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Further extensions of characterizations of chaotic order associated with Kantorovich type inequalities

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Abstract

This report is based on the following papers:

[HYM] M.Hashimoto and T.Yamazaki, *Further extensions of characterizations of chaotic order associated with Kantorovich type inequalities*, Scientiae Mathematicae, **3** (2000), 127–136.

[HYN] M.Hashimoto and M.Yanagida, *Further characterizations of chaotic order associated with Kantorovich type inequalities via Furuta inequality*, preprint.

We showed characterizations of chaotic order via Kantorovich inequality in [33]. Recently as a nice application of generalized Furuta inequality, Furuta and Seo showed an extension of one of our results and a related result on operator equations. In this report, by using essentially the same idea as theirs, we shall show further extensions of both their results and our another previous result which is a characterization of chaotic order via Specht's ratio. Moreover we shall show further extensions of our results.

1 Introduction

An operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Löwner-Heinz theorem is well known: *$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*. For the sake of convenience on application, the following Theorem F was established.

Theorem F (Furuta inequality [10]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

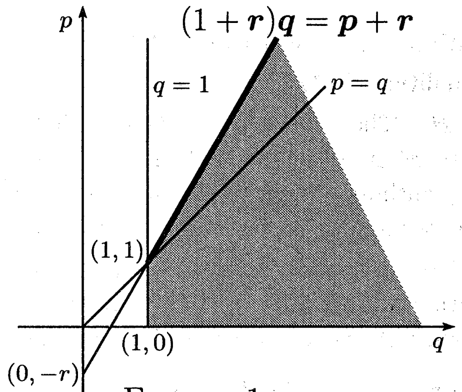

FIGURE 1

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [6][24] and also an elementary one-page proof in [11]. It is shown in [29] that the domain drawn for p, q and r in the Figure 1 is best possible one for Theorem F.

As an extension of Theorem F, the following Theorem G was obtained in [15].

Theorem G ([15]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$, and $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$, that is, for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for any $s \geq 1$ and $r \geq t$.

Ando-Hiai [2] established excellent log majorization results and proved the following useful inequality equivalent to the main log majorization theorem: *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geq 1$ and $r \geq 1$. Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself, and also extends results of [7][12] and [13]. A nice mean theoretic proof of Theorem G is shown in [8] and one-page proof of (1.1) is shown in [18]. In [21], we showed equivalence relation among the inequality (1.1), monotonicity of the function $F_{p,t}(A,B,r,s)$ in Theorem G and related results. The best possibility of the outside exponents of both sides in (1.1) is shown in [30] and its simplified proofs are shown in [9] and [32].

On the other hand, related to Löwner-Heinz theorem, the following proposition is also well known: *$A \geq B \geq 0$ does not always assure $A^\alpha \geq B^\alpha$ for any $\alpha > 1$.* As a way to settle this inconvenient, the following result is given in [17].

Theorem A.1 ([17]). *If $A \geq B \geq 0$ and $MI \geq A \geq mI > 0$, then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m,M,p)A^p \geq B^p \text{ for } p \geq 1,$$

where

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)\,(mM^p-m^pM)^{p-1}}. \tag{1.2}$$

We remark that Theorem A.1 is related to both Hölder-McCarthy inequality [25] and Kantorovich inequality: *If $MI \geq A \geq mI > 0$, then $\left(A^{-1}x,x\right)(Ax,x) \leq \frac{(m+M)^2}{4mM}$ holds for every unit vector x in H.* The number $\frac{(m+M)^2}{4mM}$ is called Kantorovich constant and $K_+(m,M,2) = \frac{(m+M)^2}{4mM}$ where $K_+(m,M,p)$ is stated in (1.2), so that $K_+(m,M,p)$ is a generalization of Kantorovich constant. Many authors have been investigating Kantorovich inequality, among others, there is a long research series of Mond-Pečarić, some of them are [26] and [27].

The order between positive invertible operators A and B defined by $\log A \geq \log B$ is said to be chaotic order which is a weaker order than usual order $A \geq B$. As an application of Theorem F, the following characterization of chaotic order is well known.

Theorem A.2 ([7][13]). *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $A^p \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ *for all $p \geq 0$.*

(iii) $A^u \geq (A^{\frac{u}{2}}B^pA^{\frac{u}{2}})^{\frac{u}{p+u}}$ *for all $p \geq 0$ and $u \geq 0$.*

(i)$\Leftrightarrow$(ii) of Theorem A.2 is shown in [1]. Recently a simple and excellent proof of (i)$\Rightarrow$(iii) is shown in [31] by only applying Theorem F, and a simplified proof of (ii)$\Rightarrow$(i) is shown in [22].

We prove the following two other characterizations of chaotic order in [33] as applications of Theorem A.1 and Theorem A.2.

Theorem B.1 (**[33]**). *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $\dfrac{(m^p + M^p)^2}{4m^p M^p} A^p \geq B^p$ *for all* $p \geq 0$.

Theorem B.2 (**[33]**). *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $M_h(p)A^p \geq B^p$ *for all* $p \geq 0$, *where* $h = \frac{M}{m} > 1$ *and*

$$M_h(p) = \frac{h^{\frac{p}{h^p - 1}}}{e \log h^{\frac{p}{h^p - 1}}}. \tag{1.3}$$

Theorem B.2 gives a more precise sufficient condition for chaotic order than Theorem B.1 since $\frac{(m^p+M^p)^2}{4m^p M^p} \geq M_h(p)$ holds for all $p \geq 0$ by the following lemma.

Lemma B.3 (**[33]**). *Let $K_+(m, M, p)$ be defined in* (1.2). *Then*

$$F(p, r, m, M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

is an increasing function of p, r and M, and also a decreasing function of m for $p > 0$, $r > 0$ and $M > m > 0$. Moreover,

$$\lim_{r \to +0} K_+\left(m^r, M^r, \frac{p+r}{r}\right) = M_h(p),$$

and

$$\left(\frac{M}{m}\right)^p \geq K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geq M_h(p) \geq 1. \tag{1.4}$$

hold for $p > 0$, $r > 0$ and $M > m > 0$, where $h = \frac{M}{m} > 1$ and $M_h(p)$ be defined in (1.3).

We remark that $M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$ is called Specht's ratio [4][28], which is the best upper bound of the ratio of the arithmetic mean to the geometric mean of numbers x_i satisfying $M \geq x_i \geq m > 0$ ($i = 1, 2, \cdots, n$), that is, the following inequality holds:

$$\frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

In [3], we showed a simplified proof of Theorem B.2 by using determinant for positive operators defined in [4] and [5]. Moreover we showed the following result which interpolates (i)$\Rightarrow$(ii) of both Theorem B.1 and Theorem B.2 in [33].

Theorem B.4 (**[33]**). *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. If $\log A \geq \log B$, then*

$$K_+\left(m^r, M^r, \frac{p+r}{r}\right) A^p \geq B^p \qquad \text{holds for } p > 0 \text{ and } r > 0,$$

where $K_+(m, M, p)$ is defined in (1.2).

As a nice application of Theorem G, Furuta and Seo established the following result in [22].

Theorem C.1 **([22]).** *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$ and $s \geq 1$ such that $(p+\alpha u)s \geq (1-\alpha)u$, there exists the unique invertible positive contraction T satisfying*

$$TA^{(p+\alpha u)s}T = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iii) *For each $\alpha \in [0,1]$, $p \geq u \geq 0$ and $s \geq 1$, there exists the unique invertible positive contraction T satisfying*

$$TA^{(p+\alpha u)s}T = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iv) *For each $p \geq 0$, there exists the unique invertible positive contraction T satisfying*

$$TA^pT = B^p.$$

Moreover as an extension of Theorem B.1, Furuta and Seo also showed the following result based on Theorem C.1 in [22].

Theorem C.2 **([22]).** *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,*

$$\frac{\left(m^{(p+\alpha u)s} + M^{(p+\alpha u)s}\right)^2}{4m^{(p+\alpha u)s}M^{(p+\alpha u)s}}A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$ such that $(p+\alpha u)s \geq (1-\alpha)u$.

(iii) *For each $\alpha \in [0,1]$ and $p \geq u \geq 0$,*

$$\frac{\left(m^{(p+\alpha u)s} + M^{(p+\alpha u)s}\right)^2}{4m^{(p+\alpha u)s}M^{(p+\alpha u)s}}A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$.

(iv) $\dfrac{(m^p + M^p)^2}{4m^pM^p}A^p \geq B^p$ *holds for all $p \geq 0$.*

In this report, we shall show a further extension of Theorem C.1. And also, by using Theorem G, we shall show a further extension of Theorem C.2 which interpolates both Theorem B.1 and Theorem B.2. Moreover we shall attempt to extend Theorem C.1 and Theorem C.2 by using Theorem F.

2 Extensions of the results by Furuta and Seo

Firstly, as an extension of Theorem C.1, we have the following characterization of chaotic order via operator equations.

Theorem 1. *Let A and B be positive invertible operators. Then for each natural number n, the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each* $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$ *and* $r \geq 1-\alpha$ *such that* $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, *there exists the unique invertible positive contraction* $T = T(n,\alpha,p,u,s,r)$ *satisfying*

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}. \tag{2.1}$$

(iii) *For each* $\alpha \in [0,1]$, $p \geq nu \geq 0$, $s \geq 1$ *and real numbers* r *such that* $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, *there exists the unique invertible positive contraction* $T = T(n,\alpha,p,u,s,r)$ *satisfying*

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}. \tag{2.1}$$

(iv) *For each* $p \geq 0$, *there exists the unique invertible positive contraction* $T = T(n,p)$ *satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

The following Corollary 2 is easily obtained by Theorem 1.

Corollary 2. *Let A and B be positive invertible operators. Then for each natural number n, the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each* $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$ *and* $s \geq 1$ *such that* $(p+\alpha u)s \geq n(1-\alpha)u$, *there exists the unique invertible positive contraction* $T = T(n,\alpha,p,u,s)$ *satisfying*

$$T(A^{\frac{(p+\alpha u)s}{n}}T)^n = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iii) *For each* $\alpha \in [0,1]$, $p \geq nu \geq 0$ *and* $s \geq 1$, *there exists the unique invertible positive contraction* $T = T(n,\alpha,p,u,s)$ *satisfying*

$$T(A^{\frac{(p+\alpha u)s}{n}}T)^n = (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s.$$

(iv) *For each* $p \geq 0$, *there exists the unique invertible positive contraction* $T = T(n,p)$ *satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

Remark 1. Corollary 2 implies Theorem C.1 when we put $n = 1$, that is, Theorem 1 includes Theorem C.1 as a special case.

Secondly, as an extension of Theorem C.2, we have the following Kantorovich type characterization of chaotic order.

Theorem 3. *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$ and $K_+(m,M,p)$ be defined in (1.2). Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number* n, $\alpha \in [0,1]$, $p \geq 0$ *and* $u \geq 0$,

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$ *and* $r \geq 1-\alpha$ *such that* $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$.

(iii) *For each natural number n, $\alpha \in [0,1]$ and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s \tag{2.2}$$

holds for all $s \geq 1$ and real number r such that $\{nr + (n+1)\alpha\}u \geq (p+\alpha u)s$.

(iv) *For each natural number n and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{p+ru}{n+1}}, M^{\frac{p+ru}{n+1}}, n+1\right) A^p \geq B^p$$

holds for all real number r such that $nru \geq p$.

Remark 2. Theorem 3 implies Theorem C.2 as follows. We have (ii) [resp. (iii)] of Theorem C.2 when we put $n = 1$ and $r = \frac{(p+\alpha u)s}{u}$ in (ii) [resp. (iii)] of Theorem 3. And put $n = 1$ and $r = \frac{p}{u}$ in (iv) of Theorem 3, then we have (iv) of Theorem C.2.

As mentioned above, Theorem 3 yields Theorem C.2 and Theorem C.2 yields Theorem B.1. Moreover Theorem 3 also yields the following Theorem 4 and Theorem 4 yields Theorem B.2, which is a more precise estimation than Theorem B.1.

Theorem 4. *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$, and $K_+(m,M,p)$ and $M_h(p)$ be defined in* (1.2) *and* (1.3), *respectivery. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$*

$$K_+\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$ such that $(p+\alpha u)s \geq (n+\alpha)u$.

(iii) *For each natural number n, $\alpha \in [0,1]$ and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1\right) A^{(p+\alpha u)s} \geq (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s$$

holds for all $s \geq 1$.

(iv) *$M_h(p)\, A^p \geq B^p$ holds for all $p \geq 0$, where $h = \frac{M}{m} > 1$.*

3 Proofs of results

In order to prove Theorem 1, we prepare the following result which is an application of Theorem G.

Proposition 5. *Let A and B be positive invertible operators. If $\log A \geq \log B$, then*

$$A^{\frac{(p+\alpha u)s+ru}{q}} \geq \{A^{\frac{ru}{2}} (A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s A^{\frac{ru}{2}}\}^{\frac{1}{q}} \tag{3.1}$$

holds for any $u \geq 0$, $p \geq 0$, $\alpha \in [0,1]$, $s \geq 1$, $r \geq 1-\alpha$ and $q \geq 1$ with $u(\alpha + r)q \geq (p+\alpha u)s + ru$.

We remark that Proposition 5 is a part of [16, Theorem 2.2]. For the sake of later argument, we recall the proof of Proposition 5.

Proof of Proposition 5. Both sides of (3.1) equal I in case $u=0$, so that we have only to consider the case $u>0$. By Theorem A.2, $\log A \geq \log B$ implies the following (3.2):

$$A^u \geq (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{u}{p+u}} \quad \text{for } p \geq 0 \text{ and } u > 0. \tag{3.2}$$

Put $A_1 = A^u$ and $B_1 = (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{u}{p+u}}$, then $A_1 \geq B_1 > 0$. By (1.1) of Theorem G,

$$A_1^{\frac{(p_1-t)s+r}{q}} \geq \{A_1^{\frac{r}{2}}(A_1^{\frac{-t}{2}} B_1^{p_1} A_1^{\frac{-t}{2}})^s A_1^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{3.3}$$

holds for $p_1 \geq 1$, $t \in [0,1]$, $s \geq 1$, $r \geq t$ and $q \geq 1$ with $(1-t+r)q \geq (p_1-t)s+r$. (3.3) is equivalent to the following (3.4):

$$A^{\frac{u(p_1-t)s+ru}{q}} \geq [A^{\frac{ru}{2}}\{A^{\frac{-ut}{2}}(A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{up_1}{p+u}} A^{\frac{-ut}{2}}\}^s A^{\frac{ru}{2}}]^{\frac{1}{q}}. \tag{3.4}$$

Put $p_1 = \frac{p+u}{u} \geq 1$ and $\alpha = 1-t \in [0,1]$ in (3.4), then we have the following (3.1):

$$A^{\frac{(p+\alpha u)s+ru}{q}} \geq \{A^{\frac{ru}{2}}(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s A^{\frac{ru}{2}}\}^{\frac{1}{q}} \tag{3.1}$$

for $u>0$, $p \geq 0$, $\alpha \in [0,1]$, $s \geq 1$, $r \geq 1-\alpha$ and $q \geq 1$ with $u(\alpha+r)q \geq (p+\alpha u)s+ru$.

Consequently, the proof of Proposition 5 is complete. □

Proof of Theorem 1. Let n be a natural number. We shall show (i)⇒(ii)⇒(iii)⇒(iv)⇒(i) as follows:
(i)⟹(ii): In case $u=0$, (ii) holds obviously since the assumption of (ii) ensures $ps=0$ and (2.1) turns out to be $T^{n+1}=I$, so that we have only to show the case $u>0$ as follows: By putting $q=n+1 \geq 1$ in (3.1) of Proposition 5, $\log A \geq \log B$ implies the following (3.5):

$$A^{\frac{(p+\alpha u)s+ru}{n+1}} \geq \{A^{\frac{ru}{2}}(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s A^{\frac{ru}{2}}\}^{\frac{1}{n+1}} \tag{3.5}$$

for $\alpha \in [0,1]$, $p \geq 0$, $u>0$, $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$. (3.5) implies the following (3.6):

$$I \geq A^{\frac{-\{(p+\alpha u)s+ru\}}{2(n+1)}}(A^{\frac{ru}{2}} D^s A^{\frac{ru}{2}})^{\frac{1}{n+1}} A^{\frac{-\{(p+\alpha u)s+ru\}}{2(n+1)}} > 0, \tag{3.6}$$

where $D = A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}}$. Let $T = T(n,\alpha,p,u,s,r)$ be defined as follows:

$$T = A^{\frac{-\{(p+\alpha u)s+ru\}}{2(n+1)}}(A^{\frac{ru}{2}} D^s A^{\frac{ru}{2}})^{\frac{1}{n+1}} A^{\frac{-\{(p+\alpha u)s+ru\}}{2(n+1)}}. \tag{3.7}$$

Then it turns out that T is an invertible positive contraction by (3.6) and

$$A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} T A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} = (A^{\frac{ru}{2}} D^s A^{\frac{ru}{2}})^{\frac{1}{n+1}} \tag{3.8}$$

holds by (3.7). Taking the (n+1)-th power of both sides of (3.8), we obtain

$$(A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} T A^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1} = A^{\frac{ru}{2}} D^s A^{\frac{ru}{2}}, \tag{3.9}$$

(3.9) is equivalent to

$$A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} T (A^{\frac{(p+\alpha u)s+ru}{n+1}} T)^n A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} = A^{\frac{ru}{2}}(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s A^{\frac{ru}{2}},$$

that is, we have (2.1).

Uniqueness of T can be shown as follows: Assume that for each $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$

and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, there exists an invertible positive contraction $S = S(n,\alpha,p,u,s,r)$ satisfying:

$$S(A^{\frac{(p+\alpha u)s+ru}{n+1}}S)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}. \tag{3.10}$$

By (2.1) and (3.10), we have

$$S(A^{\frac{(p+\alpha u)s+ru}{n+1}}S)^n = T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n. \tag{3.11}$$

(3.11) is equivalent to

$$(A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}SA^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1} = (A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}TA^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1}.$$

Then we have $S=T$. Hence the proof of (i)$\Longrightarrow$(ii) is complete.

(ii)$\Longrightarrow$(iii): (iii) holds in case $u=0$ obviously by the same discussion as (ii). Let $p \geq nu > 0$ in (ii), then the condition $r \geq 1-\alpha$ follows from $p \geq nu > 0$ and the other assumptions of (ii) since

$$r \geq \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha \geq \frac{p+\alpha u}{nu} - \frac{n+1}{n}\alpha = \frac{p}{nu} - \alpha \geq 1-\alpha,$$

so that we have (iii).

(iii)$\Longrightarrow$(iv): Put $r = \frac{(p+\alpha u)s}{nu}$, $\alpha = 0$ and $s=1$ in (iii), then we have

$$T(A^{\frac{p}{n}}T)^n = B^p$$

holds for each $p \geq nu > 0$, i.e., $p > 0$. (iv) holds in case $p=0$ obviously, so that the proof of (iii)$\Longrightarrow$(iv) is complete.

(iv)$\Longrightarrow$(i): Assume (iv). Then we have

$$(A^{\frac{p}{2n}}TA^{\frac{p}{2n}})^{n+1} = A^{\frac{p}{2n}}T(A^{\frac{p}{n}}T)^nA^{\frac{p}{2n}} = A^{\frac{p}{2n}}B^pA^{\frac{p}{2n}} \quad \text{by (iv)}. \tag{3.12}$$

By taking the $\frac{1}{n+1}$-th power of both sides of (3.12), we have the following (3.13):

$$A^{\frac{p}{n}} \geq A^{\frac{p}{2n}}TA^{\frac{p}{2n}} = (A^{\frac{p}{2n}}B^pA^{\frac{p}{2n}})^{\frac{1}{n+1}} \tag{3.13}$$

holds for any $p \geq 0$ since $I \geq T > 0$. Put $X = (A^{\frac{p}{2n}}B^pA^{\frac{p}{2n}})^{\frac{1}{n+1}}$, then we have

$$\begin{aligned}\frac{A^{\frac{p}{n}}-I}{p} &\geq \frac{(A^{\frac{p}{2n}}B^pA^{\frac{p}{2n}})^{\frac{1}{n+1}}-I}{p} \qquad \text{by (3.13)}\\ &= \frac{X-I}{p} = \frac{(X^{n+1}-I)(X^n+X^{n-1}+\cdots+X+I)^{-1}}{p}\\ &= \left(\frac{A^{\frac{p}{2n}}(B^p-I)A^{\frac{p}{2n}}}{p}+\frac{A^{\frac{p}{n}}-I}{p}\right)(X^n+X^{n-1}+\cdots+X+I)^{-1}.\end{aligned} \tag{3.14}$$

Tending $p \to +0$ in (3.14), we have

$$\frac{1}{n}\log A \geq \frac{1}{n+1}\left(\log B + \frac{1}{n}\log A\right)$$

since $X = (A^{\frac{p}{2n}}B^pA^{\frac{p}{2n}})^{\frac{1}{n+1}} \to I$ as $p \to +0$, so that $\log A \geq \log B$.

Consequently the proof of Theorem 1 is complete. □

We remark that a proof of (iv)$\Rightarrow$(i) has been already shown in [14, Theorem 2.1], and the idea of factorization which we use in the above proof of (iv)$\Rightarrow$(i) is due to Furuta [20][19].

Proof of Corollary 2. Put $r = \frac{(p+\alpha u)s}{nu}$ in (ii) and (iii) of Theorem 1, then the condition $\{nr + (n+1)\alpha\}u \geq (p+\alpha u)s$ in (ii) is satisfied and $r \geq 1-\alpha$ can be rewritten as $(p+\alpha u)s \geq n(1-\alpha)u$. Then we have Corollary 2. $\square$

In order to prove Theorem 3, we prepare the following lemma.

Lemma 6. *Let A be a positive invertible operator satisfying $MI \geq A \geq mI > 0$ and T be an invertible positive contraction. Then*

$$K_{+}(m, M, p+1)\, A^{p} \geq T^{\frac{1}{2}}(T^{\frac{1}{2}}AT^{\frac{1}{2}})^{p}T^{\frac{1}{2}}$$

holds for $p > 0$, where $K_{+}(m, M, p)$ is defined in (1.2).

We need the following Lemma D.1 to prove Lemma 6.

Lemma D.1 ([15]). *Let A be a positive invertible operator and B be an invertible operator. Then*

$$(BAB^{*})^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^{*}BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^{*}$$

holds for any real number λ.

Proof of Lemma 6. The condition $I \geq T > 0$ asserts $A \geq A^{\frac{1}{2}}TA^{\frac{1}{2}} > 0$. Put $A_1 = A$ and $B_1 = A^{\frac{1}{2}}TA^{\frac{1}{2}}$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$ with $MI \geq A_1 \geq mI > 0$. Applying Theorem A.1,

$$K_{+}(m, M, p+1){A_1}^{p+1} \geq {B_1}^{p+1} \tag{3.15}$$

holds for $p > 0$, where $K_{+}(m, M, p)$ is defined in (1.2). (3.15) is equivalent to the following by Lemma D.1.

$$\begin{aligned} K_{+}(m, M, p+1)A^{p+1} &\geq (A^{\frac{1}{2}}TA^{\frac{1}{2}})^{p+1} \\ &= A^{\frac{1}{2}}T^{\frac{1}{2}}(T^{\frac{1}{2}}AT^{\frac{1}{2}})^{p}T^{\frac{1}{2}}A^{\frac{1}{2}}. \end{aligned} \tag{3.16}$$

Multiplying $A^{\frac{-1}{2}}$ on both sides of (3.16), the proof is complete. $\square$

Proof of Theorem 3.
(i)$\Longrightarrow$(ii): Let n be a nutural number, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr + (n+1)\alpha\}u \geq (p+\alpha u)s$. By (i)$\Rightarrow$(ii) of Theorem 1, there exists the unique invertible positive contraction T satisfying the following (2.1):

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^{n} = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}(A^{\frac{\alpha u}{2}}B^{p}A^{\frac{\alpha u}{2}})^{s}A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}. \tag{2.1}$$

By scrutinizing the proof of Theorem 1, (2.1) is equivalent to the following (3.9):

$$(A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}TA^{\frac{(p+\alpha u)s+ru}{2(n+1)}})^{n+1} = A^{\frac{ru}{2}}D^{s}A^{\frac{ru}{2}}, \tag{3.9}$$

where $D = A^{\frac{\alpha u}{2}}B^{p}A^{\frac{\alpha u}{2}}$. (3.9) can be rewritten as

$$A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{n+1}}T^{\frac{1}{2}})^{n}T^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} = A^{\frac{ru}{2}}D^{s}A^{\frac{ru}{2}}. \tag{3.17}$$

Let $A_1 = A^{\frac{(p+\alpha u)s+ru}{n+1}}$. Then $MI \geq A \geq mI > 0$ ensures $M^{\frac{(p+\alpha u)s+ru}{n+1}}I \geq A_1 \geq m^{\frac{(p+\alpha u)s+ru}{n+1}}I > 0$ and

$$K_{+}\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right){A_1}^{n} \geq T^{\frac{1}{2}}(T^{\frac{1}{2}}A_1T^{\frac{1}{2}})^{n}T^{\frac{1}{2}} \tag{3.18}$$

holds for each nutural number n by Lemma 6. (3.18) can be rewritten as

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right) A^{\frac{\{(p+\alpha u)s+ru\}n}{n+1}} \geq T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{n+1}}T^{\frac{1}{2}})^n T^{\frac{1}{2}}. \tag{3.19}$$

Multiplying $A^{\frac{(p+\alpha u)s+ru}{2(n+1)}}$ on both sides of (3.19), we have

$$\begin{aligned} &K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right) A^{(p+\alpha u)s+ru} \\ &\geq A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+\alpha u)s+ru}{n+1}}T^{\frac{1}{2}})^n T^{\frac{1}{2}} A^{\frac{(p+\alpha u)s+ru}{2(n+1)}} \\ &= A^{\frac{ru}{2}} D^s A^{\frac{ru}{2}}. \end{aligned} \tag{3.20}$$

Hence the proof of (i)$\Longrightarrow$(ii) is complete.

(ii)$\Longrightarrow$(iii): (iii) holds in case $u=0$ since the assumption of (iii) ensures $ps=0$ and (2.2) turns out to be $K_+(1,1,n+1)I \geq I$ by (1.4) in Lemma B.3. Let $p \geq nu > 0$ in (ii), then the condition $r \geq 1-\alpha$ follows from $p \geq nu > 0$ and the other assumption of (ii) since

$$r \geq \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha \geq \frac{p+\alpha u}{nu} - \frac{n+1}{n}\alpha = \frac{p}{nu} - \alpha \geq 1-\alpha,$$

so that we have (iii).

(iii)$\Longrightarrow$(iv): Put $\alpha = 0$ and $s=1$ in (iii).

Proof of (iv)$\Longrightarrow$(i). Put $n=1$ and $r=\frac{p}{u}$ in (iv). Then $K_+(m^p, M^p, 2) = \dfrac{(M^p+m^p)^2}{4m^pM^p}$ by (1.2), so that

$$\frac{(m^p+M^p)^2}{4m^pM^p} A^p \geq B^p \tag{3.21}$$

holds for all $p \geq u > 0$, i.e., $p > 0$. By Theorem B.1, (3.21) implies (i).

Whence the proof of Theorem 3 is complete. □

Proof of Theorem 4. In case $u=0$, (i)$\Rightarrow$(ii)$\Rightarrow$(iii) holds by Theorem B.4, because (ii) and (iii) can be rewritten as follows: For each nutural number n,

$$K_+\left(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, \frac{ps+\frac{ps}{n}}{\frac{ps}{n}}\right) A^{ps} \geq B^{ps}$$

holds for $ps \geq 0$.

(i)$\Longrightarrow$(ii): In (i)$\Rightarrow$(ii) of Theorem 3, we can put $r = \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha$ since $(p+\alpha u)s \geq (n+\alpha)u$ yields $r = \frac{(p+\alpha u)s}{nu} - \frac{n+1}{n}\alpha \geq 1-\alpha$. Hence the proof of (i)$\Longrightarrow$(ii) is complete.

(ii)$\Longrightarrow$(iii): Put $p \geq nu \geq 0$, then the required condition $(p+\alpha u)s \geq (n+\alpha)u$ is satisfied.

(iii)$\Longrightarrow$(iv): Put $u=0$ in (iii), we have, for each nutural number n,

$$K_+\left(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, n+1\right) A^{ps} \geq B^{ps} \tag{3.22}$$

holds for $ps \geq 0$. (3.22) is equivalent to

$$K_+\left(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, \frac{ps+\frac{ps}{n}}{\frac{ps}{n}}\right) A^{ps} \geq B^{ps}.$$

Tending $n \to \infty$ (i.e., $\frac{ps}{n} \to 0$), we have (iv) by Lemma B.3.

(iv)$\Rightarrow$(i) has been already shown in Theorem B.2.

Hence the proof of Theorem 4 is complete. □

4 Some consideration on the results

In this section, we shall rewrite the results shown in Section 2 into more simple form by expressing them without one of the parameter u. We recall that in order to give their proofs in Section 3, we used the following result which is an application of Theorem G.

Proposition 5. *Let A and B be positive invertible operators. If* $\log A \geq \log B$, *then*

$$A^{\frac{(p+\alpha u)s+ru}{q}} \geq \{A^{\frac{ru}{2}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{ru}{2}}\}^{\frac{1}{q}} \tag{3.1}$$

holds for any $u \geq 0$, $p \geq 0$, $\alpha \in [0,1]$, $s \geq 1$, $r \geq 1-\alpha$ and $q \geq 1$ with $u(\alpha+r)q \geq (p+\alpha u)s+ru$.

In (3.1), the parameter u does not appear by itself, but appears only in the form of αu and ru. Put $\alpha_1 = \alpha u$ and $r_1 = ru$ in Proposition 5, then (3.1) can be rewritten as follows:

$$A^{\frac{(p+\alpha_1)s+r_1}{q}} \geq \{A^{\frac{r_1}{2}}(A^{\frac{\alpha_1}{2}}B^pA^{\frac{\alpha_1}{2}})^sA^{\frac{r_1}{2}}\}^{\frac{1}{q}}.$$

Here we consider the conditions of the parameters α_1 and r_1. We recall that the conditions of the parameters α, r and u are as follows:

$$\alpha \in [0,1],\ r \geq 1-\alpha \text{ and } u \geq 0. \tag{4.1}$$

(4.1) is equivalent to the following (4.2):

(4.2)
$\alpha_1 = \alpha u \in [0,u]$, $r_1 = ru \geq u - \alpha u = u - \alpha_1$ and $u \geq 0$.

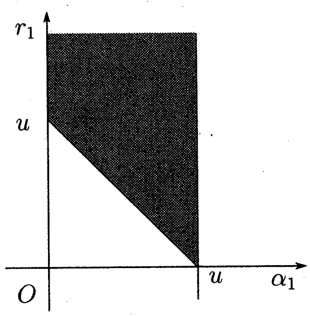

FIGURE 2

Figure 2 expresses the domain of α_1 and r_1 for a fixed $u \geq 0$ in (4.2). Since the parameter u does not appear in the statement any longer, we can choose the value of u arbitrarily. α_1 and r_1 can attain any positive real numbers by choosing the value of u appropriately, so that (4.2) implies the following (4.3):

$$\alpha_1 \geq 0 \text{ and } r_1 \geq 0. \tag{4.3}$$

Hence Proposition 5 can be rewritten as follows:

Proposition 5'. *Let A and B be positive and invertible operators. If* $\log A \geq \log B$, *then*

$$A^{\frac{(p+\alpha)s+r}{q}} \geq \{A^{\frac{r}{2}}(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{4.4}$$

holds for any $p \geq 0$, $\alpha \geq 0$, $s \geq 1$, $r \geq 0$ and $q \geq 1$ with $(\alpha+r)q \geq (p+\alpha)s+r$.

By using Proposition 5' instead of Proposition 5 in their proofs, our previous results in Section 2 can be rewritten as follows. Here we omit to describe the proofs.

Theorem 1'. *Let A and B be positive invertible operators. Then for each natural number n, the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \geq 0$, $p \geq 0$, $s \geq 1$ and $r \geq \max\{0, \frac{1}{n}(p+\alpha)s - \frac{(n+1)}{n}\alpha\}$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,s,r)$ satisfying*

$$T(A^{\frac{(p+\alpha)s+r}{n+1}}T)^n = A^{\frac{-(p+\alpha)s+nr}{2(n+1)}}\left(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}}\right)^sA^{\frac{-(p+\alpha)s+nr}{2(n+1)}}.$$

(iii) *For each* $\alpha \geq 0$, $p \geq n\alpha$, $s \geq 1$ *and* $r \geq \frac{1}{n}(p+\alpha)s - \frac{n+1}{n}\alpha$, *there exists the unique invertible positive contraction* $T = T(n,\alpha,p,s,r)$ *satisfying*

$$T(A^{\frac{(p+\alpha)s+r}{n+1}}T)^n = A^{\frac{-(p+\alpha)s+nr}{2(n+1)}}\left(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}}\right)^s A^{\frac{-(p+\alpha)s+nr}{2(n+1)}}.$$

(iv) *For each* $p \geq 0$, *there exists the unique invertible positive contraction* $T = T(n,p)$ *satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

Corollary 2'. *Let* A *and* B *be positive invertible operators. Then for each natural number* n, *the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each* $\alpha \geq 0$, $p \geq 0$ *and* $s \geq 1$, *there exists the unique invertible positive contraction* $T = T(n,\alpha,p,s)$ *satisfying*

$$T(A^{\frac{(p+\alpha)s}{n}}T)^n = \left(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}}\right)^s.$$

(iii) *For each* $\alpha \geq 0$, $p \geq n\alpha$ *and* $s \geq 1$, *there exists the unique invertible positive contraction* $T = T(n,\alpha,p,s)$ *satisfying*

$$T(A^{\frac{(p+\alpha)s}{n}}T)^n = \left(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}}\right)^s.$$

(iv) *For each* $p \geq 0$, *there exists the unique invertible positive contraction* $T = T(n,p)$ *satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

Theorem 3'. *Let* A *and* B *be positive invertible operators satisfying* $MI \geq A \geq mI > 0$, *and let* $K_+(m,M,p)$ *be defined in* (1.2). *Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number* n, $\alpha \geq 0$ *and* $p \geq 0$,

$$K_+\left(m^{\frac{(p+\alpha)s+r}{n+1}}, M^{\frac{(p+\alpha)s+r}{n+1}}, n+1\right)A^{(p+\alpha)s} \geq \left(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}}\right)^s$$

holds for all $s \geq 1$ *and* $r \geq \max\{0, \frac{1}{n}(p+\alpha)s - \frac{n+1}{n}\alpha\}$.

(iii) *For each natural number* n, $\alpha \geq 0$ *and* $p \geq n\alpha$,

$$K_+\left(m^{\frac{(p+\alpha)s+r}{n+1}}, M^{\frac{(p+\alpha)s+r}{n+1}}, n+1\right)A^{(p+\alpha)s} \geq \left(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}}\right)^s$$

holds for all $s \geq 1$ *and* $r \geq \frac{1}{n}(p+\alpha)s - \frac{n+1}{n}\alpha$.

(iv) *For each natural number* n *and* $p \geq 0$,

$$K_+\left(m^{\frac{p+r}{n+1}}, M^{\frac{p+r}{n+1}}, n+1\right)A^p \geq B^p$$

holds for all $r \geq \frac{p}{n}$.

Theorem 4'. *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$, and let $K_+(m, M, p)$ and $M_h(p)$ be defined in (1.2) and (1.3), respectively. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \geq 0$ and $p \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha)s-\alpha}{n}}, M^{\frac{(p+\alpha)s-\alpha}{n}}, n+1\right) A^{(p+\alpha)s} \geq (A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^s$$

holds for all $s \geq 1$ and $(p+\alpha)s \geq (n+1)\alpha$.

(iii) *For each natural number n, $\alpha \geq 0$ and $p \geq n\alpha$,*

$$K_+\left(m^{\frac{(p+\alpha)s-\alpha}{n}}, M^{\frac{(p+\alpha)s-\alpha}{n}}, n+1\right) A^{(p+\alpha)s} \geq (A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^s$$

holds for all $s \geq 1$.

(iv) *$M_h(p)A^p \geq B^p$ holds for all $p \geq 0$, where $h = \frac{M}{m} > 1$.*

5 Further extentions of our results

In the previous section, we rewrote our results into more simple form. In this section, we consider whether the domain of s can be extended or not. In Theorem 1, Corollary 2, Theorem 3 and Theorem 4, the parameter s is restricted to $s \geq 1$. Even after rewriting into simple form, this restriction does not be relaxed. Practically, we can find that this restriction derives from Proposition 5 or Proposition 5'. In other words, it derives from Theorem G.

Contrary to Proposition 5', we have the following result as an application of Theorem F.

Proposition 7. *Let A and B be positive invertible operators. If $\log A \geq \log B$, then*

$$A^{\frac{(p+\alpha)s+r}{q}} \geq \{A^{\frac{r}{2}}(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{4.4}$$

holds for any $p \geq 0$, $\alpha \geq 0$, $s \geq 0$, $r \geq 0$ and $q \geq 1$ such that $(\alpha + r)q \geq (p+\alpha)s + r$.

We remark that Proposition 7 is an immediate corollary of [23, Theorem 1], which is a function version of Proposition 7.

Proof of Proposition 7. (i) Case $\alpha > 0$. By Theorem A.2, $\log A \geq \log B$ implies the following (5.1):

$$A^\alpha \geq (A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^{\frac{\alpha}{p+\alpha}} \quad \text{for } p \geq 0 \text{ and } \alpha > 0. \tag{5.1}$$

Put $A_1 = A^\alpha$ and $B_1 = (A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^{\frac{\alpha}{p+\alpha}}$, then $A_1 \geq B_1 > 0$ by (5.1). By Theorem F,

$$A_1^{\frac{p_1+r_1}{q}} \geq (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{1}{q}} \tag{5.2}$$

holds for $p_1 \geq 0$, $r_1 \geq 0$ and $q \geq 1$ with $(1+r_1)q \geq p_1 + r_1$. (5.2) is equivalent to the following (5.3):

$$A^{\frac{(p_1+r_1)\alpha}{q}} \geq \left\{A^{\frac{r_1\alpha}{2}} (A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^{\frac{p_1\alpha}{p+\alpha}} A^{\frac{r_1\alpha}{2}}\right\}^{\frac{1}{q}}. \tag{5.3}$$

Put $s = \frac{p_1\alpha}{p+\alpha}$ and $r = r_1\alpha$, then the conditions $p_1 = \frac{(p+\alpha)s}{\alpha} \geq 0$, $r_1 = \frac{r}{\alpha} \geq 0$ and $(1+r_1)q \geq p_1 + r_1$ are equivalent to $s \geq 0$, $r \geq 0$ and $(\alpha + r)q \geq (p+\alpha)s + r$, respectively, and (5.3) can be rewritten as follows:

$$A^{\frac{(p+\alpha)s+r}{q}} \geq \{A^{\frac{r}{2}}(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{4.4}$$

for $p \geq 0$, $\alpha > 0$, $s \geq 0$, $r \geq 0$ and $q \geq 1$ with $(\alpha+r)q \geq (p+\alpha)s+r$.
(ii) Case $\alpha = 0$. (4.4) can be rewritten as follows:

$$A^{\frac{ps+r}{q}} \geq (A^{\frac{r}{2}} B^{ps} A^{\frac{r}{2}})^{\frac{1}{q}}. \tag{5.4}$$

(5.4) holds for $p \geq 0$, $s \geq 0$, $r \geq 0$ and $q \geq 1$ such that $rq \geq ps+r$ by Theorem A.2.
Consequently, the proof of Proposition 7 is complete. □

By comparing Proposition 5', an application of Theorem G, with Proposition 7, an application of Theorem F, we can find that Proposition 7 is an extension of Proposition 5' since the inequalities are the same but the domain $s \geq 0$ of Proposition 7 includes the domain $s \geq 1$ of Proposition 5'.

We have the following results which are extensions of Theorem 1' and Theorem 3' by using Proposition 7 instead of Proposition 5',

Theorem 8. *Let A and B be positive invertible operators. Then for each natural number n, the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $\alpha \geq 0$, $p \geq 0$, $s \geq 0$ and $r \geq \max\{0, \frac{1}{n}(p+\alpha)s - \frac{n+1}{n}\alpha\}$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,s,r)$ satisfying*

$$T(A^{\frac{(p+\alpha)s+r}{n+1}}T)^n = A^{\frac{-(p+\alpha)s+nr}{2(n+1)}} \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s A^{\frac{-(p+\alpha)s+nr}{2(n+1)}}. \tag{5.5}$$

(iii) *For each $\alpha \geq 0$, $p \geq 0$ and $s \geq 0$, there exists the unique invertible positive contraction $T = T(n,\alpha,p,s)$ satisfying*

$$T(A^{\frac{(p+\alpha)s}{n}}T)^n = \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s.$$

(iv) *For each $p \geq 0$, there exists the unique invertible positive contraction $T = T(n,p)$ satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

Theorem 9. *Let A and B be positive invertible operators satisfying $MI \geq A \geq mI > 0$, and let $K_+(m,M,p)$ and $M_h(p)$ be defined in (1.2) and (1.3), respectively. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \geq 0$ and $p \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha)s+r}{n+1}}, M^{\frac{(p+\alpha)s+r}{n+1}}, n+1\right) A^{(p+\alpha)s} \geq \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s \tag{5.6}$$

holds for $s \geq 0$ and $r \geq \max\{0, \frac{1}{n}(p+\alpha)s - \frac{n+1}{n}\alpha\}$.

(iii) *For each natural number n, $\alpha \geq 0$ and $p \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha)s-\alpha}{n}}, M^{\frac{(p+\alpha)s-\alpha}{n}}, n+1\right) A^{(p+\alpha)s} \geq \left(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}}\right)^s$$

holds for $s \geq 0$ such that $(p+\alpha)s \geq (n+1)\alpha$.

(iv) *For each natural number n and $p \geq 0$,*

$$K_+\left(m^{\frac{p}{n}}, M^{\frac{p}{n}}, n+1\right) A^p \geq B^p$$

holds.

(v) $\dfrac{(m^p+M^p)^2}{4m^pM^p}A^p \geq B^p$ *holds for all* $p \geq 0$.

(vi) $M_h(p)A^p \geq B^p$ *holds for all* $p \geq 0$, *where* $h = \frac{M}{m} > 1$.

Proofs of Theorem 8 and Theorem 9 are slight modification of proofs of Theorem 1 and Theorem 3, respectivery. So that we omit describe their proofs.

By Comparing the new results Theorem 8 and Theorem 9 with the refined former results Theorem 1', Corollary 2', Theorem 3' and Theorem 4', it turns out that the new results are extensions of the former results since the domain $s \geq 0$ of the new results includes the domain $s \geq 1$ of the former results. This fact is based on Proposition 5' and Proposition 7 which are used in the proofs of the former and new results, respectively.

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