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Generalizations of the results on powers of p-hyponormal operators

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M.Ito, *Several properties on class A including p-hyponormal and log-hyponormal operators*, Math. Inequal. Appl., **2** (1999), 569–578.

M.Ito, *Generalizations of the results on powers of p-hyponormal operators*, to appear in J. Inequal. Appl.

Abstract

We shall show that *"if T is a p-hyponormal operator for $p > 0$, then T^n is $\min\{1, \frac{p}{n}\}$-hyponormal for any positive integer n"* and related results as generalizations of the results by Aluthge-Wang [2] and Furuta-Yanagida [11].

1 Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

An operator T is said to be *p-hyponormal* for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$. p-Hyponormal operators were defined as an extension of hyponormal ones, i.e., $T^*T \geq TT^*$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p \geq q > 0$ by Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$,"* and it is well known that there exists a hyponormal operator T such that T^2 is not hyponormal [13], but paranormal [7], i.e., $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. We remark that every p-hyponormal operator for $p > 0$ is paranormal [3] (see also [1][5][10]).

Recently, Aluthge and Wang [2] showed the following results on powers of p-hyponormal operators.

Theorem A.1 ([2]). *Let T be a p-hyponormal operator for $p \in (0, 1]$. The inequalities*

$$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}$$

hold for all positive integer n.

Corollary A.2 ([2]). *If T is a p-hyponormal operator for $p \in (0, 1]$, then T^n is $\frac{p}{n}$-hyponormal for any positive integer n.*

By Corollary A.2, if T is a hyponormal operator, then T^2 belongs to the class of $\frac{1}{2}$-hyponormal operators which is smaller than that of paranormal operators.

As a more precise result than Theorem A.1, Furuta and Yanagida [11] obtained the following result.

Theorem A.3 **([11, Theorem 1]).** *Let T be a p-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n^*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1} \text{ and } (TT^*)^{p+1} \geq (T^nT^{n^*})^{\frac{p+1}{n}}$$

hold for all positive integer n.

Theorem A.3 asserts that the first and third inequalities of Theorem A.1 hold for the larger exponents $\frac{p+1}{n}$ than $\frac{p}{n}$ in Theorem A.1. In fact, Theorem A.3 ensures Theorem A.1 by Löwner-Heinz theorem for $\frac{p}{p+1} \in (0,1)$ and p-hyponormality of T.

On the other hand, Fujii and Nakatsu [6] showed the following result.

Theorem A.4 **([6]).** *For each positive integer n, if T is an n-hyponormal operator, then T^n is hyponormal.*

We remark that Theorem A.1, Corollary A.2 and Theorem A.3 are results on p-hyponormal operators for $p \in (0,1]$, and Theorem A.4 is a result on n-hyponormal operators for positive integer n. In this report, more generally, we shall discuss powers of p-hyponormal operators for all positive real number $p > 0$.

2 Main results

Theorem 1. *Let T be a p-hyponormal operator for $p > 0$. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq (T^*T)^n$ *and* $(TT^*)^n \geq T^nT^{n^*}$ *hold for positive integer n such that $n < p+1$.*

(2) $(T^{n^*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1}$ *and* $(TT^*)^{p+1} \geq (T^nT^{n^*})^{\frac{p+1}{n}}$ *hold for positive integer n such that $n \geq p+1$.*

Corollary 2. *Let T be a p-hyponormal operator for $p > 0$. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq T^nT^{n^*}$ *holds for positive integer n such that $n < p$.*

(2) $(T^{n^*}T^n)^{\frac{p}{n}} \geq (T^nT^{n^*})^{\frac{p}{n}}$ *holds for positive integer* n *such that* $n \geq p$.

In other words, if T *is a p-hyponormal operator for* $p > 0$*, then* T^n *is* $\min\{1, \frac{p}{n}\}$*-hyponormal for any positive integer* n.

In case $p \in (0, 1]$, Theorem 1 (resp. Corollary 2) means Theorem A.3 (resp. Corollary A.2). Corollary 2 also yields Theorem A.4 in case $p = n$. Theorem 1 and Corollary 2 can be rewritten into the following Theorem 1' and Corollary 2', respectively. We shall prove Theorem 1' and Corollary 2'.

Theorem 1'. *For some positive integer* m*, let* T *be a p-hyponormal operator for* $m-1 < p \leq m$*. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq (T^*T)^n$ *and* $(TT^*)^n \geq T^nT^{n^*}$ *hold for* $n = 1, 2, \cdots, m$.

(2) $(T^{n^*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1}$ *and* $(TT^*)^{p+1} \geq (T^nT^{n^*})^{\frac{p+1}{n}}$ *hold for* $n = m+1, m+2, \cdots$.

Corollary 2'. *For some positive integer* m*, let* T *be a p-hyponormal operator for* $m-1 < p \leq m$*. Then the following assertions hold:*

(1) $T^{n^*}T^n \geq T^nT^{n^*}$ *holds for* $n = 1, 2, \cdots, m-1$.

(2) $(T^{n^*}T^n)^{\frac{p}{n}} \geq (T^nT^{n^*})^{\frac{p}{n}}$ *holds for* $n = m, m+1, \cdots$.

We need the following theorem in order to give a proof of Theorem 1'.

Theorem B.1 (Furuta inequality [8]).

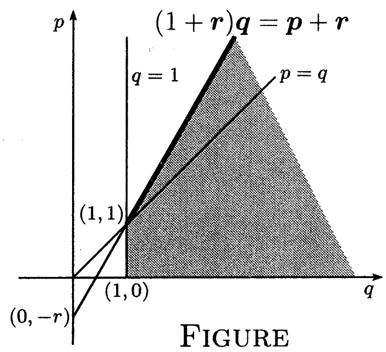
If $A \geq B \geq 0$*, then for each* $r \geq 0$*,*

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.



FIGURE

We remark that Theorem B.1 yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem B.1 are given in [4] and [14] and also an elementary one page proof in [9]. It is shown in [15] that the domain drawn for p, q and r in the Figure is the best possible one for Theorem B.1.

Proof of Theorem 1'. We shall prove Theorem 1' by induction.

Proof of (1). We shall prove

$$T^{n^*}T^n \geq (T^*T)^n \tag{2.1}$$

and

$$(TT^*)^n \geq T^nT^{n^*} \tag{2.2}$$

for $n = 1, 2, \cdots, m$. (2.1) and (2.2) always hold for $n = 1$. Assume that (2.1) and (2.2) hold for some $n \leq m-1$. Then we have

$$T^{n^*}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{n^*} \tag{2.3}$$

since the second inequality holds by p-hyponormality of T and Löwner-Heinz theorem for $\frac{n}{p} \in (0, 1]$. By (2.3), we have

$$T^{n^*}T^n \geq (TT^*)^n \tag{2.4}$$

and

$$(T^*T)^n \geq T^nT^{n^*}. \tag{2.5}$$

(2.4) ensures

$$T^{n+1^*}T^{n+1} = T^*(T^{n^*}T^n)T \geq T^*(TT^*)^nT = (T^*T)^{n+1},$$

and (2.5) ensures

$$(TT^*)^{n+1} = T(T^*T)^nT^* \geq T(T^nT^{n^*})T^* = T^{n+1}T^{n+1^*}.$$

Hence (2.1) and (2.2) hold for $n+1$, so that the proof of (1) is complete.

Proof of (2). We shall prove

$$(T^{n^*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1} \tag{2.6}$$

and

$$(TT^*)^{p+1} \geq (T^nT^{n^*})^{\frac{p+1}{n}} \tag{2.7}$$

for $n = m+1, m+2, \cdots$. Let $T = U|T|$ be the polar decomposition of T where $|T| = (T^*T)^{\frac{1}{2}}$ and put $A_n = |T^n|^{\frac{2p}{n}}$ and $B_n = |T^{n^*}|^{\frac{2p}{n}}$ for each positive integer n. We remark that $T^* = U^*|T^*|$ is also the polar decomposition of T^*.

(a) Case $n = m+1$. (2.1) and (2.2) for $n = m$ ensure

$$(T^{m^*}T^m)^{\frac{p}{m}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^mT^{m^*})^{\frac{p}{m}} \tag{2.8}$$

since the first and third inequalities hold by (2.1), (2.2) and Löwner-Heinz theorem for $\frac{p}{m} \in (0,1]$, and the second inequality holds by p-hyponormality of T. (2.8) ensures the following (2.9) and (2.10).

$$A_m = (T^{m^*}T^m)^{\frac{p}{m}} \geq (TT^*)^p = B_1. \tag{2.9}$$

$$A_1 = (T^*T)^p \geq (T^m T^{m^*})^{\frac{p}{m}} = B_m. \tag{2.10}$$

By using (i) of Theorem B.1 for $\frac{m}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$\begin{aligned}
(T^{m+1*}T^{m+1})^{\frac{p+1}{m+1}} &= (U^*|T^*|T^{m*}T^m|T^*|U)^{\frac{p+1}{m+1}} \\
&= U^*(|T^*|T^{m*}T^m|T^*|)^{\frac{p+1}{m+1}}U \\
&= U^*(B_1^{\frac{1}{2p}}A_m{}^{\frac{m}{p}}B_1^{\frac{1}{2p}})^{\frac{1+\frac{1}{p}}{\frac{m}{p}+\frac{1}{p}}}U \\
&\geq U^*B_1^{1+\frac{1}{p}}U \\
&= U^*|T^*|^{2(p+1)}U \\
&= |T|^{2(p+1)} \\
&= (T^*T)^{p+1},
\end{aligned}$$

so that (2.6) holds for $n = m+1$.

By using (ii) of Theorem B.1 for $\frac{m}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$\begin{aligned}
(T^{m+1}T^{m+1*})^{\frac{p+1}{m+1}} &= (U|T|T^mT^{m*}|T|U^*)^{\frac{p+1}{m+1}} \\
&= U(|T|T^mT^{m*}|T|)^{\frac{p+1}{m+1}}U^* \\
&= U(A_1^{\frac{1}{2p}}B_m{}^{\frac{m}{p}}A_1^{\frac{1}{2p}})^{\frac{1+\frac{1}{p}}{\frac{m}{p}+\frac{1}{p}}}U^* \\
&\leq UA_1^{1+\frac{1}{p}}U^* \\
&= U|T|^{2(p+1)}U^* \\
&= |T^*|^{2(p+1)} \\
&= (TT^*)^{p+1},
\end{aligned}$$

so that (2.7) holds for $n = m+1$.

(b) Assume that (2.6) and (2.7) hold for some $n \geq m+1$. Then (2.6) and (2.7) for n ensure

$$(T^{n^*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n^*})^{\frac{p}{n}} \tag{2.11}$$

since the first and third inequalities hold by (2.6) and (2.7) for n and Löwner-Heinz theorem for $\frac{p}{p+1} \in (0,1)$, and the second inequality holds by p-hyponormality of T. (2.11) ensures the following (2.12) and (2.13).

$$A_n = (T^{n^*}T^n)^{\frac{p}{n}} \geq (TT^*)^p = B_1. \tag{2.12}$$

$$A_1 = (T^*T)^p \geq (T^nT^{n^*})^{\frac{p}{n}} = B_n. \tag{2.13}$$

By using (i) of Theorem B.1 for $\frac{n}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$\begin{aligned}
(T^{n+1^*}T^{n+1})^{\frac{p+1}{n+1}} &= (U^*|T^*|T^{n^*}T^n|T^*|U)^{\frac{p+1}{n+1}} \\
&= U^*(|T^*|T^{n^*}T^n|T^*|)^{\frac{p+1}{n+1}}U \\
&= U^*(B_1^{\frac{1}{2p}}A_n{}^{\frac{n}{p}}B_1^{\frac{1}{2p}})^{\frac{1+\frac{1}{p}}{\frac{n}{p}+\frac{1}{p}}}U \\
&\geq U^*B_1^{1+\frac{1}{p}}U \\
&= U^*|T^*|^{2(p+1)}U \\
&= |T|^{2(p+1)} \\
&= (T^*T)^{p+1},
\end{aligned}$$

so that (2.6) holds for $n+1$.

By using (ii) of Theorem B.1 for $\frac{n}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$\begin{aligned}
(T^{n+1}T^{n+1^*})^{\frac{p+1}{n+1}} &= (U|T|T^nT^{n^*}|T|U^*)^{\frac{p+1}{n+1}} \\
&= U(|T|T^nT^{n^*}|T|)^{\frac{p+1}{n+1}}U^* \\
&= U(A_1^{\frac{1}{2p}}B_n{}^{\frac{n}{p}}A_1^{\frac{1}{2p}})^{\frac{1+\frac{1}{p}}{\frac{n}{p}+\frac{1}{p}}}U^* \\
&\leq UA_1^{1+\frac{1}{p}}U^* \\
&= U|T|^{2(p+1)}U^* \\
&= |T^*|^{2(p+1)} \\
&= (TT^*)^{p+1},
\end{aligned}$$

so that (2.7) holds for $n+1$.

By (a) and (b), (2.6) and (2.7) hold for $n = m+1, m+2, \cdots$, that is, the proof of (2) is complete.

Consequently the proof of Theorem 1' is complete. □

Proof of Corollary 2'.

Proof of (1). By (1) of Theorem 1', for $n = 1, 2, \cdots, m-1$,

$$T^{n^*}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{n^*}$$

hold since the second inequality holds by p-hyponormality of T and Löwner-Heinz theorem for $\frac{n}{p} \in (0,1)$. Therefore $T^{n^*}T^n \geq T^nT^{n^*}$ holds for $n = 1, 2, \cdots, m-1$.

Proof of (2). By (1) of Theorem 1' and Löwner-Heinz theorem for $\frac{p}{m} \in (0,1]$ in case $n = m$, and by (2) of Theorem 1' and Löwner-Heinz theorem for $\frac{p}{p+1} \in (0,1)$ in case $n = m+1, m+2, \cdots$, we have

$$(T^{n^*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n^*})^{\frac{p}{n}}$$

since the second inequality holds by p-hyponormality of T. Therefore $(T^{n^*}T^n)^{\frac{p}{n}} \geq (T^nT^{n^*})^{\frac{p}{n}}$ holds for $n = m, m+1, \cdots$. □

3 Best possibilities of Theorem 1 and Corollary 2

Furuta and Yanagida [11] discussed the best possibilities of Theorem A.3 and Corollary A.2 on p-hyponormal operators for $p \in (0,1]$. In this section, more generally, we shall discuss the best possibilities of Theorem 1 and Corollary 2 on p-hyponormal operators for $p > 0$.

Theorem 3. *Let n be a positive integer such that $n \geq 2$, $p > 0$ and $\alpha > 1$.*

(1) In case $n < p+1$, the following assertions hold:

(i) There exists a p-hyponormal operator T such that $(T^{n*}T^n)^\alpha \ngeq (T^*T)^{n\alpha}$.

(ii) There exists a p-hyponormal operator T such that $(TT^*)^{n\alpha} \ngeq (T^nT^{n*})^\alpha$.

(2) In case $n \geq p+1$, the following assertions hold:

(i) There exists a p-hyponormal operator T such that $(T^{n*}T^n)^{\frac{(p+1)\alpha}{n}} \ngeq (T^*T)^{(p+1)\alpha}$.

(ii) There exists a p-hyponormal operator T such that $(TT^*)^{(p+1)\alpha} \ngeq (T^nT^{n*})^{\frac{(p+1)\alpha}{n}}$.

Theorem 4. *Let n be a positive integer such that $n \geq 2$, $p > 0$ and $\alpha > 1$.*

(1) In case $n < p$, there exists a p-hyponormal operator T such that $(T^{n*}T^n)^\alpha \ngeq (T^nT^{n*})^\alpha$.

(2) In case $n \geq p$, there exists a p-hyponormal operator T such that $(T^{n*}T^n)^{\frac{p\alpha}{n}} \ngeq (T^nT^{n*})^{\frac{p\alpha}{n}}$.

Theorem 3 (resp. Theorem 4) asserts the best possibility of Theorem 1 (resp. Corollary 2). We need the following results to give proofs of Theorem 3 and Theorem 4.

Theorem C.1 ([16][18]). *Let $p > 0$, $q > 0$, $r > 0$ and $\delta > 0$. If $0 < q < 1$ or $(\delta + r)q < p + r$, then the following assertions hold:*

(i) There exist positive invertible operators A and B on $\mathbb{R}^2$ such that $A^\delta \geq B^\delta$ and

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \ngeq B^{\frac{p+r}{q}}.$$

(ii) There exist positive invertible operators A and B on $\mathbb{R}^2$ such that $A^\delta \geq B^\delta$ and

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}.$$

Lemma C.2 ([11]). *For positive operators A and B on H, define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B^{\frac{1}{2}} & 0 & & & & \\ & & B^{\frac{1}{2}} & \boxed{0} & & & \\ & & & A^{\frac{1}{2}} & 0 & & \\ & & & & A^{\frac{1}{2}} & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{3.1}$$

where $\square$ shows the place of the $(0,0)$ matrix element. Then the following assertion holds:

(i) T is p-hyponormal for $p > 0$ if and only if $A^p \geq B^p$.

Furthermore, the following assertions hold for $\beta > 0$ and integers $n \geq 2$:

(ii) $(T^{n*}T^n)^{\frac{\beta}{n}} \geq (T^*T)^\beta$ if and only if

$$(B^{\frac{k}{2}} A^{n-k} B^{\frac{k}{2}})^{\frac{\beta}{n}} \geq B^\beta \text{ holds for } k = 1, 2, \ldots, n-1. \tag{3.2}$$

(iii) $(TT^*)^\beta \geq (T^n T^{n*})^{\frac{\beta}{n}}$ if and only if

$$A^\beta \geq (A^{\frac{k}{2}} B^{n-k} A^{\frac{k}{2}})^{\frac{\beta}{n}} \text{ holds for } k = 1, 2, \ldots, n-1. \tag{3.3}$$

(iv) $(T^{n*}T^n)^{\frac{\beta}{n}} \geq (T^n T^{n*})^{\frac{\beta}{n}}$ if and only if

$$\begin{cases} A^\beta \geq B^\beta \text{ holds and} \\ \qquad (B^{\frac{k}{2}} A^{n-k} B^{\frac{k}{2}})^{\frac{\beta}{n}} \geq B^\beta \text{ and } A^\beta \geq (A^{\frac{k}{2}} B^{n-k} A^{\frac{k}{2}})^{\frac{\beta}{n}} \text{ hold for } k = 1, 2, \ldots, n-1. \end{cases} \tag{3.4}$$

Proof of Theorem 3. Let $n \geq 2$, $p > 0$ and $\alpha > 1$.
Proof of (1). Put $p_1 = n - 1 > 0$, $q_1 = \frac{1}{\alpha} \in (0, 1)$, $r_1 = 1 > 0$ and $\delta = p > 0$.
Proof of (i). By (i) of Theorem C.1, there exist positive operators A and B on H such that $A^\delta \geq B^\delta$ and $(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{1}{q_1}} \not\geq B^{\frac{p_1+r_1}{q_1}}$, that is,

$$A^p \geq B^p \tag{3.5}$$

and

$$(B^{\frac{1}{2}}A^{n-1}B^{\frac{1}{2}})^{\alpha} \ngeq B^{n\alpha}. \tag{3.6}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.5) and (i) of Lemma C.2, and $(T^{n*}T^n)^{\alpha} \ngeq (T^*T)^{n\alpha}$ by (ii) of Lemma C.2 since the case $k=1$ of (3.2) does not hold for $\beta = n\alpha$ by (3.6).

Proof of (ii). By (ii) of Theorem C.1, there exist positive operators A and B on H such that $A^{\delta} \geq B^{\delta}$ and $A^{\frac{p_1+r_1}{q_1}} \ngeq (A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1}{q_1}}$, that is,

$$A^p \geq B^p \tag{3.7}$$

and

$$A^{n\alpha} \ngeq (A^{\frac{1}{2}}B^{n-1}A^{\frac{1}{2}})^{\alpha}. \tag{3.8}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.7) and (i) of Lemma C.2, and $(TT^*)^{n\alpha} \ngeq (T^nT^{n*})^{\alpha}$ by (iii) of Lemma C.2 since the case $k=1$ of (3.3) does not hold for $\beta = n\alpha$ by (3.8).

Proof of (2). Put $p_1 = n-1 > 0$, $q_1 = \frac{n}{(p+1)\alpha} > 0$, $r_1 = 1 > 0$ and $\delta = p > 0$, then we have $(\delta + r_1)q_1 = \frac{n}{\alpha} < n = p_1 + r_1$.

Proof of (i). By (i) of Theorem C.1, there exist positive operators A and B on H such that $A^{\delta} \geq B^{\delta}$ and $(B^{\frac{r_1}{2}}A^{p_1}B^{\frac{r_1}{2}})^{\frac{1}{q_1}} \ngeq B^{\frac{p_1+r_1}{q_1}}$, that is,

$$A^p \geq B^p \tag{3.9}$$

and

$$(B^{\frac{1}{2}}A^{n-1}B^{\frac{1}{2}})^{\frac{(p+1)\alpha}{n}} \ngeq B^{(p+1)\alpha}. \tag{3.10}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.9) and (i) of Lemma C.2, and $(T^{n*}T^n)^{\frac{(p+1)\alpha}{n}} \ngeq (T^*T)^{(p+1)\alpha}$ by (ii) of Lemma C.2 since the case $k=1$ of (3.2) does not hold for $\beta = (p+1)\alpha$ by (3.10).

Proof of (ii). By (ii) of Theorem C.1, there exist positive operators A and B on H such that $A^{\delta} \geq B^{\delta}$ and $A^{\frac{p_1+r_1}{q_1}} \ngeq (A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1}{q_1}}$, that is,

$$A^p \geq B^p \tag{3.11}$$

and

$$A^{(p+1)\alpha} \ngeq (A^{\frac{1}{2}}B^{n-1}A^{\frac{1}{2}})^{\frac{(p+1)\alpha}{n}}. \tag{3.12}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.11) and (i) of Lemma C.2, and $(TT^*)^{(p+1)\alpha} \not\geq (T^nT^{n*})^{\frac{(p+1)\alpha}{n}}$ by (iii) of Lemma C.2 since the case $k = 1$ of (3.3) does not hold for $\beta = (p+1)\alpha$ by (3.12). □

Proof of Theorem 4. Let $n \geq 2$, $p > 0$ and $\alpha > 1$.

Proof of (1). Put $p_1 = n - 1 > 0$, $q_1 = \frac{1}{\alpha} \in (0,1)$, $r_1 = 1 > 0$ and $\delta = p > 0$. By (i) of Theorem C.1, there exist positive operators A and B on H such that $A^\delta \geq B^\delta$ and $(B^{\frac{r_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{1}{q_1}} \not\geq B^{\frac{p_1+r_1}{q_1}}$, that is,

$$A^p \geq B^p \tag{3.13}$$

and

$$(B^{\frac{1}{2}} A^{n-1} B^{\frac{1}{2}})^\alpha \not\geq B^{n\alpha}. \tag{3.14}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.13) and (i) of Lemma C.2, and $(T^{n*}T^n)^\alpha \not\geq (T^nT^{n*})^\alpha$ by (iv) of Lemma C.2 since the case $k = 1$ of the second inequality of (3.4) does not hold for $\beta = n\alpha$ by (3.14).

Proof of (2). It is well known that there exist positive operators A and B on H such that

$$A^p \geq B^p \tag{3.15}$$

and

$$A^{p\alpha} \not\geq B^{p\alpha}. \tag{3.16}$$

Define an operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then T is p-hyponormal by (3.15) and (i) of Lemma C.2, and $(T^{n*}T^n)^{\frac{p\alpha}{n}} \not\geq (T^nT^{n*})^{\frac{p\alpha}{n}}$ by (iv) of Lemma C.2 since the first inequality of (3.4) does not hold for $\beta = p\alpha$ by (3.16). □

4 Concluding remarks

Remark 1. An operator T is said to be *log-hyponormal* if T is invertible and $\log T^*T \geq \log TT^*$. It is easily obtained that every invertible p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function, and Ando [3] showed that every log-hyponormal operator is paranormal. We remark that log-hyponormal can be regarded as 0-hyponormal since $(T^*T)^p \geq (TT^*)^p$ approaches $\log T^*T \geq \log TT^*$ as $p \to +0$.

As an extension of Theorem A.1, Yamazaki [17] obtained the following Theorem D.1 and Corollary D.2 on log-hyponormal operators.

Theorem D.1 ([17]). *Let T be a log-hyponormal operator. Then the following inequalities hold for all positive integer n:*

(1) $T^*T \le (T^{2^*}T^2)^{\frac{1}{2}} \le \cdots \le (T^{n^*}T^n)^{\frac{1}{n}}$.

(2) $TT^* \ge (T^2T^{2^*})^{\frac{1}{2}} \ge \cdots \ge (T^nT^{n^*})^{\frac{1}{n}}$.

Corollary D.2 ([17]). *If T is a log-hyponormal operator, then T^n is also log-hyponormal for any positive integer n.*

The best possibilities of Theorem D.1 and Corollary D.2 are discussed in [12].

As a parallel result to Theorem D.1, Furuta and Yanagida [12] showed the following Theorem D.3 on p-hyponormal operators for $p \in (0,1]$.

Theorem D.3 ([12]). *Let T be a p-hyponormal operator for $p \in (0,1]$. Then the following inequalities hold for all positive integer n:*

(1) $(T^*T)^{p+1} \le (T^{2^*}T^2)^{\frac{p+1}{2}} \le \cdots \le (T^{n^*}T^n)^{\frac{p+1}{n}}$.

(2) $(TT^*)^{p+1} \ge (T^2T^{2^*})^{\frac{p+1}{2}} \ge \cdots \ge (T^nT^{n^*})^{\frac{p+1}{n}}$.

In fact, Theorem D.3 in the case $p \to +0$ corresponds to Theorem D.1.

As a further extension of Theorem D.3, we obtain the following Theorem 5 on p-hyponormal operators for $p > 0$.

Theorem 5. *For some positive integer m, let T be a p-hyponormal operator for $m-1 < p \le m$. Then the following inequalities hold for $n = m+1, m+2, \cdots$:*

(1) $(T^*T)^{p+1} \le (T^{m+1^*}T^{m+1})^{\frac{p+1}{m+1}} \le (T^{m+2^*}T^{m+2})^{\frac{p+1}{m+2}} \le \cdots \le (T^{n^*}T^n)^{\frac{p+1}{n}}$.

(2) $(TT^*)^{p+1} \ge (T^{m+1}T^{m+1^*})^{\frac{p+1}{m+1}} \ge (T^{m+2}T^{m+2^*})^{\frac{p+1}{m+2}} \ge \cdots \ge (T^nT^{n^*})^{\frac{p+1}{n}}$.

We remark that Theorem 5 yields Theorem D.3 by putting $m = 1$.

Remark 2. Recently, in [10], we introduced a new class of operators as follows: An operator T belongs to *class A* if $|T^2| \ge |T|^2$. We call an operator T "class A operator" briefly if T belongs to class A. In [10], we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal. It turns out that these results contain another proof of Ando's result [3] which states that every log-hyponormal operator is paranormal. We remark that class A is defined by an operator inequality and paranormal is defined by a norm inequality, and their definitions appear to be similar forms.

We obtain the following Theorem 6 on class A.

Theorem 6. *Let T be an invertible and class A operator. Then the following inequalities hold for all positive integer n:*

(1) $|T|^2 \leq |T^2| \leq \cdots \leq |T^n|^{\frac{2}{n}}$, *i.e.*, $T^*T \leq (T^{2^*}T^2)^{\frac{1}{2}} \leq \cdots \leq (T^{n^*}T^n)^{\frac{1}{n}}$.

(2) $|T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^{\frac{2}{n}}$, *i.e.*, $TT^* \geq (T^2T^{2^*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n^*})^{\frac{1}{n}}$.

Theorem 6 is an extension of Theorem D.1 since every log-hyponormal operator belongs to class A.

Related to Theorem 6, we have the following Proposition 7 on paranormal operators as a variant from the result in [7].

It is interesting to point out the contrast between Theorem 6 and Proposition 7.

Proposition 7. *Let T be a paranormal operator. Then*

$$\|Tx\| \leq \|T^2x\|^{\frac{1}{2}} \leq \cdots \leq \|T^nx\|^{\frac{1}{n}}$$

hold for every unit vector $x \in H$ and all positive integer n.

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