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N person stopping game with players given priority randomly

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Abstract

In the paper a construction of Nash equilibria for a random priority finite horizon N-person stopping game is given. The normal form of the game is formulated. The random priority scheme for the players is modeled by division of the unit interval and a sequence of random variables with uniform distribution on it. The strategies of the players are triples of randomized stopping times. A recursive procedure is developed to calculate the Nash value and the equilibrium strategies.

1 Introduction

In the paper the following N person stopping game is considered. At each moment $t = 1, 2, \ldots, T$ the decision makers (henceforth called players) are able to observe sequentially the homogeneous Markov process $(X_t, \mathcal{F}_t, \mathbf{P}_x)_{t=0}^{T}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ with state space $(\mathbb{E}, \mathcal{B})$. The players have utility functions $g_i : \mathbb{E} \to \Re$, $i = 1, 2, \ldots, N$ and at each moment t each decides separately whether to accept the realization x_t of X_t or not. If it happens that more than one player has selected the same moment t to accept the state, then a lottery decides which player gets the right (priority) of acceptance. According to the lottery, at moment τ, if players $\{i_1, i_2, \ldots, i_l\}$ would like to accept x_τ then Player r is chosen with probability proportional to $p_{r,\tau}$, $r \in \{i_1, i_2, \ldots, i_l\}$. The players rejected

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by the lottery may select any other realization x_t at a later moment t, $\tau < t \leq T$. Once accepted a realization cannot be rejected, once rejected it cannot be reconsidered. If a player has not chosen any realization of the Markov process, he gets $g_i^* = \inf_{x \in \mathbb{E}} g_i(x)$. The aim of each player is to choose the realization which maximizes his utility function. A non-zero-sum game approach is used. A formalization of the model is given and a construction of Nash equilibria for a finite horizon game is given. The model is a generalization of the two person games considered by Szajowski [14] and N person game with fixed priority scheme solved by Enns and Ferenstein in [1]. Such games are also strictly connected with the optimal stopping of stochastic processes. The ideas of Kuhn [4] and Rieder [8] as well as Yasuda [15] and Ohtsubo [6] are adopted to this random priority game model. The inspiration for these game models is the secretary problem. For the original secretary problem and its extension the reader is referred to Gilbert & Mosteller [3], Freeman [2] or Rose [9]. Related games can be found in the papers by Majumdar [5], Sakaguchi [10, 11], Ravindran and Szajowski [7] and Szajowski [12], where non-zero sum versions of the games have been investigated. A review of these problems can be found in Ravindran and Szajowski [7].

2 Normal form of the game

The basic class of strategies $\mathcal{T}^T$ in optimal stopping problems are Markov stopping times with respect to σ-fields $\{\mathcal{F}_t\}_{t=1}^T$. We permit $\mathbf{P}(\tau \leq T) < 1$ for some $\tau \in \mathcal{T}^T$.

Definition 1 *(see Yasuda [15]) A random stopping time is a sequence $\bar{p} = (p_t)_{t=1}^T$ such that, for each t: (i) p_t is adapted to $\mathcal{F}_t$ and (ii) $0 \leq p_t \leq 1$ a.s..*

The set of all such strategies will be denoted $\mathcal{P}^T$. Let $A_1^i, A_2^i, \ldots, A_T^i$ be i.i.d.r.v. from the uniform distribution on $[0,1]$ and independent of the observed Markov process. A randomized Markov time $\tau(\bar{p}^i)$ for strategy $\bar{p}^i = (p_t^i) \in \mathcal{P}^{T,i}$ is defined by $\tau(\bar{p}^i) = \inf\{T \geq t \geq 1 : A_t \leq p_t^i\}$. We denote by $\mathbf{M}_i^T$, $i = 1, 2, \ldots, N$, the sets of all randomized strategies of the i-th Player.

The random assignment of priority to a player requires us to consider modified strategies. Denote $\mathcal{T}_k^T = \{\tau \in \mathcal{T}^T : \tau \geq k\}$ and $\mathcal{P}_k^T = \{\bar{p} \in \mathcal{P}^T : p_i = 0 \text{ for } i = 1, 2, \ldots, k-1\}$.

One can define the sets of strategies $\tilde{\mathbf{M}}^{T,i} = \{(\bar{p}^i, \{\bar{p}_t^{i,N-1}\}, \{\bar{p}_t^{i,N-2}\}, \dots, \{\bar{p}_t^{i,2}\}, \{\tau_t^i\}) : \bar{p}^i \in \mathcal{P}^{T,i}, \bar{p}_t^{i,j} \in \mathcal{P}_t^{T,i}, \tau_t^i \in \mathcal{T}_{t+1}^T$ for every t, $j = 2, \dots, N-1\}$ for Player i.

Let $\xi_1, \xi_2, \dots$ be i.i.d.r.v. uniformly distributed on $[0,1]$ and independent of $\bigvee_{n=1}^N \mathcal{H}_n$, where $\mathcal{H}_n = \sigma\{\mathcal{F}_n, A_1, A_2, \dots, A_n\}$ and the lottery be given by $\bar{\gamma}^j = (\gamma_1^j, \gamma_2^j, \dots, \gamma_T^j)$, $j = 0, 1, 2, \dots, N$. We assume $\gamma_i^0 = 0$ and $\gamma_i^N = 1$ for every $i = 1, 2, \dots, T$. Denote $\tilde{\mathcal{H}}_n = \sigma\{\mathcal{H}_n, \xi_1, \xi_2, \dots, \xi_n\}$ and let $\tilde{\mathcal{T}}^T$ be the set of Markov times with respect to $(\tilde{\mathcal{H}}_n)_{n=0}^N$. For the lottery given $\{\bar{\gamma}^j\}_{j=1}^N$ we assume that $\delta_i^j = \gamma_i^j - \gamma_i^{j-1} > 0$, $i = 1, 2, \dots, T$ and $j = 1, 2 \dots, N$. Let $\bar{\lambda} = \min_{i \in I}\{\lambda_i(\bar{p}^i)\}$, where $I = \{1, 2, \dots, N\}$. For every N-tuple $\bar{s}$ such that $s^i \in \tilde{\mathbf{M}}^{T,i}$, we define

$$\begin{aligned}
\tau_i(\bar{s}) \;=\; & \lambda_i(p^i)\mathbb{I}_{\{\lambda_i(p^i) < \min_{l \neq i} \lambda_l(p^l)\}} \\
& + \sum_{l=1}^{N-1} \sum_{\{i, i_1, i_2, \dots, i_l\} \subset I} \Bigg(\lambda_i(p^i)\mathbb{I}_{\{\xi_{\lambda_i(p^i)} \leq \frac{\delta_i}{\delta_i + \sum_{r=1}^{l} \delta_{i_r}}\}} \\
& \quad + \tau_i^{I \setminus \{i_l\}}(\bar{s}^{I \setminus \{i_l\}})\mathbb{I}_{\{\frac{\delta_i + \sum_{r=1}^{l-1} \delta_{i_r}}{\delta_i + \sum_{r=1}^{N-1} \delta_{i_r}} \leq \xi_{\lambda_{(p^i)}} < \frac{\delta_i + \sum_{r=1}^{l} \delta_{i_r}}{\delta_i + \sum_{r=1}^{N-1} \delta_{i_r}}\}} \Bigg) \mathbb{I}_{\{\lambda_i(p^i) = \lambda_{i_1}(p^{i_1}) = \dots = \lambda_{i_l}(p^{i_l}) = \bar{\lambda}\}} \\
& + \sum_{l=1}^{N-1} \sum_{\substack{\{i_1, i_2, \dots, i_l\} \subset I \\ \{i_1, i_2, \dots, i_l\} \not\ni i}} \Bigg(\tau_i^{I \setminus \{i_l\}}(\bar{s}^{I \setminus \{i_l\}})\mathbb{I}_{\{\frac{\sum_{r=1}^{l-1} \delta_{i_r}}{\delta_i + \sum_{r=1}^{N-1} \delta_{i_r}} \leq \xi_{\lambda_{(p^i)}} < \frac{\sum_{r=1}^{l} \delta_{i_r}}{\sum_{r=1}^{N-1} \delta_{i_r}}\}} \Bigg) \mathbb{I}_{\{\lambda_{i_1}(p^{i_1}) = \dots = \lambda_{i_l}(p^{i_l}) = \bar{\lambda}\}}
\end{aligned}$$

The random variables $\tau_i(\bar{s}) \in \tilde{\mathcal{T}}^T$ for $i = 1, 2, \dots, N$ and every $\bar{s} \in \prod_{i=1}^N \tilde{\mathbf{M}}^{T,i}$, .

For each $\bar{s} \in \prod_{i=1}^N \tilde{\mathbf{M}}^{T,i}$ and given lottery the payoff function for the i-th player is defined as $f_i(\bar{s}) = g_i(X_{\tau_i(\bar{s})})$. Let $\tilde{R}_i(x, \bar{s}) = \mathbf{E}_x f_i(\bar{s}) = \mathbf{E}_x g_i(X_{\tau_i(\bar{s})})$ be the expected gain of the Player i-th, if the players use $\bar{s}$. We have defined the game in normal form $(\tilde{\mathbf{M}}^{T,1}, \tilde{\mathbf{M}}^{T,2}, \dots, \tilde{\mathbf{M}}^{T,N}, \tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_N)$. This random priority game will be denoted $\mathcal{G}_{rp}$. Let $\bar{s}^{\{N^*\}} = (\bar{s}^{1^*}, \dots, \bar{s}^{N^*})$, $\tilde{R}_i^N(x, \bar{s}^i) = \tilde{R}_i^N(x, \bar{s}^{1*}, \dots, \bar{s}^i, \dots, \bar{s}^{N*})$ and $\tilde{R}_i^N(x, \bar{s}^{i*}) = \tilde{R}_i^N(x, s^{1*}, \dots, s^{i*}, \dots, s^{N*})$.

Definition 2 *A strategy $\bar{s}$ such that $\bar{s}^{i^*} \in \tilde{\mathbf{M}}^{T,i}$, $i = 1, 2, \dots, N$ is called a Nash equilibrium in $\mathcal{G}_{rp}$, if for all $x \in \mathbb{E}$ and $i = 1, 2, \dots, N$*

$$v_i(x) \;=\; \tilde{R}_i^N(x, \bar{s}^{i^*}) \geq \tilde{R}_i^N(x, \bar{s}^i) \textit{ for every } s^i \in \tilde{\mathbf{M}}^{T,i}$$

The N-tuple $(v_1(x), v_2(x), \dots, v_N(x))$ will be called the Nash value.

In Szajowski [13] the two person non zero sum game has been solved. After the first successful acceptance of a state in the N person game the players who have not accepted a state, are playing as $N-1$ person game. Taking into account the above definition of $\mathcal{G}_{rp}$ one can conclude that the Nash values of this game are the same as in the auxiliary game $\mathcal{G}_{wp}$ with the sets of strategies $\prod_{i=1}^{N}\mathcal{P}^{T,i}$.

$$
\begin{aligned}
\varphi_i(\tilde{p}) &= g_i(X_{\lambda_i(p^i)})\mathbb{I}_{\{\lambda_i(p^i)<\min_{l\neq i}\lambda_l(p^l)\}} \\
&+ \sum_{l\neq i} R_i^{I\setminus\{l\}}(X_{\lambda_l(p^l)},\tilde{p}^l)\mathbb{I}_{\{\lambda_l(p^l)<\min_{j\neq l}\lambda_j(p^j)\}} \\
&+ \sum_{l=1}^{N-1}\sum_{\{i,i_1,i_2,\dots,i_l\}\subset I}\left(g_i(X_{\lambda_i(p^i)})\frac{\delta_i}{\delta_i+\sum_{r\in\{i_1,i_2,\dots,i_l\}}\delta_r}\right. \\
&+ \left.\sum_{r\in\{i_1,\dots,i_l\}} R_i^{I\setminus\{r\}}(X_{\lambda_i(p^i)},\tilde{p}^r)\frac{\delta_r}{\delta_i+\sum_{r\in\{i_1,i_2,\dots,i_l\}}\delta_r}\right)\mathbb{I}_{\{\lambda_i(p^i)=\lambda_{i_1}(p^{i_1})=\dots=\lambda_{i_l}(p^{i_l})=\bar{\lambda}\}} \\
&+ \sum_{l=1}^{N-1}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset I\\ \{i_1,i_2,\dots,i_l\}\not\ni i}}\left(\sum_{r\in\{i_1,\dots,i_l\}} R_i^{I\setminus\{r\}}(X_{\lambda_r(p^r)},\tilde{p}^r)\frac{\delta_r}{\sum_{r\in\{i_1,i_2,\dots,i_l\}}\delta_r}\right)\mathbb{I}_{\{\lambda_{i_1}(p^{i_1})=\dots=\lambda_{i_l}(p^{i_l})=\bar{\lambda}\}}
\end{aligned}
$$

3 Solution of the game

After the first successful acceptance of a state in the N–person game the players, who have not accepted state, play a similar $N-1$–person game. Taking into account the definition of $\mathcal{G}_{rp}$, one can conclude that the Nash values of this game are the same as in the auxiliary game $\mathcal{G}_{wp}$, with the sets of strategies $\prod_{i=1}^{N}\mathcal{P}^{T,i}$ and the gain function $\varphi_i(\tilde{p})$, $\tilde{p}=(p^1,p^2,\dots,p^N)$, defined in an apropriate way.

Let $\mathcal{P}_t^{T,i}=\{\bar{p}=(p_t^i)\in\mathcal{P}:p_1^i=\dots=p_{t-1}^i=0,p_T^i=1\}$, $i=1,2,\dots,N$. If $\bar{p}^i\in\mathbb{P}^{T,i}$, then $(p_t^i,\bar{p}^i)$ is the strategy belonging to $\mathcal{P}^{T,i}$ in which the t-th coordinate is changed to p_t^i.

Definition 3 *A N -tuple $\bar{p}\in\prod_{i=1}^{N}\mathcal{P}_t^{T,i}$ is called an equilibrium point of $\mathcal{G}_{wp}$ at t, if for $i\in\{1,2,\dots,N\}$*

$$v_i(t,X_t) = \mathbf{E}_{X_t}\varphi_i(\tilde{p}^{i^*})\geq\mathbf{E}_{X_t}\varphi_i(\tilde{p}^i)\ \textit{for every}\ \bar{p}^i\in\mathcal{P}_t^{T,i},\ \mathbf{P}_x\textit{-a.s.}$$

A Nash equilibrium point at $t=0$ is a solution of $\mathcal{G}_{wp}$. The N-tuple

$$(v_1(0,x),v_2(0,x),\dots,v_N(0,x))$$

is the Nash value corresponding to $\bar{p} \in \prod_{i=1}^{N} \mathcal{P}^{T,i}$.

In the construction of the solution of the game a sequence of matrix games are solved. Assume that up to moment n the players have not accepted any state and $X_n = x$. In this case the gain matrix $V(n,x) = [(v_i^{k_1,k_2,\dots,k_N})_{i=1}^{N}]_{\substack{k_j=0,1\\ j=1,2\dots,N}}$, is of the form:

$$
v_i^{k_1k_2\dots k_N} = \begin{cases}
g_i(n,x) & k_i = 1,\ k_j = 0,\\
 & j \neq i,\ j = 1,2,\dots,N\\[2ex]
g_i(n,x)\dfrac{\delta_i}{\delta_i + \sum\limits_{r\in\{i_1,i_2,\dots,i_l\}} \delta_r} & k_r = 1 \text{ for } r \in \{i,i_1,i_2,\dots,i_l\}\\
\quad + \sum\limits_{r\in\{i_1,\dots,i_l\}} R_i^{I\setminus\{r\}}((n,x),\bar{p}^r)\dfrac{\delta_r}{\delta_i + \sum\limits_{r\in\{i_1,i_2,\dots,i_l\}} \delta_r} & \text{and } k_r = 0 \text{ otherwise,}\\[2ex]
\sum\limits_{r\in\{i_1,\dots,i_l\}} R_i^{I\setminus\{r\}}((n,x),\bar{p}^r)\dfrac{\delta_r}{\sum\limits_{r\in\{i_1,i_2,\dots,i_l\}} \delta_r} & k_r = 1 \text{ for } r \in \{i_1,i_2,\dots,i_l\}\\
 & \text{and } k_r = 0 \text{ otherwise,}\\[2ex]
Tv_i(n,x) & k_j = 0,\ i,j = 1,2,\dots,N
\end{cases} \tag{1}
$$

Recursive solution of these matrix games (1) gives the following result.

Theorem 1 *There exists a Nash equilibrium $\bar{p}$ in the game $\mathcal{G}_{wp}$. The Nash value and equilibrium point can be calculated recursively.*

The solution of the game $\mathcal{G}_{rp}$ can be constructed from the solution $\bar{p}$ of the corresponding game $\mathcal{G}_{wp}$.

As examples of such games one can consider the related secretary problem.

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