On the cohomology of finite Chevalley groups

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Introduction

Let $G(\mathbb{F}_q)$ be a finite Chevalley group defined over the finite field $\mathbb{F}_q$ with $q$ elements and $l$ a prime number with $(\text{ch}(\mathbb{F}_q), l) = 1$. In this note, we consider the cohomology $H^*(G(\mathbb{F}_q), \mathbb{Z}/l)$ by the étale method inaugurated by a mile-stone paper of Quillen [Q1, Q2]. Friedlander has developed and published a book [F1].

Let $G$ be a Chevalley $\mathbb{Z}$-scheme and $G_k$ a scalar extension by an algebraically closed field $k$ with ch$(k) = p$. Let $X$ be a $k$-scheme equipped with an $G_k$-action and $B(X, G_k)_\bullet$ a classifying simplicial scheme. Then the Deligne spectral sequence [D] is of the form

$$E_2 = \text{Cotor}_{H^*(G_k, \mathbb{Z}/l)}(H^*_\text{et}(X), \mathbb{Z}/l) \Rightarrow H^*(B(X, G)_\bullet, \mathbb{Z}/l))$$

using the Lang isogeny [L]: $G(\mathbb{F}_q) \backslash G_k \simeq G_k$, the right $G_k$-action of $G$ is given by the $F$-conjugation where $F$ is the $q$-th Frobenius map. Then the above spectral sequence takes the form

$$E_2 = \text{Cotor}_{H^*(G_k, \mathbb{Z}/l)}(H^*_\text{et}(G_k, \mathbb{Z}/l), \mathbb{Z}/l) \Rightarrow H^*(B(G(\mathbb{F}_q) \backslash G_k, G)_\bullet, \mathbb{Z}/l)).$$

We prove it in §1. The Lang map $L$ induces a Galois covering $G_k \to G(\mathbb{F}_q) \backslash G_k$ and $B(G_k, G_k)_\bullet \to B(G(\mathbb{F}_q) \backslash G_k, G_k)_\bullet$ is considered as a Galois covering between the simplicial schemes. Generally, let $p : Y_\bullet \to X_\bullet$ be a Galois covering with its Galois group. Then we construct the Hochschild-Serre spectral sequence. In the above case, there is a spectral sequence such that

$$E_2 = H^p(G(\mathbb{F}_q), H^q(B(G_k, G_k)_\bullet, \mathbb{Z}/l)) \Rightarrow H^{p+q}(G(\mathbb{F}_q), \mathbb{Z}/l).$$

Applying the Deligne spectral sequence, it is easily shown that $H^*(B(G_k, G_k)_\bullet, \mathbb{Z}/l)$ is acyclic. Hence the above spectral sequence collapses at the $E_2$-term. After all, we get a spectral sequence which converges to the cohomology of a finite Chevalley group. We call the spectral sequence Deligne-Eilenberg-Moore spectral sequence.
1. Deligne-Eilenberg-Moore spectral sequence

In this section, we introduce a spectral sequence of Eilenberg-Moore type converging to $H^*(G(F_q);\mathbb{Z}/l)$. For general arguments, we refer to Friedlander [F1].

First let us recall the simplicial scheme $B(X, G)$ from [F1, 1. Example 2]. Let $S$ be a scheme, $G$ a group scheme over a scheme $S$ and $X$ a scheme over $S$ equipped with a right $G$-action $X \times_S G \rightarrow X$. Then the simplicial scheme $B(X, G)$ is defined by

\[(1.1) \quad B(X, G)_n = X \times_S G \times_S \cdots \times_S G.\]

We define the face operators $d_i : B(X, G)_n \rightarrow B(X, G)_{n-1}$ for $0 \leq i \leq n$ by

\[(1.2) \quad \begin{align*}
d_0(x, g_1, \ldots, g_n) &= (x, g_1, g_2, \ldots, g_n) \quad (i = 0), \\
d_i(x, g_1, \ldots, g_n) &= (x, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) \quad (1 \leq i \leq n-1), \\
d_n(x, g_1, \ldots, g_n) &= (x, g_1, \ldots, g_{n-1}) \quad (i = n),
\end{align*}\]

for $x \in X(T)$, $g_i \in G(T)$, where $T$ is a scheme over $S$ and $X(T)$ and $G(T)$ are $T$-valued points defined by $X(T) = \text{Hom}_S(X, T)$ and $G(T) = \text{Hom}_S(G, T)$.

Let $X$, $T$ be schemes over $S$. Then we denote $T \times_S X$ by $X_T$, which is considered as a scheme over $T$.

Now we recall here the definition of the Lang map (see [L]). Let $G_{F_q}$ be a linear algebraic group over $F_q$ and $\phi$ the Frobenius automorphism defined by $\phi(x) = x^q$ for $x \in k$, where $k$ is an algebraically closed field of $F_q$. Then $\phi$ can be considered as a morphism of $G_{F_q}$ and $G_k$ which is a linear algebraic group over $k$. We consider the Lang map $\mathcal{L} : G_k \rightarrow G_k$ which can be defined by $\mathcal{L}(x) = \phi^{-1}(x)x$ for $x \in G(k)$.

**Lemma 1.1** (Lang [L]). There holds $\mathcal{L}(x) = \mathcal{L}(y)$ for $x, y \in G(k)$ if and only if $y = ax$ for some $a \in G(F_q)$.

**Lemma 1.2** (Lang [L]). (1) The map $\mathcal{L} : G(k) \rightarrow G(k)$ is surjective. Hence $G_k$ is a principal left $G(F_q)$-space over $G_k$ and $\mathcal{L}$ induces an isomorphism $G(F_q) \backslash G_k \cong G_k$.

(2) If we define a right $G_{F_q}$-action on $G_k$ by

\[(1.3) \quad z \cdot x = \phi(x)^{-1}zx \quad \text{for } x, z \in G(k),\]

then there holds

$z \cdot (xy) = (z \cdot x) \cdot y, \quad z \cdot 1 = z.$

Moreover, the induced isomorphism

$\mathcal{L} : G(F_q) \backslash G_k \cong G_k$

is a right equivariant $G_{F_q}$-map, that is, there holds

$\mathcal{L}([z]x) = z \cdot x$.
for \( z \in (G(F_{q}) \backslash G_{k})(k) \) and \( x \in G(k) \).

**Corollary 1.3.** The isomorphism in the above lemma induces an isomorphism

\[
B(G(F_{q}) \backslash G_{k}, G_{k}) \cong B(G_{k}, G_{k})
\]
as a simplicial scheme, where the right \( G_{k} \)-action on \( G_{k} \) is given by \( z \cdot x = \phi(x)^{-1}zx \) for \( x, y \in G_{k} \).

**Theorem 1.4.** Let \( G_{\mathbb{Z}} \) be a Chevalley group scheme of Lie type over \( \mathbb{Z} \). Then we have a spectral sequence \( \{E_{r}\} \) of Eilenberg-Moore type such that

\[
E_{2} = \text{Cotor}_{H_{et}^{*}(G_{k}; \mathbb{Z}/l)}(H_{et}^{*}(G_{k}; \mathbb{Z}/l), \mathbb{Z}/l),
\]

\[
E_{\infty} = \text{gr}H^{*}(c(F_{q}; \mathbb{Z}/l), \mathbb{Z}/l),
\]

where \( l \) is a prime such that \((l, q) = 1\) and \( k \) is an algebraically closed field of \( F_{q} \). The comodule structure of \( H_{et}^{*}(G_{k}; \mathbb{Z}/l) \) is induced from (1.3) in Lemma 1.2.

The Eilenberg-Moore spectral sequence of a simplicial scheme for complex algebraic groups is given by Deligne [D]. However, as his proof seems not to be appropriate in our context, we give here a proof following Friedlander [F1], and so we use his notations.

We recall a constant sheaf \( \mathbb{Z}/l \) on the étale site \( \text{Et}(B(Y, G)) \). If we denote by \( (\mathbb{Z}/l)_{n} \) a constant sheaf \( \mathbb{Z}/l \) on \( \text{Et}(B(Y, G)) \), then a constant sheaf \( \mathbb{Z}/l \) is a collection of \( (\mathbb{Z}/l)_{n} \) for \( n \geq 0 \) satisfying the following property: if \( \alpha^{*}: X_{m} \to X_{n} \) is a map induced from a simplicial map \( \alpha: \Delta(n) \to \Delta(m) \), then it induces the identification \( \alpha^{*}(\mathbb{Z}/l)_{m} = (\mathbb{Z}/l)_{n} \).

**Proposition 1.5 ([F1] Proposition 2.2).** Let \( X_{*} \) be a simplicial scheme and \( F \) an abelian sheaf on \( \text{Et}(X_{*}) \). Then \( F \to \prod_{n=0}^{\infty} R_{n}(I_{n}^{*}) \) is an injective resolution in \( \text{Absh}(X_{*}) \), where the function

\[
R_{n}(\ ) : \text{Absh}(X_{n}) \to \text{Absh}(X_{*})
\]
is defined by

\[
(R_{n}(G))_{m} = \prod_{\Delta(m)} \alpha^{*}G,
\]
such that each restriction \( F_{n} \to I_{n}^{*} \) is an injective resolution on \( \text{Absh}(X_{n}) \).

Moreover we have

\[
\text{Hom}_{X_{*}}(R_{n}(G), F) \cong \text{Hom}_{X_{n}}(G, F_{n}).
\]

**Proof of Theorem.** Let us recall the complex defined in [F1, Proposition 2.4]; let \( L^{n}(\ ) : \text{Absh}(X_{n}) \to \text{Absh}(X_{*}) \) be defined by

\[
(L^{n}(G))_{m} = \bigoplus_{\alpha \in \Delta[m]} \alpha^{*}G, \ n \geq 0
\]
for $G \in \text{Absh}(X_n)$. From the definition of a sheaf on a simplicial scheme, we see that
\[
\text{Hom}_{X_\bullet}(L^n(G), E) \cong \text{Hom}_{X_\bullet}(G, F_n)
\]
for $F \in \text{Absh}(X_\bullet)$. We set
\[
L^n(Z|_{X_m}) = Z\langle m \rangle
\]
for $Z|_{X_m} \in \text{Absh}(X_m)$. From the definition of $L^n$, we have
\[
(Z\langle m \rangle)_n = \bigoplus_{\Delta[n]_m} Z
\]
on $X_n$. We define the augmented complex of sheaves
\[
\{C(\cdot) = \bigoplus_{m=0} \mathbb{Z}\langle m \rangle, \partial\langle m \rangle : \mathbb{Z}\langle m \rangle \to \mathbb{Z}\langle m - 1 \rangle\}
\]
in the following manner. Restricting to $X_n$, an augmentation and a boundary operator
\[
(\varepsilon)_n : (\mathbb{Z}\langle 0 \rangle)_n \to (\mathbb{Z})_n,
\partial\langle n \rangle_n : (\mathbb{Z}\langle m \rangle)_n \to (\mathbb{Z}\langle m - 1 \rangle)_n
\]
are given by the summation
\[
\bigoplus_{\Delta[n]_0} \mathbb{Z}(U) \to \mathbb{Z}(U),
\sum_{i=0}^{m} (-1)^i \partial_i : \bigoplus_{\Delta[n]_m} \mathbb{Z}(U) \to \bigoplus_{\Delta[n]_{m-1}} \mathbb{Z}(U)
\]
for $U \to X_n$ in $\text{Et}(X_\bullet)$ respectively.

When we restrict the complex to $X_n$, we see that
\[
C(\cdot)_n \simeq C_\bullet(\Delta[n]),
\]
where $C_\bullet(\Delta[n])$ is the augmented chain complex of a simplex $\Delta[n]$. Since the restriction functor $(\cdot)_n$ (see [F1]) is exact and since $C_\bullet(\Delta[n])$ is acyclic, the complex $C(\cdot)$ is acyclic in $\text{Absh}(X_\bullet)$. We denote for simplicity $C(m)$ and $\partial(m)$ by $C^{-m}$ and $\partial^{-m}$ respectively.

Let $F \to I^\bullet$ be an injective resolution of $F$ in $\text{Absh}(X_\bullet)$ and $\delta^i : I^{i+1} \to I^{i+1}$ a difference. We denote
\[
\prod_{q \geq 0} \text{Hom}_{\text{Absh}(X_\bullet)}^n(C^{-q}, I^{-q+n})
\]
simply by $\text{Hom}_{\text{Absh}(X_\bullet)}^n(C^\bullet, I^\bullet)$. We define that
\[
\text{Hom}^\bullet(C^\bullet, I^\bullet) = \bigoplus_{n \geq 0} \text{Hom}_{\text{Absh}(X_\bullet)}^n(C^\bullet, I^\bullet)
\]
and that
\[
(\delta^n f)^{-q} = \delta^{-q+n} f^{-q} + (-1)^{n+1} f^{-q+1} \partial^{-q}
\]
for $f = (f^{-q}) \in \text{Hom}^n(C^{-q}, C^{-q+n}) = \text{Hom}^n(C^\bullet, I^\bullet)$. 

We consider a spectral sequence associated with the double complex defined as follows. We define the first filtration by

\[ F^I = \text{Hom}^*(C^*, \bigoplus_{n \leq p} I^n), \]

where we define two kinds of differentials \( \delta_I \) and \( \delta_{II} \) respectively by

\[
(\delta_I f)^{-q} = \delta^{-q+n} f^{-q},
\]

\[
(\delta_{II} f)^{-q} = (-1)^{n+1} f^{-q+1} \partial^{-q}
\]

and define \( \delta = \delta_I + \delta_{II} \).

Since \( C^* \) is acyclic and since \( I^* \) is injective, we see that

\[
E_1^{Ip} = H(F_p^I / F_{p-1}^I, \delta_2) = \begin{cases} 0 & (p \geq 0) \\ \text{Hom}(\mathbb{Z}, I^*) & (p = 0). \end{cases}
\]

From the definition of the cohomology, we have

\[
E_2^{lp} = H^p(\text{Hom}(\mathbb{Z}, I), \delta_1) = H^p(X_*, F).
\]

We see immediately that \( E_2^p = E_{\infty}^p \), which implies that

\[
H^n(\text{Hom}^*(C^*, I^*) \delta) = H^n(X_*, F).
\]

We define the second filtration by

\[ F_p^{II} = \text{Hom}^*\big(\bigoplus_{m \leq p} C^{-m}, I^*\big). \]

Then we see that (where \( q = \deg f - p \)):

\[
E_1^{pq} = H^q(\text{Hom}^*(C^{-p}, I^*), \delta_1) = H^q(\text{Hom}^*(I^p(\mathbb{Z}_{X_p}), I^*), \delta_I)
\]

\[
= H^q(\text{Hom}^*_{X_p}(\mathbb{Z}, I^*[p]), \delta_{II}|_{X_p}) = H^{p+q}(X_p, F_p).
\]

**Proposition 1.6** ([F1], Proposition 2.4). We have a spectral sequence \( \{E_r^{pq}\} \) such that

\[
E_1^{pq} = H^{p+q}(X_p; F_p),
\]

\[
E_\infty^{pq} = \text{gr} H^{p+q}(X_*, F).
\]

We apply this spectral sequence to

\[
X_\ast = B(G_k, G_k) \cong B(G(F_q) \setminus G_k, G_k)
\]

with \( F = \mathbb{Z}/l \).

**Lemma 1.7.** We have a spectral sequence \( \{E_r\} \) such that

\[
E_1^{p,*} = H^*_\text{et}(G_k; \mathbb{Z}/l) \otimes \bigotimes_{\mathbb{Z}/l} (H^*_\text{et}(G_k; \mathbb{Z}/l)[1]),
\]

\[
E_\infty = \text{gr} H^*(B(G(F_q) \setminus G_k, G_k); \mathbb{Z}/l).
\]

**Proposition 1.8.** We have

\[
E_2^{p,*} = \text{Cotor}^{p}_{H^*_\text{et}(G_k; \mathbb{Z}/l)}(H^*_\text{et}(X_\ast; \mathbb{Z}/l), \mathbb{Z}/l).
\]
Proof. The non-decreasing function $\delta_{1}^{*} : [p] \rightarrow [p+1]$ such that $i \not\in \text{Im } d_{1}$ induces a simplicial map $\partial_{i} : \Delta[p+1] \rightarrow \Delta[p]$ defined by

$$\partial_{i}[0,1,\ldots,p+1] = [0,1,\ldots,i,\ldots,p]$$

and the morphism $d_{i} : X_{p+1} \rightarrow X_{p}$ defined by (1.2). So, the morphism $\delta_{1}^{*} : \text{Hom}(C_{1,X_{p}}, I_{1,X_{p}}^{*}) \rightarrow \text{Hom}(C_{1,X_{p+1}}, I_{1,X_{p+1}}^{*})$ defined by $f \partial_{i}$ is induced from the inverse image of a sheaf $\mathbb{Z}/l$ by $d_{i} : X_{p+1} \rightarrow X_{p}$. Hence we have

$$E_{2} = H(E_{1}, \delta_{1})$$

and

$$\delta_{1} = (1)^{p+1} \sum_{i=0}^{p} (-1)^{i} \delta_{i}^{*} = (1)^{p+1} \sum_{i=0}^{p} (-1)^{i} d_{i}^{*} : E_{1}^{p,*} \rightarrow E_{1}^{p,*}.$$  

In this case, we can give an explicit representation of $d_{i}^{*}$ as follows; let

$$\Delta_{X} : H_{\text{et}}^{*}(X; \mathbb{Z}/l) \rightarrow H_{\text{et}}^{*}(X; \mathbb{Z}/l) \otimes H_{\text{et}}^{*}(G_{k}; \mathbb{Z}/l)$$

and

$$\Delta : H^{*}(G_{k}; \mathbb{Z}/l) \rightarrow H_{\text{et}}^{*}(G_{k}; \mathbb{Z}/l) \otimes H^{*}(G_{k}; \mathbb{Z}/l)$$

be the comodule map and the coalgebra map respectively induced from a right $G$-action $X \times G_{k} \rightarrow X$ and a multiplication $G_{k} \times G_{k} \rightarrow G_{k}$. Then we obtain

$$d_{i}^{*}(m \otimes x_{1} \otimes \cdots \otimes x_{p}) = \begin{cases} \Delta_{X}(m) \otimes x_{1} \otimes \cdots \otimes x_{p} & (i = 0) \\ m \otimes x_{1} \otimes \cdots \otimes x_{i-1} \otimes \Delta(x_{i}) \otimes x_{i+1} \otimes \cdots \otimes x_{p} & (1 \leq i \leq p-1) \\ m \otimes x_{1} \otimes \cdots \otimes x_{p} \otimes 1 & (i = p). \end{cases}$$

Therefore we have shown that

$$E_{2}^{p,*} = \text{Cotor}_{H_{\text{et}}^{*}(G_{k}; \mathbb{Z}/l)}^{p}(H_{\text{et}}^{*}(X; \mathbb{Z}/l), \mathbb{Z}/l).$$

$\square$
2. Hochschild-Serre spectral sequence

In this section, we construct the Hochschild-Serre spectral sequence for simplicial schemes in a little more direct way than in Milne [Mi].

Let $X_*$ and $Y_*$ be simplicial schemes over a field $k$. Then we call $\pi_* : Y_* \to X_*$ a finite Galois cover with Galois group $G$ if $\pi_n : Y_n \to X_n$ is a finite Galois cover with Galois group $G$ for all $n$ and if $\pi_*$ is compatible with the face and degeneracy operators.

**Theorem 2.1.** Let $\pi_* : Y_* \to X_*$ be a finite Galois cover with Galois group $G$ for simplicial schemes. Let $F$ be an abelian sheaf on $\text{Et}(X_*)$. Then we have a Hochschild-Serre spectral sequence $\{E_r^{p,q}\}$ such that

$$E_2^{p,q} = H^p(G, H^q(X_*; F)),$$

$$E_\infty = \text{gr} H^{p+q}(Y_*; F).$$

To prove the theorem, we prepare some notations.

Let $(B_*(G, G), \partial_*, \sigma_*)$ and $(Y_*, d_*, s_*)$ be simplicial schemes defined in the section 1. Then we define a double simplicial scheme $B(G, G)_\bullet \boxtimes Y_\bullet$ as follows; as schemes, we set

$$B(G, G)_p \boxtimes Y_q = \prod_{g_I \in G^{p+1}} Y_{q, g_I},$$

where $Y_{q, g_I}$ is indexed by $g_I \in G^{p+1} = B(G, G)_p$ and we have $Y_{q, g_I} \cong Y_q$ as schemes.

We denote $Y_{q, g_I}$ by $g_I \boxtimes Y_t$. Then we define two kinds of face operators

$$\delta_p^i \boxtimes 1_q : B(G, G)_p \boxtimes Y_q \to B(G, G)_{p-1} \boxtimes Y_q,$$

$$1_p \boxtimes d_q^i : B(G, G)_p \boxtimes Y_q \to B(G, G)_p \boxtimes Y_{q-1}$$

by

$$\delta_p^i \boxtimes 1_q((g_0, g_1, \ldots, g_p, y)) = \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_p, y) & (0 \leq i \leq p - 1) \\ (g_0, g_1, \ldots, g_{p-1}, y) & (i = p), \end{cases}$$

$$1_p \boxtimes d_q^i((g_0, g_1, \ldots, g_p, y)) = (g_0, g_1, \ldots, g_p, d_q^i(y))$$

respectively, where we identify $B(G, G)_p \boxtimes Y_q(S)$ with $G^{(p+1)} \times Y_q(S)$ for a $k$-scheme $S$. Similarly we define two kinds of degeneracy operators

$$\sigma^i_p \boxtimes 1_q : B(G, G)_p \boxtimes Y_q \to B(G, G)_{p+1} \boxtimes Y_q,$$

$$1_p \boxtimes s_q^i : B(G, G)_p \boxtimes Y_q \to B(G, G)_p \boxtimes Y_{q+1}$$

by

$$\sigma^i_p \boxtimes 1_q((g_0, g_1, \ldots, g_p, y)) = (g_0, \ldots, g_i, e, g_{i+1}, \ldots, g_p, y)$$

$$1_p \boxtimes s_q^i((g_0, g_1, \ldots, g_p, y)) = (g_0, g_1, \ldots, g_p, s_q^i(y)).$$

By abuse of notation we put

$$\delta_p^i \boxtimes 1_q = \delta^i_p, \quad 1_p \boxtimes d_q^i = d_q^i, \quad \sigma^i_p \boxtimes 1_q = \sigma^i_p, \quad 1_p \boxtimes s_q^i = s_q^i.$$
We define a $G$-action on $B(G, G)_* \boxtimes Y_*$ by

$$g((g_0, g_1, \ldots, g_p, y)) = (gg_0, g_1, \ldots, g_p, y), \quad g \in G.$$  

Clearly we see that the $G$-action is compatible with all the face and degeneracy operators, and we have the identities

$$\partial_p^i s_q^i = s_q^i \sigma_p^i \quad \sigma_p^i \partial_q^i = \partial_q^i \sigma_p^i.$$

**Remark 2.2.** Let $F$ be an abelian sheaf on $Y_{\text{set}}$. Then we can consider $F$ as a sheaf on $X_{\text{set}}$, because $Y_*$ is a Galois cover over $X_*$. The Galois group $G$ acts on $F$ from the right hand side and on $Y_*$ from the left one. Moreover $\prod_{g_I \in G^{p+1}} F_{g_I}$ is a sheaf on $B(G, G)_p \boxtimes (Y_{\text{set}})$.

$$B(G, G)_p \boxtimes (Y_{\text{set}}) = \prod_{g_I \in G^{p+1}} (Y_{\text{set}})_{g_I},$$

where $F_{g_I}$ is a sheaf $F$ indexed by $g_I \in G^{p+1}$.

In the similar manner to before, we can associate the sheaf on $B(G, G)_p \boxtimes (Y_{\text{set}})$ to a sheaf $F$ on $Y_{\text{set}}$ and denote its sheaf by $\mathfrak{B} = (\mathfrak{B}_{g_I})$.

We also have

$$F(B(G, G)_p \boxtimes U) = \prod_{g_I \in G^{p+1}} F(U_{g_I}),$$

and denote its section by $s = (s_{g_I})$. The face operator

$$\partial_p^i : F(B(G, G)_p \boxtimes U) \to F(B(G, G)_{p+1} \boxtimes U)$$

is given by

$$\partial_p^i s_{g_0, \ldots, g_p} = \begin{cases} s(g_0, \ldots, g_i g_{i+1}, \ldots, g_p) & (0 \leq i \leq p - 1) \\ s(g_0, g_1, \ldots, g_{p-1}) & (i = p). \end{cases}$$

We consider an injective resolution

$$0 \to F \to \mathcal{I}^*$$

of $F$ on $X_{\text{set}}$ and define the sheaf complex

$$(C^*, d^*) = (\bigoplus_{n \geq 0} \oplus C^{p,q}, \bigoplus_{n \geq 0} \oplus d^{p,q})$$
by
\[ C^{p,q} = B(G, G) \otimes I^q, \]
\[ d^{p,q} = (-1)^i \partial^i_p + (-1)^p d^q_p. \]

Then we have

**Lemma 2.3.** The \( G \)-free complex \( C^* \) gives rise to also an injective resolution on \( X_{\text{et}} \):

\[ 0 \to F \xrightarrow{d^*} C^*. \]

**Proof.** Since \( I^q \) is injective and since \( B(G, G)_p \otimes I^q \) is a direct product of \( I^q \), we see that \( B(G, G)_p \otimes I^q \) is injective and \( C^n \) is injective on \( X_{\text{et}} \). Hence we will show that \( 0 \to F \to C^* \) is acyclic. For a fixed geometric point \( \overline{x} \), it is enough to show that \( 0 \to F_{\overline{x}} \to C^*_{\overline{x}} \) is acyclic.

We calculate the homology of the double complex \((C_i, d^\cdot)\) by using a spectral sequence. We introduce filtration \( F^n C^*_{\overline{x}} \) by \( \bigoplus_{n \geq 0} C_{\varpi}^p \) for \( n \) and consider the associated spectral sequence. From the injective resolution of \( F \), we see that

\[ E_1^{p,q} = H^q(B(G, G)_p \otimes I^q_{s}, (-1)^p d^q_p) \]
\[ = \begin{cases} 0 & (q \geq 0) \\ B(G, G)_p \otimes F_{\overline{x}} & (q = 0). \end{cases} \]

The differential \( d_1 \) is given by \( \sum_{i=0}^p (-1)^i \partial^i_p \) from (2.2) in Remark 2.2.

Forgetting the \( G \)-action on \( F_{\overline{x}} \), we obtain

\[ E_1^{p,0} = \text{Hom}_Z(Z[G] \otimes B^p(G), Z) \otimes Z F_{\overline{x}}, \]

where \( Z[G] \otimes B^*(G) \) is the standard bar complex of \( G \) over \( Z \) and \( Z[G] \) is a group ring over \( Z \) [Mc]. Hence we obtain that

\[ E_2^{p,0} = \begin{cases} 0 & p > 0 \\ F_{\overline{x}} & p = 0. \end{cases} \]

It is easy to prove that \( H^*(C, d^*) = F_{\overline{x}}. \)

**Lemma 2.4.** Let \( \Gamma_X \) and \( \Gamma_Y \) be the section functors of \( X_* = \{X_n\} \) and \( Y_* = \{Y_n\} \) respectively. Then for a sheaf \( F \) on \( Y_{\text{et}} \), we have

\[ \Gamma_X(F) = \Gamma_Y(F)^G. \]

**Proof.** From the definition of the section functor [F1, D] we recall that

\[ \Gamma_Y(F) = \text{Ker}(F(Y_0) \xrightarrow{d_0^1} F(Y_1)). \]

Since \( Y_i/X_i \) is a Galois cover with the same Galois group \( G \), we have

\[ F(X_i) = F(Y_i/G) = F(Y_i)^G. \]
Observing that the face operators are compatible with the $G$-action, we have
\[ \Gamma_{X_{*}}(F) = \Gamma_{Y_{*}}(F) \cap \Gamma(X_{0}, F) = \Gamma_{Y_{*}}(F) \cap \Gamma(Y_{*}, F)^{G} = \Gamma_{Y_{*}}(F)^{G}. \]
From Lemmas 1.1 and 1.2, we summarize that
\[ H^{n}(X_{*}; F) = H^{n}(\Gamma_{Y_{*}}(C_{*})^{G}). \]
\[ \square \]

**Lemma 2.5.** We have
\[ \Gamma_{Y_{*}}(C^{p,q}) = \text{Hom}_{Z}(\mathbb{Z}[G] \otimes B^{p}(G), \Gamma_{Y_{*}}^{q}(I^{q})), \]
\[ \Gamma_{Y_{*}}(C^{p,q})^{G} \cong \text{Hom}_{Z}(B^{p}(G), \Gamma_{Y_{*}}^{q}(I^{q})), \]
where $\mathbb{Z}[G] \otimes B^{p}(G)$ is the standard bar complex of $G$ over $\mathbb{Z}$.

**Proof.** From the construction of $C^{p,q}$, we have
\[ \Gamma_{Y_{*}}(C^{p,q}) = \prod_{g_{I} \in G_{p+1}} \Gamma_{Y_{*}}(I_{g_{I}}^{q}) = \prod_{g_{I} \in G_{p+1}} \Gamma_{Y_{*}}(I_{g_{I}}^{q}), \]
where $I_{g_{I}}^{q} \cong I^{q}$. Noting Remark 2.2, we see that $I^{*}$ is a right $G$-module and so the left $G$-action is given by
\[ g^{-1}s = sg \]
for $g \in G(k) = G_{Z}(k)$ and $s \in I^{*}(U)$, where $U \rightarrow X_{n}$ is any étale map.

Hence the left action of $G$ on $\prod_{g_{I} \in G_{p+1}} \Gamma_{Y_{*}}(I_{g_{I}}^{q})$ is given by
\[ g * s_{(g_{0}, g_{1}, \ldots, g_{p})} = g^{-1}(s_{(g_{0}, g_{1}, \ldots, g_{p})}) \]
for $(s_{g_{I}}) \in \prod_{g_{I} \in G_{p+1}} \Gamma_{Y_{*}}(I_{g_{I}}^{q})$, $g_{I} = (g_{0}, g_{1}, \ldots, g_{p})$ and $g, g_{i} \in G$.

When we identify $\prod_{g_{I} \in G_{p+1}} \Gamma_{Y_{*}}(I_{g_{I}}^{q})$ with $\text{Hom}_{Z}(\mathbb{Z}[G] \otimes B^{p}(G), \Gamma_{Y_{*}}(I^{q}))$ by
\[ f(g_{0}, g_{1}, \ldots, g_{p}) = s_{(g_{0}, g_{1}, \ldots, g_{p})} \]
for $f \in \text{Hom}_{Z}(\mathbb{Z}[G] \otimes B^{p}(G), \Gamma_{Y_{*}}(I^{q}))$, the $G$-module structure is given by
\[ (gf)(g_{0}, g_{1}, \ldots, g_{p}) = g^{-1}f(gg_{0}, g_{1}, \ldots, g_{p}). \]
Therefore we see that
\[ \Gamma_{Y_{*}}(C^{p,q})^{G} = \text{Hom}_{Z[G]}(\mathbb{Z}[G] \otimes B^{p}(G), \Gamma_{Y_{*}}(I^{q})) \]
\[ \cong \text{Hom}_{Z}(B^{p}(G), \Gamma_{Y_{*}}(I^{q})). \]
\[ \square \]

Under these preparations, the construction of the spectral sequence is a routine argument from the double complex. We define the filtration
of the complex $\Gamma_Y(C^*)^G$ by $F^n = \bigoplus_{p \geq n} \Gamma_Y(C^p)^G$. From Lemma 2.5, it follows that

$$E_1^{p,q} \cong \text{Hom}_{Z[G]}(Z[G] \otimes B^p(G), H^q(\Gamma_Y(I^*), d_F))$$

$$= \text{Hom}_{Z}(Z[G] \otimes B^p(G), H^q(Y_*, F)).$$

As shown in the proof of Lemma 2.3, the differential $d_1$ is given by

$$d_1 = \sum_{i=0}^{p} (-1)^i \partial_i^*.$$

Hence we obtain that

$$E_2^{p,q} = H^p(G, H^q(Y_*, F)).$$

Thus we have the spectral sequence which converges to

$$E_\infty^{*,*} = \text{gr}H^*(X_*, F).$$

Now we apply the Hochschild-Serre spectral sequence in the following form. Let $G_k$ be an algebraic group defined over a prime field $\mathbb{F}_p$ and $G(\mathbb{F}_q)$ the finite group consisting of its $\mathbb{F}_q$-rational points with $q = p^n$. Then according to Lang [L], the left coset $G(\mathbb{F}_q) \backslash G_k$ is a $k$-affine scheme [Se, III, 12] and $G(\mathbb{F}_q)$ is a finite Galois cover over $G(\mathbb{F}_q) \backslash G_k$ with Galois group $G(\mathbb{F}_q)$. So we can take $B_*(G_k, G_k)$ and $B_*(G(\mathbb{F}_q) \backslash G_k, G_k)$ as $Y_*$ and $X_*$ in the above argument. Under the present context, the spectral sequence $\{E_r^{p,q}\}$ takes the form

$$E_2^{p,q} = H^p(G(\mathbb{F}_q), H^q(B_*(G_k, G_k); F))$$

$$\Rightarrow H^{p+q}(B_*(G(\mathbb{F}_q) \backslash G_k, G_k); F).$$

Lemma 2.6 (Friedlander [F2]). For a reductive algebraic group $G_k$ defined and split over $\mathbb{F}_p$, we have

$$H^n(B_*(G_k, G_k); \mathbb{Z}/l) = 0 \quad \text{for } n > 0.$$

Proof. We consider the Deligne-Eilenberg-Moore spectral sequence

$$E_1^{n,*} = H^*(B_n(G_k, G_k); \mathbb{Z}/l) \Rightarrow H^*(B_*(G_k, G_k); \mathbb{Z}/l).$$

From Friedlander-Parshall [FP], we can apply the Künneth formula to $H^*(B_n(G_k, G_k); \mathbb{Z}/l)$. We have

$$H^*(B_n(G_k, G_k); \mathbb{Z}/l) \cong H_{et}^*(G_k; \mathbb{Z}/l)^{\otimes n},$$

which implies that the $E_1$-term is the cobar complex of $H_{et}^*(G_k; \mathbb{Z}/l)$ over $\mathbb{Z}/l$. Hence we have

$$E_2^{p,q} = 0 \quad \text{except } p = q = 0.$$

Theorem 2.7. For a reductive algebraic group $G_k$ defined and split over $\mathbb{F}_q$, we have

$$H^*(G(\mathbb{F}_q); \mathbb{Z}/l) \cong H^*(B(G(\mathbb{F}_q) \backslash G_k, G_k); \mathbb{Z}/l).$$
Proof. That the spectral sequence (2.4) collapses follows from Lemma 2.6. Then the rest of the assertion can be proved straightforwardly. □

Together with the Deligne-Eilenberg-Moore spectral sequence, we can now state the main theorem.

Theorem 2.8. For a reductive algebraic group $G_k$ defined and split over $\mathbb{F}_q$, we obtain the spectral sequence $\{E_r\}$ such that

\[
E_2 = \text{Cotor}_{H_{et}^* (G; \mathbb{Z}/l)}(H_{et}^* (G; \mathbb{Z}/l), \mathbb{Z}/l),
\]

\[
E_{\infty} = \text{gr} H^* (G(\mathbb{F}_q), \mathbb{Z}/l).
\]
References


