数理解析研究所講究録
1144 巻 2000 年 64-69



# Normal Toeplitz matrices on $\mathbb{C}^n$

東北大学理学研究科　吉野　崇 (Takashi Yoshino)

In this lecture, we gave an another proof of the characterization given in [1] and also in [2] of the normal Toeplitz matrices on $\mathbb{C}^n$.

For any positive integer $n$ and for any $a_{j,k} \in \mathbb{C}$ $(j, k = 1, 2, \cdots, n)$, let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} = (a_{j,k}) \quad \text{on} \quad \mathbb{C}^n$$

and let

$$W = \overbrace{\begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \cdot & 1 & 0 \\ \vdots & \cdot & \cdot & \cdot & \vdots \\ 0 & 1 & \cdot & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}}^{n \text{ times}}.$$

Then

$$WAW = \begin{pmatrix} a_{n,n} & \cdots & a_{n,2} & a_{n,1} \\ \vdots & & \vdots & \vdots \\ a_{2,n} & \cdots & a_{2,2} & a_{2,1} \\ a_{1,n} & \cdots & a_{1,2} & a_{1,1} \end{pmatrix}. \tag{1}$$

Let ${}_tA = (a_{n+1-k,n+1-j})$ and let ${}^tA = (a_{k,j})$. Then, by (1), we have

$$WAW = {}^t({}_tA) = {}_t({}^tA) \tag{2}$$

and

$$\begin{aligned} {}_t(AB) &\underset{(2)}{=} {}^t(WABW) = {}^t\{(WAW)(WBW)\} \\ &= {}^t(WBW)\,{}^t(WAW) \underset{(2)}{=} {}_tB\,{}_tA. \end{aligned} \tag{3}$$

If $T$ is a Toeplitz matrix on $\mathbb{C}^n$, then

$$ {}_tT = T \tag{4}$$

and $$ {}_t(T^*T) \underset{(3)}{=} {}_tT \; {}_t(T^*) \underset{(4)}{=} TT^* \tag{5}$$

and, in particular, we have

$$\begin{aligned} T^*T \; ; \text{ Toeplitz matrix } &\underset{(4)}{\rightarrow} \quad {}_t(T^*T) = T^*T \\ &\underset{(5)}{\rightleftarrows} \quad T \; ; \text{ normal} \end{aligned}$$

Since ${}_t(T^*T - TT^*) = {}_t(T^*T) - {}_t(TT^*) \underset{(5)}{=} TT^* - T^*T$, we have the following by (4).

**Theorem 1.** For a Toeplitz matrix $T$ on $\mathbb{C}^n$, $T$ is normal if and only if $T^*T - TT^*$ is a Toeplitz matrix.

Henceforth we assume that the Toeplitz matrix $T$ on $\mathbb{C}^n$ is represented as follows ;

$$T = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-n+1} \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix},$$

where $a_j$, $(j = 0, \pm 1, \pm 2, \cdots, \pm(n-1))$ are some complex numbers. Then $T^*T = (t_{j,k})$, where $t_{j,k} = \sum_{m=1}^{n} \overline{a_{m-j}} a_{m-k}$, and $T^*T - TT^* = (\alpha_{j,k})$, where

$$\alpha_{j,k} = \sum_{m=1}^{n} \overline{a_{m-j}} a_{m-k} - \sum_{m=1}^{n} \overline{a_{k-m}} a_{j-m}$$

and hence, by Theorem 1, we have the following.

**Theorem 2.** The Toeplitz matrix $T$ on $\mathbb{C}^n$ is normal if and only if, for any $j, k = 1, 2, \cdots, n-1$,

$$\overline{a_{-j}} a_{-k} - \overline{a_k} a_j = \overline{a_{n-j}} a_{n-k} - \overline{a_{-(n-k)}} a_{-(n-j)}. \tag{6}$$

Let $n-k$ instead of $k$ in (6). Then, for any $j, k = 1, 2, \cdots, n-1$, we have

$$\overline{a_{-j}}a_{-(n-k)} - \overline{a_{n-k}}a_j = \overline{a_{n-j}}a_k - \overline{a_{-k}}a_{-(n-j)}. \tag{6-1}$$

Let $k = j$ in (6) and (6-1) respectively. Then, for any $j = 1, 2, \cdots, n-1$, we have

$$|a_{-j}|^2 - |a_j|^2 = |a_{n-j}|^2 - |a_{-(n-j)}|^2 \tag{6-2}$$

$$\text{and} \quad \overline{a_{-j}}a_{-(n-j)} = \overline{a_{n-j}}a_j \tag{6-3}$$

and hence $|a_{-j}| = |a_{n-j}|$ or $|a_{-j}| = |a_j|$

$$\text{i.e.,} \begin{cases} a_{-j} = e^{i\varphi_j}\overline{a_j} \\ \text{or} \quad a_{-j} = e^{i\psi_j}a_{n-j} \end{cases} \quad \text{for some } \varphi_j,\ \psi_j \in [0, 2\pi). \tag{6-4}$$

If $a_{-j} = e^{i\varphi_j}\overline{a_j}$, then, by (6-3), we have

$$0 = e^{-i\varphi_j}a_j a_{-(n-j)} - \overline{a_{n-j}}a_j = e^{-i\varphi_j}a_j(a_{-(n-j)} - e^{i\varphi_j}\overline{a_{n-j}})$$

and

$$a_{-j} = e^{i\varphi_j}\overline{a_j}\ (a_j \neq 0) \quad \text{implies} \quad a_{-(n-j)} = e^{i\varphi_j}\overline{a_{n-j}} \tag{6-5}$$

And if $a_{-j} = e^{i\psi_j}a_{n-j}$, then, by (6-3), we have

$$0 = e^{-i\psi_j}\overline{a_{n-j}}a_{-(n-j)} - \overline{a_{n-j}}a_j = e^{-i\psi_j}\overline{a_{n-j}}(a_{-(n-j)} - e^{i\psi_j}a_j)$$

and

$$a_{-j} = e^{i\psi_j}a_{n-j}\ (a_{n-j} \neq 0) \quad \text{implies} \quad a_{-(n-j)} = e^{i\psi_j}a_j \tag{6-6}$$

In particular, if $\begin{cases} a_{-j} = e^{i\varphi_j}\overline{a_j} & (a_j \neq 0) \\ a_{-k} = e^{i\psi_k}a_{n-k} & (a_{n-k} \neq 0) \end{cases}$, then, by (6), (6-5) and (6-6), we have

$$\begin{cases} a_{-j} = e^{i\varphi_j}\overline{a_j} = e^{i\varphi_j}e^{-i(\varphi_j - \psi_k)}a_{n-j} = e^{i\psi_k}a_{n-j} \\ \text{or} \quad a_{-k} = e^{i\psi_k}a_{n-k} = e^{i\psi_k}e^{-i(\psi_k - \varphi_j)}\overline{a_k} = e^{i\varphi_j}\overline{a_k} \end{cases}$$

and

$$\begin{cases} a_{-j} = e^{i\varphi_j}\overline{a_j} & (a_j \neq 0) \\ a_{-k} = e^{i\psi_k}a_{n-k} & (a_{n-k} \neq 0) \end{cases} \quad \text{implies} \quad \begin{cases} a_{-j} = e^{i\psi_k}a_{n-j} \\ \text{or} \quad a_{-k} = e^{i\varphi_j}\overline{a_k}. \end{cases} \tag{6-7}$$

**Lemma 1.** Under the condition (6), if $a_j \neq 0,\ a_k \neq 0,\ a_{n-j} \neq 0,\ a_{n-k} \neq 0$ and if $|\frac{a_{n-j}}{a_j}| \neq |\frac{a_{n-k}}{a_k}|$, then, for some $\varphi,\ \psi \in [0, 2\pi)$,

$$\begin{cases} a_{-j} = e^{i\varphi}\overline{a_j} \\ a_{-k} = e^{i\varphi}\overline{a_k} \end{cases} \quad \text{or} \quad \begin{cases} a_{-j} = e^{i\psi}a_{n-j} \\ a_{-k} = e^{i\psi}a_{n-k}. \end{cases}$$

**Lemma 2.** Under the condition (6), if $a_j \neq 0,\ a_k \neq 0,\ a_{n-j} \neq 0,\ a_{n-k} \neq 0$ and if $\overline{\frac{a_{n-j}}{a_j}} \neq \overline{\frac{a_{n-k}}{a_k}}$, then

$$\begin{cases} a_{-j} = e^{i\varphi}\overline{a_j} \\ a_{-k} = e^{i\varphi}\overline{a_k} \end{cases} \quad \left(\text{and also} \quad \begin{cases} a_{-j} = e^{i\psi}a_{n-j} \\ a_{-k} = e^{i\psi}a_{n-k} \end{cases}\right)$$

implies that, for any $m = 1, 2, \cdots, n-1$,

$$a_{-m} = e^{i\varphi}\overline{a_m} \quad (\text{and} \quad a_{-m} = e^{i\psi}a_{n-m} \quad \text{respectively}).$$

**Lemma 3.** Under the condition (6), if $a_j \neq 0,\ a_k \neq 0,\ a_{n-j} \neq 0,\ a_{n-k} \neq 0$ and if $|\frac{a_{n-j}}{a_j}| = |\frac{a_{n-k}}{a_k}| = |\frac{a_k}{a_{n-k}}|$, then

$$\begin{cases} a_{-j} = e^{i\varphi_j}\overline{a_j} = e^{i\psi_j}a_{n-j} \\ a_{-k} = e^{i\varphi_k}\overline{a_k} = e^{i\psi_k}a_{n-k}, \end{cases} \quad \begin{cases} a_{n-j} = e^{i(\varphi_j - \psi_j)}\overline{a_j} \\ a_{n-k} = e^{i(\varphi_k - \psi_k)}\overline{a_k} \end{cases} \quad \text{and} \quad \begin{cases} \varphi_j = \varphi_k \\ \text{or} \quad \psi_j = \psi_k. \end{cases}$$

In particular, if $\frac{a_{n-j}}{\overline{a_j}} = \frac{a_{n-k}}{\overline{a_k}}$, then

$$\begin{cases} a_{-j} = e^{i\varphi_j}\overline{a_j} = e^{i\psi_j}a_{n-j} \\ a_{-k} = e^{i\varphi_j}\overline{a_k} = e^{i\psi_j}a_{n-k}. \end{cases}$$

**Lemma 4.** Under the condition (6), if $a_{-j} = e^{i\varphi}\overline{a_j}$ $(a_j \neq 0)$ and if $a_{n-j} = 0$, then

$$a_{-m} = e^{i\varphi}\overline{a_m} \quad (m = 1, 2, \cdots, n-1).$$

**Lemma 5.** Under the condition (6), if $a_{-j} = e^{i\psi} a_{n-j}$ $(a_{n-j} \neq 0)$ and if $a_j = 0$, then

$$a_{-m} = e^{i\psi} a_{n-m} \quad (m = 1, 2, \cdots, n-1).$$

**Theorem 3.** ([1], [2]) The Toeplitz matrix $T$ on $\mathbb{C}^n$ is normal if and only if, for some $\theta \in [0, 2\pi)$,

$$\begin{cases} \quad a_{-m} = e^{i\theta}\overline{a_m} \quad (m = 1, 2, \cdots, n-1) \quad \text{(type I)} \\ \text{or} \quad a_{-m} = e^{i\theta} a_{n-m} \quad (m = 1, 2, \cdots, n-1) \quad \text{(type II)}. \end{cases}$$

**Proof.** By Theorem 2, we have only to prove that the equation (6) implies (type I) or (type II) because (6) is satisfied clearly for each case of $a_{-k}$'s such as (type I) and (type II).

**(case 1)** If $a_j = 0$ $(j = 1, 2, \cdots, n-1)$, then $a_{-j} = 0$ $(j = 1, 2, \cdots, n-1)$ by (6-4) and hence (type I) and (type II) occur at the same time.

**(case 2)** If there exists $j$ such that $a_{-j} = e^{i\varphi}\overline{a_j}$ $(a_j \neq 0)$ and that $a_{n-j} = 0$, then, by Lemma 4, (type I) occurs.

**(case 3)** If there exists $j$ such that $a_{-j} = e^{i\psi} a_{n-j}$ $(a_{n-j} \neq 0)$ and that $a_j = 0$, then, by Lemma 5, (type II) occurs.

If $a_j = 0$, then, by (6-3), $\overline{a_{-j}} a_{-(n-j)} = 0$ and we have $a_{-j} = 0$ or $a_{-(n-j)} = 0$. In the case where $a_{-j} \neq 0$, by (6-4), $a_{-j} = e^{i\varphi_j}\overline{a_j} = 0$ or $a_{-j} = e^{i\psi_j} a_{n-j}$ and we have $a_{(n-j)} \neq 0$ and hence (case 3) occurs. In the case where $a_{-(n-j)} \neq 0$, by (6-4), $a_{-(n-j)} = e^{i\varphi_{n-j}}\overline{a_{n-j}}$ or $a_{-(n-j)} = e^{i\psi_{n-j}} a_j = 0$ and we have $a_{(n-j)} \neq 0$ and hence (case 2) occurs. Therefore we may consider the case that $a_{-j} = a_{-(n-j)} = 0$. And then, by (6), we have $\overline{a_{n-j}} a_{n-k} = 0$ $(k = 1, 2, \cdots, n-1)$ and we may assume $a_{n-j} = 0$ because, in the other case, (case 1) occurs. And henceforth we may assume that

$$a_j = 0 \quad \text{implies} \quad a_{n-j} = 0 \quad \text{and then} \quad a_{-j} = a_{-(n-j)} = 0 \qquad (\sharp)$$

by (6-4). And hence

$$\begin{array}{l} \text{the equations of (type I) and (type II)} \\ \text{are satisfied for each } j \text{ such as } a_j = 0. \end{array} \qquad (\flat)$$

And henceforth we may assume that there exists some $j_0$ such that $a_{j_0} \neq 0$ and $a_{n-j_0} \neq 0$. Then, by (6-4), (6-5) and (6-6), we have

$$\begin{cases} a_{-j_0} = e^{i\varphi_{j_0}} \overline{a_{j_0}} \\ a_{-(n-j_0)} = e^{i\varphi_{j_0}} \overline{a_{n-j_0}} \end{cases} \quad \text{or} \quad \begin{cases} a_{-j_0} = e^{i\psi_{j_0}} a_{n-j_0} \\ a_{-(n-j_0)} = e^{i\psi_{j_0}} a_{j_0}. \end{cases}$$

(case 4) If $a_j = 0$ for all $j \neq j_0$, then $a_{n-j} = a_{-j} = a_{-(n-j)} = 0$ for all $j \neq j_0$ by ($\sharp$) and (type I) or (type II) occurs by ($\flat$).

(case 5) If there exists some $j_1$ such that $a_{j_1} \neq 0$ and that $|\frac{a_{n-j_0}}{a_{j_0}}| \neq |\frac{a_{n-j_1}}{a_{j_1}}|$, then, by Lemmas 1 and 2, (type I) or (type II) occurs because $a_{j_1} \neq 0$ implies $a_{n-j_1} \neq 0$ by ($\sharp$).

(case 6) If there exists some $j_2$ such that $a_{j_2} \neq 0$, $|\frac{a_{n-j_0}}{a_{j_0}}| = |\frac{a_{n-j_2}}{a_{j_2}}| = |\frac{a_{j_2}}{a_{n-j_2}}|$ and that $\frac{a_{n-j_0}}{\overline{a_{j_0}}} \neq \frac{a_{n-j_2}}{\overline{a_{j_2}}}$, then, by Lemmas 3 and 2, (type I) or (type II) occurs because $a_{j_2} \neq 0$ implies $a_{n-j_2} \neq 0$ by ($\sharp$).

(case 7) If $|\frac{a_{n-j_0}}{a_{j_0}}| = |\frac{a_{n-j}}{a_j}| = |\frac{a_j}{a_{n-j}}|$ and $\frac{a_{n-j_0}}{\overline{a_{j_0}}} = \frac{a_{n-j}}{\overline{a_j}}$ for all $j \neq j_0$ such as $a_j \neq 0$, then, by Lemma 3 and by ($\flat$), (type I) and (type II) occur at the same time because $a_j \neq 0$ implies $a_{n-j} \neq 0$ by ($\sharp$).

## References

[1] Farenik, D. R., Krupnik, M., Krupnik, N. and Lee, W. Y., *Normal Toeplitz matrices*, SIAM J. Matrix Anal. Appl., 17(1996), pp. 1037–1043.

[2] Ito, T., *Every normal Toeplitz matrix is either of type I or of type II*, SIAM J. Matrix Anal. Appl., 17(1996), pp. 998–1006.