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Lie-Drach-Vessiot Theory and Painlevé Equations

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1. Lie was inspierd of the idea of Galois and Abel for algebraic equations. Therefore much of Lie's work was motivated by the dream of exending the Galois Theory of Algebraic Equation to the Galois Theory of Differential Equation. In 1895, on the centenary of l'Ecole nornarle supérieure of Paris, Lie writes

> ...depuis vingt-cinq ans, je me suis tout particulièrement efforcé d'étendre à d'autres domaines de la science mathématique ses idées sur les équations algébriques, si originales et si fécondes. (InΩuence de Galois sur le développement des Mathématiques, Le Centenaire de l'École normale 1795-1895)

There are important contributions to the realization of Lie's dream. In 1896, E. Picard presented Galois Theory of linear ordinary differntial equation called today Picard-Vessiot Theory. The Picard-Vessiot Theory is finite dimensional, while Lie's goal is essentially infinite dimensional. The first trial of infinite dimensional Differential Galois Theory was done in 1989 by J. Drach in his thesis, Essai sur une théorie générale de l'intégration et sur la classification des transcedentes. As E. Vessiot pointed out, Drach's thesis contains errors. Vessiot devoted years of research to try to construct a Differential Galois Theory of infinite dimansion on a rigorous foundation. Despite of his efforts, there remained still some obscurities and incomprehensivity in Differential Galois Theory of infinite dimension. Kolchin is famous for his Differential Galois Theory. But his theory is finite dimensional. His major contributions to Differential Galois Theory are foudation of Differential Algebra and formulation of Differential Galois Theory of finite dimension in the language of algebraic geometry. In spired of one the last papers of Vessiot, we proposed an infinite dimensional Differential Galois theory ([2], [3]). We do not forget J.-F. Pommaret's interesting book, Differential Galois Theory published in 1983.

The Painlevé equations were discoverd 100 years ago in the pursuit of special functions. The mathematicians of the 19th century wanted to generalize the Weierstrass $\wp$ function that satisfies the algebraic differential equation

$$y'^2 = 4y^3 - g_2 y - g_3.$$

So Painlevé classified algebraic differential equation

$$y'' = R(t, y, y')$$

of the seond order without movable singular points, where $R(t, y, y')$ is a rational function. Then he passes through a refinemnt procedure of throwing away those differential equation that he could inetgrate by the so far known functions. This is a natural procedure for the pursuit of new special functions. After this refinement procedure, they arrived at the list of 6 Painlevé equations. Therefore it is natural to expect that they are irreducible to the so far known functions or to the classical functions.

In fact just after the discovery of the Painlevé equations, Painlevé announced the irreducibility of the first Painleveé equation P_I. R. Liouville was against Painlevé's opinion and an unusual dispute in mathematics took place between Liouville and Painlevé. In the final stage of the dispute, Painlevé went to the rescue of the Differential Galois Theory of Drach, of which he recognized well high incompleteness. Finally in 1988, a rigorous proof of the irreducibility of the first Painlevé equation was given. Contrary to Painlevé guess, the proof does not depend on the Differential Galois Theory of infinite dimension. Such a theory did not exist in 1988.

In this note, we discuss about the calculation of the Galois group of our Differential Galois Theory for the Painlevé equations. The Galois group is in general a formal group of infinite variables and it is not easy to determine the Galois group. The calculation leads us to a new and better proof of the irreducibility ot the Painlevé equations. We have no idea of claculating the Differential Galois group for the first Painlevé equation P_I. Calculation for the second Painlevé equation P_{II} seems manageable. For it contains a parameter and it has the symmetries, the Bäcklund transformations.

2. Let us recall brieflly the definition of our Galois group **Inf-gal**(L/K), which is a formal group of infinite dimension in general([2], [3]).

We assume that all the fields are of characteristic 0. For a differential field M, its underlying abstract field is denoted by $M^\natural$. Let L/K be an ordinary differential field extension that is finitely generated as an abstract field extension. The derivation $\delta : L \to L$ defines the universal Taylor morphism

$$i : L \to L^{\natural}[[t]], \quad a \mapsto \sum_{n=0}^{\infty} \frac{\delta^n a}{n!} t^n, \tag{1}$$

which is the differential field morphism of $(L, \delta) \to (L^{\natural}[[t]], d/dt)$. Let $v_1, v_2, \ldots v_n$ be a transcendence basis of the abstract field extension $L^{\natural}/K^{\natural}$. Since $L^{\natural}$ is algebraic over $K^{\natural}(v_1, v_2, \ldots v_n)$, the derivation

$$\partial/\partial v_i : K^{\natural}(v_1, v_2, \ldots v_n) \to K^{\natural}(v_1, v_2, \ldots v_n)$$

extends to the derivation $L \to L$ for $1 \leq i \leq n$, which we denote also by $\partial/\partial v_i$. Operating on the coefficients, $\partial/\partial v_i$ is a derivation of the power series ring $L^{\natural}[[t]]$ as well as of the Laurent series field $L^{\natural}[[t]][t^{-1}]$. We define $\mathcal{L}$ as the differential subfield of $(L^{\natural}[[t]][t^{-1}], \{\partial/\partial t, \partial/\partial v_1, \partial/\partial v_2 \ldots, \partial/\partial v_n\})$ generated by $i(L)$ and $L^{\natural}$. Similarly $\mathcal{K}$ is another differential subfield of $(L^{\natural}[[t]][t^{-1}], \{\partial/\partial t, \partial/\partial v_1, \partial/\partial v_2 \ldots, \partial/\partial v_n\})$ generated by $i(K)$ and $L^{\natural}$.

3. The simplest but illustrating example is the following. Let $K = (\mathbf{C}(x), d/dx)$ and $L = K(y)$ with $y' = y$ and $y \neq 0$. Then the universal Taylor morphism $i : L \to L^{\natural}[[t]]$ sends y to

$$Y(y, t) = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)} t^n = y \exp t \in L^{\natural}[[t]]$$

because $y^{(n)} = y$ for every integer n. $i(y) = Y(y, t)$ is nothing but the generic solution of the differential equation $y' = y$. The transcendence degree of L/K is equal to 1. So we take as a transcendence basis $v_1 = y$. This means we consider the differentiation with respect to $v_1 = y$. Therefore $\mathcal{K} = L^{\natural}.\mathbf{C}(t) = L^{\natural}(t)$ and $\mathcal{L} = L^{\natural}(t, \exp t) = \mathcal{K}(\exp t)$. $\mathcal{L}/\mathcal{K}$ is a partial differential field extension with derivations $\partial/\partial t$ and $\partial/\partial y$.

4. We have the partial differential field extension $\mathcal{L}/\mathcal{K}$. They are differential subfields of $L^{\natural}[[t]][t^{-1}]$. Now $L^{\natural}$ is a partial differential field with derivations $\partial/\partial v_i$. We denote this partial differential field by $L^{\sharp}$. So we have the universal Taylor morphism

$$L^{\sharp} \to (L^{\sharp})^{\natural}[[w_1, w_2, \ldots, w_n]] = L^{\natural}[[w_1, w_2, \ldots, w_n]]$$

sending an element $a \in L^{\sharp}$ to its formal Taylor expansion

$$\sum_{\mathbf{m} \in \mathbf{N}^n} \frac{\partial^{\mathbf{m}} a}{\mathbf{m}!} \mathbf{w}^{\mathbf{m}}.$$

So using the above universal Taylor morphism, we expand the coeffcients to get the expansion

$$\iota : \mathcal{L} \to L^{\natural}[[w_1, w_2, \dots, w_n, t]][t^{-1}], \tag{2}$$

which is a morphism of partial differential ring. We consider the functor $\mathcal{F}_{\mathcal{L}/\mathcal{K}}$ of the infinitesimal deformations of ι in (2). The functor $\mathcal{F}_{\mathcal{L}/\mathcal{K}}$ is a functor of the category $(Alg/L^{\natural})$ of $L^{\natural}$-algebras to the category (Set) of sets. Namely for an $L^{\natural}$-algebra A, we set

$$\begin{aligned}\mathcal{F}_{\mathcal{L}/\mathcal{K}}(A) = \{ \quad f \ & : \mathcal{L} \to A[[w_1, w_2, \dots, w_n, t]][t^{-1}] \mid f \text{ is a morphism of} \\ & \text{differential ring such that } f \text{ coinscides with } \iota \text{ on } \mathcal{K} \\ & f \equiv \iota \ mod \ N(A)[[v_1, v_2, \dots, v_n, t]][t^{-1}]\},\end{aligned}$$

where $N(A)$ is the ideal of A consisting of all the nilpotent elements. We can show that there exists a formal group functor **Inf-gal** $(L/K) : (Alg/L^{\natural}) \to (Group)$ operating on the functor $\mathcal{F}_{\mathcal{L}/\mathcal{K}}$ such that (**Inf-gal**(L/K), $\mathcal{F}_{\mathcal{L}/\mathcal{K}}$) is a principal homogeneous space.

5. Let us see the above procedure by the Example in 3. As we have seen, in that example we have

$$\iota : \mathcal{L} = \mathcal{K}(\exp t) \to L^{\natural}[[w,\, t]][t^{-1}].$$

Hence an infinitesimal deformation

$$f : \mathcal{L} \to A[[w,\, t]][t^{-1}]$$

of ι is determined by the image $f(\exp t)$. Since

$$\frac{\partial}{\partial y} \exp t = 0, \quad \frac{\partial}{\partial t} \exp t = \exp t,$$

we have $f(\exp t) = c \exp t$ with $c \in A$. The constant $c = 1 + c'$ with $c' \in N(A)$ because the deformation f is infnitesimal. So for an $L^{\natural}$-algebra A, the multiplication of two elements $1 + c'$ nad $1 + c"$ being $(1 + c')(1 + c") = 1 + c' + c" + c'c"$. In other words, **Inf-gal**(L/K) is the formal group $\hat{\mathbf{G}}_m$ attached to the multiplicative group $\mathbf{G}_m$.

6. We interprete a Lie pseudo group as a formal group. For a ring A, we denote by $\Gamma_n(A)$ the group of all continuous automorphisms φ of the power series ring $A[[x_1, x_2, \dots, x_n]]$ such that

$$\varphi \equiv Id \quad mod\, N(A)[[x_1, x_2, \dots, x_n]],$$

where $N(A)$ is the ideal of nilpotent elements of A. So φ is given by n power series

$$(\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)) \in A[[x_1, x_2, \ldots, x_n]]^n$$

such that

$$\varphi_i \equiv x_i \quad mod\, N(A)[[x_1, x_2, \ldots, x_n]]$$

for $1 \leq i \leq n$. Consideration of nilpotent elements allows us to interprete Lie pseudo group of coordinate transformations as a true group. In fact we consider a group functor

$$\Gamma_n : (Alg/\mathbf{Z}) \rightarrow (Group).$$

A Lie pseudo group is a subgroup functor of Γ_n defined by a system of differential equations.

7. The argument in 5 allows us to prove

Theorem 1 *Let L/K be a strongly normal extension of an ordinary differential field K with Galois group G. Then we have* **Inf-gal**$(L/K) \cong \hat{G}$. *Here $\hat{G}$ denotes the formal group associated with the algebraic group G.*

Now we consider a Riccati equation.

$$q' = -q^2 - \frac{x}{2}, \tag{3}$$

the base field being $K = \mathbf{C}(x)$. The Riccati equation is linearized by the Airy equation

$$2y'' + xy = 0 \tag{4}$$

Proposition 1 *Let $L = K[q, q', \ldots]/$(differential ideal generated by $q' + q^2 + x/2$). Then we have*

$$\mathbf{Inf\text{-}gal}(L/K) \cong \widehat{SL_2}$$

Sketch of Proof. Let $Q(q,t)$ be the image of q by the universal Taylor morphism. We can show

$$2\frac{Q_{qqq}}{Q_q} - 3\left(\frac{Q_{qq}}{Q_q}\right)^2 = 0.$$

This implies **Inf-gal**$(L/K) \subset \widehat{SL_2}$. To show the opposite inclusion, we must show that tr.d.$[\mathcal{L} : \mathcal{K}] = 3$, which follows from the fact that the Airy equation has no algebraic solution.

8. The first Painlevé equation contains no parameter. The other Painlevé equations have parameters and symmetry. So they are easier to treat. We take as an example the second Painlevé equation

$$P_{II}(\alpha) \qquad\qquad q" = 2q^3 + xq + \alpha,$$

α being a parameter. Let $K = C(x)$ and q be a solution of the second Painlevé equation $P_{II}(\alpha)$ such that $tr.d.[K(q, q') : K] = 2$. We set $L = K(q, q')$.

Lemma 1 *Let Q and Q' be respectively the image of q and q' by the universal Taylor morphism i of (1) so that $Q' = dQ/dt$. Then we have*

$$\frac{J(Q, Q')}{J(q, q')} = \begin{bmatrix} Q_q & Q'_q \\ Q_{q'} & Q'_{q'} \end{bmatrix} = 1.$$

The argument in 6 for the Riccati equation gives us

Corollary 1 *We have* **Inf-Gal**$(L/K) \subset$ *Pseudo group of coordinate transformations of two variables leaving the volume invariant.*

It is natural to have the following

Conjecture 1 Inf-Gal(L/K) *coincides with the pseudo group of coordinate transformations of two variables leaving the volume invariant.*

The problem is how to prove the other inclusion or how to conclude that the Galois group is sufficiently big.

In calculation of the Galois group of algebraic extension, the following theorem is useful.

Theorem 2 *Under suitable conditions, if you specialize an equation,the Galois group becomes smaller.*

It is natural to expect the following

Hypothesis 1 *Under suitable conditions, if you specialize an extension, then our Galois group* **Inf-gal**(L/K) *becomes smaller.*

We can express Hypothesis rigorously. We are studying the second Painlevé equation. The second Painlevé equation $P_{II}(\alpha)$ is equivalent to

$$S_{II}(b) \quad \begin{cases} \dfrac{dq}{dx} &= p - q^2 - \dfrac{x}{2}, \\ \dfrac{dp}{dx} &= 2qp + b, \end{cases}$$

where $b = \alpha + 1/2$. From now on, we assume b is a variable and $K = \mathbf{C}(b,t)$, $L = K(q, q')$ such that $tr.d.[L : K] = 2$. It follows from Corollary 1 and Hypothesis that **Inf-Gal**(L/K) is a one parameter family (parameterized by b) of Lie pseudo group of coordinate transformations of two variables, which contains $\widehat{SL_2}$ when specialized to $b = o$. Because if $b = 0$, for q satisfying the Riccati equation $q' = -q^2 - 1/2$, $(q, p) = (q, 0)$ is a solution of $S_{II}(0)$. In the language of Lie algebra, the Lie algebra of Inf-gal(L/K), which is a Lie subalgebra of

$$L^{\natural}[[w_1, w_2]]\frac{\partial}{\partial w_1} + L^{\natural}[[w_1, w_2]]\frac{\partial}{\partial w_2}$$

contains

$$\left(\frac{\partial}{\partial w_1}, w_1\frac{\partial}{\partial w_1}, w_1^2\frac{\partial}{\partial w_1}\right).$$

On the other hand, we have a Bäcklund transformations

$$\begin{aligned} T_+(b, q) &= \left(b+1, -q - \frac{2b+1}{2q' + 2q^2 + x}\right), \\ T_-(b, q) &= \left(b-1, -q + \frac{2b-1}{2q' - 2q^2 - x}\right) \end{aligned}$$

(See [4], [5]). This shows that **Inf-gal**(L/K) contains many twisted $\widehat{SL_2}$ whenever b takes integral values.

Can we deduce from this fact Conjecture 1 ?

References

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