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Fast iterative solver for image-based finite element method

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1 Introduction

Recently the image-based approach was developed by Holister and Kikuchi[1] to analyze structures with complex geometry. The basic idea of this approach is to convert the bit-map information of the digital image of structures into geometrical model for the finite element analysis. Then the finite element analysis is performed by recognizing each volume element (voxel) in an image as a finite element in the uniform rectangular grid.

Our target problem is a mesoscopic simulation of concrete materials, which have the complex structure constituted by different materials such as coarse aggregate with complicated shapes and mortar. It is difficult to generate a finite element model representing the complex structure of the material precisely by conventional mesh generation techniques. Therefore the image-based approach is applied to our problem[2][3].

In the image-based approach, the complex geometry of the structure can be captured by using very fine grids, and then very large linear systems need to be solved. On the other hand, the image-based approach offers the feature of uniform rectangular grid. Using this property, sophisticated iterative solvers that are difficult to apply to the conventional finite element method using unstructured meshes can be applied to solve the linear system arising in the image-based approach.

One of them is a conjugate gradient (CG) method with an efficient preconditioner utilizing the feature of uniform rectangular grid. Such a preconditioner can be developed by using signal processing techniques such as fast Fourier transform (FFT) or wavelet transform to precondition the matrix.

Another approach is the multigrid technique. For the uniform rectangular grid, a sequence of nested finite element spaces for the multigrid method can be constructed easily.

In this work, these types of fast iterative solvers for image-based FEM are considered, and their efficiency is discussed by numerical experiments. The performance in the parallel processing is also evaluated.

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2 Iterative solvers for image-based FEM

2.1 Image-based Finite Element Method

In the image-based approach[1], the bit-map information of the digital image of structures is used as a geometrical model, and each volume element (voxel) in the image is recognized as a finite element in the uniform rectangular grid (see fig.1).

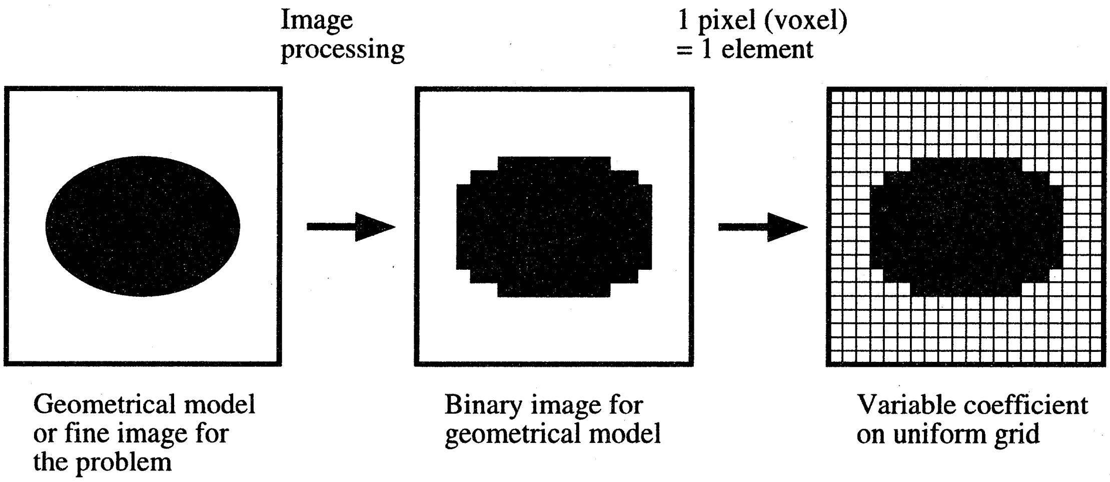


Fig. 1 Image-based Finite Element method

In the image-based approach, very fine grid is required to represent complex geometry. In our target problem of concrete materials, the number of degrees of freedom becomes 10^6 to 10^8. In the nonlinear simulation based on the image-based approach, such a huge linear system need to be solved many times. Therefore development of the fast solver for such a huge linear system is the key to apply the image-based finite element method to the nonlinear simulation.

However, the image-based approach offers the feature of uniform rectangular grid. Owing to this property, sophisticated but restrictive procedures that are difficult to apply to the conventional finite element method using unstructured meshes can be applied to solve the linear system arising in the image-based approach.

In this work, two types of iterative solvers led by the feature of uniform rectangular grid are considered as follows:

1. Conjugate gradient method with effective preconditioner using signal processing technique

2. Multigrid method

The rest of this section, actual procedures based on these approach will be discussed.

2.2 Preconditioned Conjugate Gradient Method

The best iterative solver for the elasticity should be the conjugate gradient (CG) method. As is widely known, an effective preconditioner can be designed by using the good approximation of the inverse matrix. Our problem is basically elasticity, which is of second order elliptic problem. The mathematical property of the problem is similar to the Poisson equation. Thus the Poisson solver is a candidate of the effective preconditioner.

Well-known fast Poisson solvers are based on the spectral method. In the image-based finite element method, all the unknown variable is on the uniform grid, and it can be recognized as the variable on the uniform sampling points. In such a situation, the fast Poisson solver using signal processing techniques such as fast Fourier transform (FFT) or wavelet transform to precondition the matrix can be employed. Signal processing techniques are widely used and various efficient algorithms have been developed in the field of the computer science. Therefore a fast iterative solver for the image-based FEM can be constructed by using such techniques.

In this work, the FFT is employed to construct the fast Poisson solver and its is applied to the components of the residual vector corresponding to each physical direction. In other words, the fast Poisson solver, which consists of one Fourier transform, scaling in the frequency domain and an inverse transform, are performed three times to obtain an improved direction vector from a residual vector in the three dimensional problem. In this paper, this procedure is called FFT preconditioned CG (FFT-PCG) method.

2.3 Cascadic Conjugate Gradient Method

For the uniform rectangular grid used in the image-based approach, a sequence of nested finite element spaces for the multigrid method can be constructed easily as shown in Fig. 2. Thus the multigrid method is worth considering to develop a fast iterative solver.

There are different types of iterative procedures based on the multigrid method, which are categorized by their iterative process between fine grid and coarse grid. Typical ones are known as V-cycle or W-cycle. Recently the cascadic multigrid method was developed by Deuflhard[4]. The iterative process in the cascadic multigrid method are performed from coarse grid to fine one in one way. Thus it is sometimes called one-way multigrid method. In this work, the cascadic conjugate gradient (CCG) method[4],

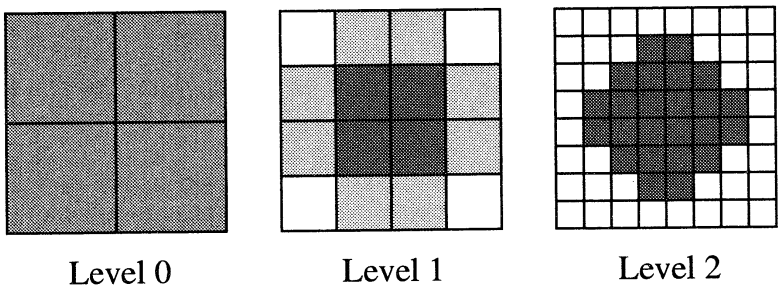


Fig. 2 Multigrid for image-based finite element method

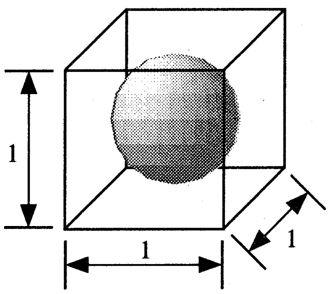
in which the conjugate gradient method is used as the basic iteration, is applied to the image-based finite element method. This method is based on the fact that both the iteration of multigrid sequence for the Galerkin finite element approximation and that of the conjugate gradient method are known to minimize the energy norm arising iteration errors. Actually the CCG method consists of the two nested iterations. The outer iteration is for the multigrid, while the inner is for the conjugate gradient method. The iteration control mechanism is based on the approximated energy error norm and it makes the CCG efficient. Although this procedure is very simple, it has been shown to be efficient owing to such mathematical background.

3 Numerical experiments

In order to evaluate numerical properties of the present methods, the problem derived from the homogenization method for composite materials is solved. A simple composite material with a spherical inclusion in the representative volume element (RVE) as shown in Fig. 2 is considered and the characteristic function corresponding to the uniform axial strain is calculated. In this problem, the periodic boundary condition is subjected to each surface on the RVE and a body force is imposed. In this work, three cases with different meshes (32^3, 64^3, 96^3) are calculated. The termination criteria for the FFT-PCG method is specified by the L^2 norms of residual $\mathbf{r}$ and external force vector $\mathbf{f}$ as

$$||\mathbf{r}||_{L^2} \leq 10^{-5}||\mathbf{f}||_{L^2}.$$

In the CCG method, 4 level multigrid is used for each calculation.



Inclusion (r=0.3725)
Young's modulus 5400
Poisson's ratio 0.15
Matrix
Young's modulus 2500
Poisson's ratio 0.19

Fig. 2 Problem definition

3.1 Convergence Property

Table 1 shows the numbers of iterations needed for the convergence in the FFT-PCG method. These numbers indicate that the number of iterations is independent to the problem size in the FFT-PCG method.

Table 2 shows the numbers of iterations for each grid level in the CCG method. Every inner CG process requires small number of iterations which are almost independent to

Table 1 Number of iterations in PCG method

	mesh		
	32^3	64^3	96^3
FFT preconditioner	47	48	49
No preconditioner	147	300	453

Table 2 Number of iterations in CCG method (4 level)

	number of iterations		
	32^3	64^3	96^3
CCG method			
level 1	8	16	24
level 2	11	22	16
level 3 (final)	18	22	34
Single level CG	147	300	453

the problem size. The numbers of iterations corresponding to the final level are smaller than those indicated in FFT-PCG method.

Overall computational costs are evaluated for the FFT-PCG and CCG methods. From Fig. 3 that shows the elapsed time, the CCG method is 50% faster that FFT-PCG method. This is due to the fact that 3 dimensional fast Fourier transform is performed in the preconditioning process of the FFT-PCG method and its computational cost is not small.

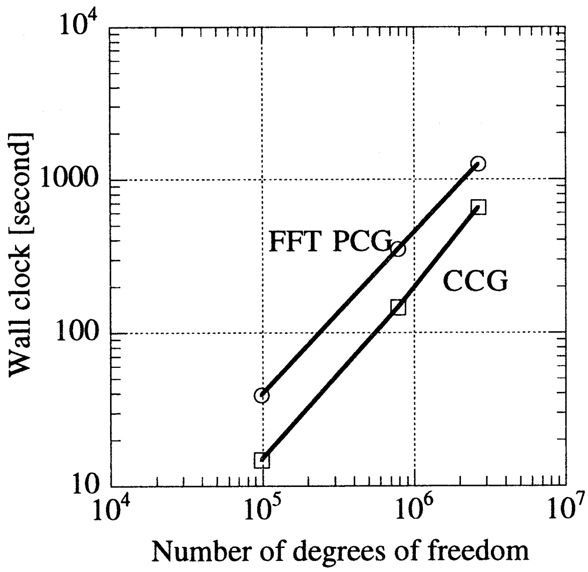


Fig. 3 Performance on Single Processor

3.2 Parallel Performance on PC Cluster

In this work, parallel performance of the present procedures is also compared by using Beowulf type PC cluster[5], which is a parallel computer consisting of personal computers connected by Ethernet. The specification of our PC cluster is shown in Table 3. To carry out parallel processing, the problem domain is divided into slabs of which number is equal to that of processors.

Table 3 Specification of PC cluster

Node(×8)	CPU: Intel Pentium II 450MHz Memory: 128MB Disk: 3.2GB
Interconnect	100base-TX PCI NIC 8 port 100base-TX Switching hub
Software	OS: Redhat Linux 5.2 (kernel 2.0.36) Compiler: gcc 2.7.2 Absoft Pro Fortran Library: MPICH 1.1.2

Figure 4 and 5 show the scalability of the present procedure on the parallel processing. The CCG method exhibits good scalability, since most of the communications between processors in the CCG method is neighboring. On the other hand, the scalability of FFT-PCG method is worth than that of the CCG method especially in the small case. In the FFT-PCG method, FFT stage requires all to all communication in which every node needs to communicate to all the rest of it. Thus the narrow band width of the network in the PC cluster decrease the scalability of FFT-PCG method.

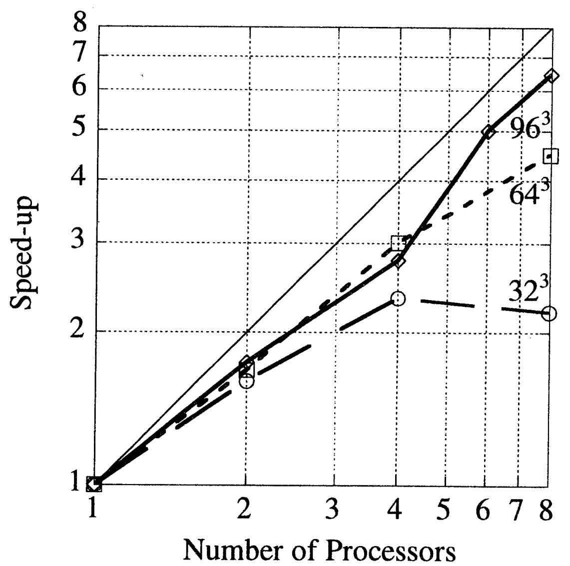


Fig. 4 Parallel performance of FFT PCG method

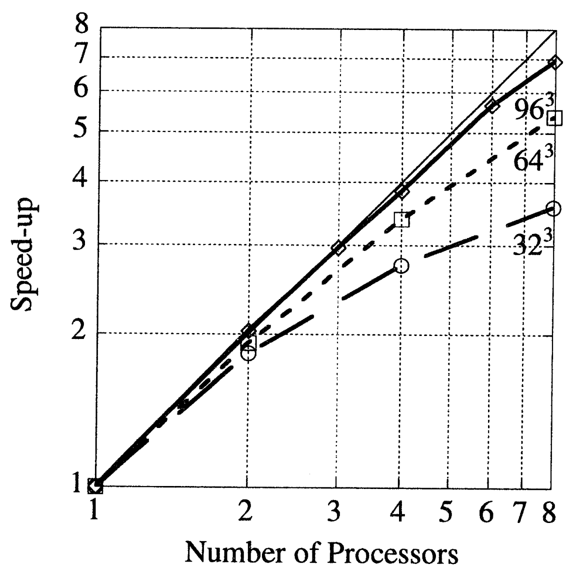


Fig. 5 Parallel performance of CCG method

4 Concluding Remarks

In this work, two different types of iterative solvers for the image-based finite element method are evaluated. Both the FFT-PCG and CCG methods exhibit almost optimal complexity in the numerical experiment. In the view of both the serial and parallel processing, the CCG method is superior to the FFT-PCG method.

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