu_1\lambda_1}[(\phi_l^* \times \phi_m^*)\iota_2((1_{\lambda_2} \times T_{g_2}^*)(T_{h_2} \times 1_{\kappa_2}))(\phi_n \times \phi_k)\iota_1((T_{h_1}^* \times 1_{\kappa_1})(1_{\lambda_1} \times T_{g_1}))]$$

up to a common factor $\sqrt{\frac{d(\lambda_2)d(\nu_2)d(\kappa_2)}{d(\theta)^2 d(\mu_2)}}d(\lambda_1)$. Summing the operators on both sides of (Q3) as above with the coefficients C_{3L}, C_{3R}, and noting that the passage from bases $\sqrt{\frac{d(\mu)}{d(\sigma)}}T_eT_f^*$ to bases $\sqrt{\frac{d(\sigma)}{d(\nu)}}(1_\lambda \times T_g^*)(T_h \times 1_\kappa)$ of $Hom(\nu\kappa, \lambda\mu)$ for any fixed $\nu, \kappa, \lambda, \mu$ is again described by a unitary matrix, we obtain equality of both sides of (Q3).

It remains to show that w_1 is an isometry, $w_1^* w_1 = 1$.

Performing the multiplication $w_1^* w_1$ yields two Kronecker delta's from the factors $W_l \times W_m$, and two more Kronecker delta's from the factors $T_{e_1} \otimes (T_{e_2}^*)^{\mathrm{opp}}$. Thus

$$w_1^* w_1 = \sum_{ns} \left(\sum_{lm,e_1 e_2} \overline{\zeta^n_{lm,e_1 e_2}} \zeta^s_{lm,e_1 e_2} \right) W_n W_s^*,$$

and we have to perform the sums over l, m, e_1, e_2 (involving, as sums over multi-indices, the summation over sectors $\nu_i, \lambda_i, \mu_i \in \Delta_i$, $i = 1, 2$).

Again, we rewrite $\zeta^n_{lm,e_1 e_2}$ as a scalar product (ϕ_m, X) within $Hom(\alpha^1_{\mu_1}, \alpha^2_{\mu_2})$ and perform the sum over m similar as before. In the resulting expression, both sums over (e_1, μ_1) and over (e_2, μ_2) can be performed after a unitary passage from the bases of orthonormal isometries T_e of $Hom(\nu, \lambda\mu)$ to the bases $\sqrt{\frac{d(\lambda)d(\nu)}{d(\mu)}}(1_\lambda \times T_{e'}^*)(\bar{R}_\lambda \times 1_\nu)$, making use of the conjugate equations between $\bar{R}_\lambda$ (contributing to the new bases) and R_λ (implementing the left-inverses Φ_λ and hence $\Phi^i_{\lambda_i}$). This produces the expression

$$\sum_{lm,e_1 e_2} \overline{\zeta^s_{lm,e_1 e_2}} \zeta^n_{lm,e_1 e_2} = \sum_{l,\lambda_1 \lambda_2} \frac{d(\lambda_2)^2}{d(\theta)} \Phi^1_{\nu_1}[\Psi^2_{\lambda_2}(\phi_l \phi_l^*) \times (\phi_s^* \phi_n)].$$

Here $\Psi^2_{\lambda_2}$ ist the standard right-inverse implemented by $\iota_2(\bar{R}_{\lambda_2})$ which coincides with $\Phi^2_{\lambda_2}$ on $Hom(\alpha^2_{\lambda_2}, \alpha^2_{\lambda_2})$, and can be evaluated by the trace property: $\Psi^2_{\lambda_2}(\phi_l \phi_l^*) = \Phi^2_{\lambda_2}(\phi_l \phi_l^*) = \frac{d(\lambda_1)}{d(\lambda_2)} \Phi^1_{\lambda_1}(\phi_l^* \phi_l) = \frac{d(\lambda_1)}{d(\lambda_2)}$, while the sum over l yields the multiplicity factor Z_{λ_1,λ_2}. Hence

$$\sum_{lm,e_1 e_2} \overline{\zeta^s_{lm,e_1 e_2}} \zeta^n_{lm,e_1 e_2} = \left(\sum_{\lambda_1,\lambda_2} \frac{d(\lambda_1) d(\lambda_2) Z_{\lambda_1,\lambda_2}}{d(\theta)} \right) \Phi^1_{\nu_1}(\phi_s^* \phi_n) = \delta_{sn},$$

and hence $w_1^* w_1 = \sum_n W_n W_n^* = 1$.

This completes the proof of the Theorem. For the detailed computations, cf. [13]. □

Proof of Proposition 1: Left multiplication of w_1 with the induced braiding operator

$$\varepsilon(\theta, \theta) = \sum_{mlm'l'} (W_{m'} \times W_{l'}) \circ (\varepsilon_1(\lambda_1, \mu_1) \otimes (\varepsilon_2(\lambda_2, \mu_2)^*)^{\mathrm{opp}}) \circ (W_l \times W_m)^*$$

amounts to a unitary passage from bases $T_e \in Hom(\nu, \lambda\mu)$ to bases $\varepsilon(\lambda, \mu) T_e \in Hom(\nu, \mu\lambda)$. But by (E3), the coefficients $\zeta^n_{lm,e_1 e_2}$ are invariant under these changes of bases. Hence $\varepsilon(\theta, \theta) w_1 = w_1$. □

Proof of Proposition 2: The proof is published in [12, Lemma 3.4 and Thm. 3.6]. □

3 Conclusion

We have shown the existence of a class of new subfactors associated with extensions of closed systems of sectors. The proof proceeds by establishing the corresponding Q-systems in terms of certain matrix elements for the transition between two extensions. The new subfactors are canonical tensor product subfactors and include the asymptotic subfactors. They may be regarded as generalized quantum doubles if they satisfy a normality condition for which a simple criterium is given. The new subfactors also include the local subfactors of two-dimensional conformal quantum field theory associated with certain modular invariants, thereby establishing the expected existence of these theories.

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