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Numerical methods for radiation and resonance problems

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1 Introduction

The main purpose of this paper is to investigate some typical problems of wave motion in unbounded region which are related to radiation or scattering phenomena. The Helmholtz equation is one of the most important mathematical models which is used to describe the time harmonic behavior of various vibration and wave propagation phenomena.

The motivation of research is to understand main characteristics of wave propagation phenomena in obstacle scattering and/or wave radiation process through its numerical computation based on its mathematical analysis.

The importance of the wave propagation resides in the fact that it transmits information and transports energy. Some examples of research fields related to the wave propagation include acoustics, elasticity, electromagnetism with various applications such as sound emission from a speaker, human speech production, sound production of musical instruments, noise reduction, diagnostics or detection by ultrasonic wave, propagation of waves in optical fiber of fiber scope, heating by wave for various kinds of materials and others. Some of the characteristic quantities to be calculated in these problems include scattering amplitudes, transmission and reflection coefficients, resonance frequencies.

To investigate numerically the wave propagation phenomena in unbounded region using computers, we have to approximate the original problem which is formulated in some infinite dimensional function space by the one in an appropriate finite dimensional linear space. For this purpose, we first use the knowledge of the analytical properties of the solution to the original problem such as the radiation condition at infinity and/or the expression of the solution by a series of special functions or by an integral involving Green's function. We then reduce the problem into the boundary value problem in a bounded region

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with some truncation error for its solution and apply a finite element discretization method to get the linear equation in a finite dimensional approximation space.

Especially, we will show the effectiveness of the radiation condition at infinity which describes the asymptotic behavior of the solution and singles out the physical solution. We then use the domain decomposition method which divides the original problem in an unbounded region into the problem in a bounded region and the one in an outer region with simple shape.

More specifically, we treat a two-dimensional wave-guide problem where we use the exact boundary condition given by the Diriclet to Neumann map on the boundary between a bounded region and an outer unbounded region which is cylindrical with a bounded cross section. We also consider a one-dimensional problem related to this original two-dimensional problem.

We will show some numerical examples, and discuss the relationship between 2D and 1D cases and show some numerical examples which indicate the efficiency of the 1D model as the good approximation of the 2D problem in the sense that it gives similar frequency response curves.

2 Mathematical Formulation

The main mathematical framework of the study consists of the scattering theory based on the perturbation theory for linear operators and the finite element method for partial differential. equations.

The first difficulty in studying the radiation or scattering problem comes from the unboundedness of the region where we consider the partial differential equation and we have to choose an appropriate function space. The second problem we have to treat appropriately is the indefiniteness of the bilinear form which appears in the weak formulation used for the finite element method in the artificial bounded region and we have to consider the problem with non-real variables as well.

In this paper, we restrict our study to the two-dimensional case although the real physical phenomena occur in three-dimensional space. However, at least the theoretical part of our study can be extended to the three-dimensional case without any essential difficulty. The main problem we may have to solve is the practical computational complexity due to the large number of unknowns in 3D case and the shortage of memory and speed of the present computers together with the human resources in programming.

2.1 Two-dimensional Wave Propagation Problem

The wave propagation phenomena in two-dimensional space R^2 can be described by the following mathematical model of the wave equation in $\Omega \subset R^2$:

$$(\frac{\partial^2}{\partial t^2} - \Delta)u(t,x,y) \quad = \quad f(t,x,y) \text{ in } (-\infty,\infty)\times\Omega, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

$$(\alpha\frac{\partial}{\partial n}+\beta)u(t,x,y) \;=\; g(t,x,y) \text{ on } (-\infty,\infty)\times\partial\Omega, \tag{2}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on the boundary $\partial\Omega$ of Ω.

In the following, we consider a stationary time harmonic solution of the problem: $u(t,x,y) = e^{i\omega t}u(x,y)$ for inhomogeneous data: $f(t,x,y) = e^{i\omega t}f(x,y)$ and $g(t,x,y) = e^{i\omega t}g(x,y)$ from which we can calculate almost every important quantity. Then u satisfies the Helmholtz equation:

$$(-\Delta-\omega^2)u(x,y) \;=\; f(x,y) \text{ in } \Omega, \tag{3}$$

$$(\alpha\frac{\partial}{\partial n}+\beta)u(x,y) \;=\; g(x,y) \text{ on } \partial\Omega \tag{4}$$

with some radiation condition at infinity ($r = (x^2+y^2)^{1/2} \to +\infty$).

We assume that the boundary $\partial\Omega$ consists of two mutually distinct parts: $\partial\Omega = \Gamma_H \cup \Gamma_S$ where $g = g_S$ on the source boundary Γ_S and $g = 0$ on the homogeneous boundary Γ_H. The existence and uniqueness of the solution to this radiation or scattering problem can be proved by the limiting absorption principle which claims that the physical solution is the limit of the solution for the problem with positive absorption when the absorption tends to zero. In case that we know Green's function of the corresponding free space problem which satisfies the radiation condition at infinity, we can construct the solution solving the integral equation on the boundary.

2.2 Reduction to a Problem in a Bounded Region

We introduce an artificial boundary in Ω which includes the source boundary Γ_S and we assume that the shape of the outside the boundary is simple. For example, it is the outside of a disk or a cylindrical region. The, using the knowledge of the solution outside the boundary we impose the boundary condition on the artificial boundary which may the Diriclet to Neumann (DtN in short) map or its approximation. We sometimes call it a radiation boundary condition (or artificial boundary condition).

In the 2D wave-guide problem with a cylindrical unbounded semi-infinite channel, the radiation condition in the cylindrical is written as:

$$\frac{\partial p}{\partial n}(=\frac{\partial p}{\partial x}) = \Lambda p \quad \text{on } \Gamma_R, \tag{5}$$

where Γ_R is an artificial boundary which is a cross section of the cylindrical region and Λ is the Dirichlet to Neumann map in the outer cylindrical region given as

$$\Lambda p = \sum_{n=0}^{\infty}\gamma_n C_n(p)\cos(\frac{n\pi}{L}y) \tag{6}$$

with

$$C_n(p) = \begin{cases} \frac{1}{L}\int_0^L p(x,y)dy & (n=0) \\ \frac{2}{L}\int_0^L p(x,y)\cos(\frac{n\pi}{L}y)dy & (n\geq 1), \end{cases} \tag{7}$$

$$\gamma_n = \begin{cases} i\zeta_n, & \zeta_n = \{\omega^2 - (\frac{n\pi}{L})^2\}^{1/2}, \quad 0 < \frac{n\pi}{L} < \omega \\ -\eta_n, & \eta_n = \{(\frac{n\pi}{L})^2 - \omega^2\}^{1/2}, \quad \omega \leq \frac{n\pi}{L}. \end{cases} \tag{8}$$

Then the Helmholtz equation in the inner domain Ω_i is given as:

$$\begin{aligned} (-\omega^2 - \Delta)p &= 0 \text{ in } \Omega_i \\ \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma_H, \quad \frac{\partial p}{\partial n} &= g_S \text{ on } \Gamma_S, \quad \frac{\partial p}{\partial n} = \Lambda p \text{ on } \Gamma_R \end{aligned} \tag{9}$$

Related to this 2D wave-guide problem, we can consider the corresponding 1D Webster's horn equation given as:

$$-\frac{\partial v}{\partial t} = \frac{A(x)}{\rho}\frac{\partial p}{\partial x}, \qquad -\frac{\partial p}{\partial t} = \frac{\rho c^2}{A(x)}\frac{\partial v}{\partial x}, \tag{10}$$

where p is the pressure and v is the velocity, and $A(x)$ denotes the area of the cross section. Eliminating v, we have the 1D approximation model called Webster's horn equation:

$$\frac{\partial^2 p}{\partial t^2} - \frac{1}{A(x)}c^2\frac{\partial}{\partial x}(A(x)\frac{\partial p}{\partial x}) = 0, \tag{11}$$

3 Week Formulation and Discretization

In this paper, we use the finite element method to dicretize the problem in the artificially truncated region with an artificial boundary condition. We start with a weak formulation of the problem in an appropriate closed subspace $\mathcal{V}$ of the Sobolev space $H^1(\Omega_i)$ defined through the boundary condition and then restrict the problem into a finite dimensional subspace of $\mathcal{V}$ which is a set of all piece-wise linear continuous functions in $\mathcal{V}$ with respect to a regular triangulation of Ω_i. We note that we have to introduce an appropriate approximation of the boundary integral which corresponds to the non-local boundary condition such as the higher order radiation boundary condition or the Dirichlet to Neumann map. In the following, we show the case of the 2D wave-guide problem in some detail.

The weak formulation for the Helmholtz problem (3) and (4) with the artificial boundary condition is given as:

Find $p \in \mathcal{V} \subset H^1(\Omega)$:

$$a(p,q) = (f,q)(= a_0(g,q)) \qquad \forall q \in \mathcal{V}$$

where, together with its approximation $a_N(\cdot,\cdot)$,

$$\begin{aligned} a(p,q) &= a_0(p,q) + b_1(p,q) + b_2(p,q), \\ a^N(p,q) &= a_0(p,q) + b_1(p,q) + b_2^N(p,q) \end{aligned}$$

with

$$
\begin{aligned}
a_0(p,q) &= \int_\Omega \nabla p \cdot \overline{\nabla q} + p\overline{q}\,dxdy \\
b_1(p,q) &= -(\omega^2+1)\int_\Omega p\overline{q}\,dxdy \\
b_2(p,q) &= -(\Lambda p(x_R,\cdot), q(x_R,\cdot)) = b_{2,i}(p,q) + b_{2,r}^\infty(p,q) \\
b_{2,i}(p,q) &= -i\omega L C_0(p)C_0(q) - i\sum_{0<\frac{n\pi}{L}<\omega} \zeta_n(\frac{L}{2})C_n(p)C_n(q) \\
b_{2,r}^\infty(p,q) &= \sum_{\omega\le\frac{n\pi}{L}} \eta_n(\frac{L}{2})C_n(p)C_n(q),
\end{aligned}
$$

where ζ_n and η_n are all nonnegative constants in (8), and

$$
\begin{aligned}
b_2^N(p,q) &= -(\Lambda^N p(x_R,\cdot), q(x_R,\cdot)) = b_{2,i}(p,q) + b_{2,r}^N(p,q), \\
b_{2,r}^N(p,q) &= \sum_{\frac{L}{\pi}\omega\le n\le N} \eta_n(\frac{L}{2})C_n(p)C_n(q).
\end{aligned}
$$

Now the finite element method is formulated as:

Find $p_h \in \mathcal{V}_h \subset H^1(\Omega)$:

$$
a(p_h,q_h) = (f,q_h)(= a_0(g,q_h)) \qquad \forall q_h \in \mathcal{V}_h.
$$

4 Error Analysis

We develop the error analysis for the finite element discretization for the Helmholtz equation with the DtN boundary condition. We give rather abstract results which is essentially known but in an operator theoretical formulation. In application to 2D wave-guide problem, we use the result of Mikhlin [2] and the results of compact perturbation theory as well as the uniqueness of the analytic solution.

4.1 Abstract Error Analysis for Finite Element Method

We consider the following four problems:

1: $(\mathrm{E})_\mathrm{w}$: Find $u \in \mathcal{V}$ such that

$$
a(u,v) = (f,v) \quad \text{for all} \quad v \in \mathcal{V}.
$$

2: $(\mathrm{E_h})_\mathrm{w}$: Find $u_h \in \mathcal{V}_h$ such that

$$
a(u_h,v_h) = (f,v_h) \quad \text{for all} \quad v_h \in \mathcal{V}_h.
$$

3: $(\mathrm{E^N})_{\mathrm{w}}$: Find $u^N \in \mathcal{V}$ such that

$$a^N(u^N, v) = (f, v) \quad \text{for all} \quad v \in \mathcal{V}.$$

4: $(\mathrm{E_h^N})_{\mathrm{w}}$: Find $u_h^N \in \mathcal{V}_h$ such that

$$a^N(u_h^N, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in \mathcal{V}_h.$$

By Riesz's representation theorem, two operators A and A_N are defined as:

$$a(u, v) = (Au, v) \quad \text{and} \quad a^N(u, v) = (A^N u, v) \quad \text{for all} \quad v \in \mathcal{V}.$$

Then, we have the above four equations are equivalent to the following operator equations respectively:

1. $(\mathrm{E})_{\mathrm{op}}$: $Au = f$

2. $(\mathrm{E_h})_{\mathrm{op}}$: $A_h u_h = f_h$ with $A_h = P_h A$, $f_h = P_h f$

3. $(\mathrm{E^N})_{\mathrm{op}}$: $A^N u^N = f$

4. $(\mathrm{E_h^N})_{\mathrm{op}}$: $A_h^N u_h^N = f_h$ with $A_h^N = P_h A^N$, $f_h = P_h f$

Using the relations $Au = A^N u^N = f$ and

$$P_h A u_h = A_h u_h = f_h = A_h^N u_h^N = P_h f = P_h A u = P_h A^N u^N,$$

we can transform the expression of the error $u - u_h^N$ as follows:

$$\begin{aligned}
u - u_h^N &= u - v_h + v_h - u_h^N \\
&= u - v_h + (A_h^N)^{-1} A_h^N v_h - u_h^N \\
&= u - v_h + (A_h^N)^{-1} A_h^N v_h - (A_h^N)^{-1} f_h \\
&= u - v_h + (A_h^N)^{-1} A_h^N v_h - (A_h^N)^{-1} P_h f \\
&= u - v_h + (A_h^N)^{-1} A_h^N v_h - (A_h^N)^{-1} P_h A u \\
&= u - v_h + (A_h^N)^{-1} \{A_h^N v_h - P_h A u\} \\
&= u - v_h + (A_h^N)^{-1} \{P_h A^N v_h - P_h A u\} \\
&= u - v_h + (A_h^N)^{-1} \{P_h A^N (v_h - u) + P_h A^N u - P_h A u\} \\
&= \{I - (A_h^N)^{-1} P_h A^N)\}(u - v_h) + (A_h^N)^{-1} P_h (A^N - A) u.
\end{aligned}$$

Hence we can estimate the above difference as:

$$\|u - u_h^N\| \leq (I + \|(A_h^N)^{-1}\| \|A^N\|) \inf_{v_h \in \mathcal{V}_h} \|u - v_h\| + \|(A_h^N)^{-1}\| \|(A^N - A) u\|$$

Therefore, our next task is to prove the followings:

1. The uniform boundedness of $\|(A_h^N)^{-1}\|$: $\|(A_h^N)^{-1}\| \le M < +\infty$ with respect to h and N.
2. The truncation error estimate: $\|(A^N - A)u\| \le \frac{C}{N^\alpha}\|u\|_{\mathcal{W}}$ under the regularity condition for u: $u \in \mathcal{W} \subset \mathcal{V}$.

In the next section, we apply the results to the wave-guide problem.

4.2 Application to Wave-Guide Problem

We can apply the abstract error estimation based on the following observations:

1. The sesquilinear form $b_{2,r}^\infty(p,q)$ is bounded and nonnegative in $\mathcal{V}$. Hence $a_{0,DN}(p,q) \equiv a_0(p,q) + b_{2,r}^\infty(p,q)$ is an inner product in $\mathcal{V}$
2. The form $b_1(p,q) + b_{2,i}(p,q)$ is compact with respect to $a_{0,DN}(p,q)$ in $\mathcal{V}$.
3. We can then apply the results by Mikhlin [2] (see also Kako [1]) and we can prove the convergence of the finite element method under some additional condition on the non-existence of a positive eigenvalue for $-\Delta$ in Ω.
4. The difference between $a(p,q)$ and $a^N(p,q)$ is written as:

$$a(p,q) - a^N(p,q) = \sum_{N<n} \eta_n(\frac{L}{2})C_n(p)C_n(q) = (\{\Lambda - \Lambda^N\}p, q),$$

   and $\|\{\Lambda - \Lambda^N\}p\|_{L^2(0,L)}$ tends to zero exponentially with respect to N or is estimated from above by $\frac{C}{N^\alpha}\|u\|_{\mathcal{W}}$ with any α and a corresponding higher order Sobolev space $\mathcal{W}$.

5 Some Numerical Examples

In this section, we show some numerical examples calculated by using the methods introduced in the previous sections.

We show a numerical example of 2D wave propagation in the vocal tract open to an infinite cylinder. The Fig.1 shows a wave profile with a time frequency 7.5 kHz for the shapes of vowels "a"(left) and "e"(right). The source is placed on the left edge and the right side is a radiation boundary. The next Fig. 2 shows a frequency response curve for "a" measured at the mid point on the radiation boundary. We can see that, as the shape of the vocal tract becomes flatter, the response curve approaches nearer to the one of 1D model.

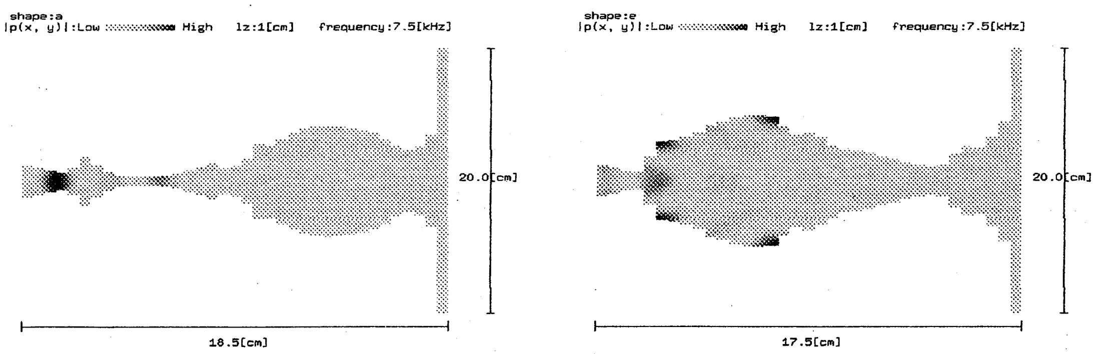

Figure 1: Wave profiles for vowels "a" and "e" in 7.5KHz

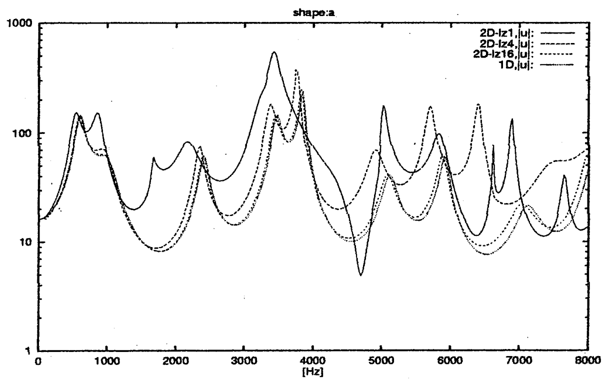

Figure 2: Comparison between 1D and 2D frequency response curves for "a"

6 Concluding Remarks

We have developed a methodology to calculate problems in unbounded regions by use of the DtN mapping or its approximations. Error analysis is given as an extension of the standard method. Application to a problem having resonance phenomena is presented and some typical phenomena have been captured in these numerical experiments. Applications to more realistic problems are future subject of study.

References

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