数理解析研究所講究録
1150 巻 2000 年 93-95

Construction of contact diffeomorphisms from Schwarzian derivatives

Hajime SATO (Nagoya University)

I talk on my joint work with Tetsuya OZAWA([O-S]).

1 Contact Schwarzian derivative

On the affine 3-space $\mathbb{K}^3$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) with the usual coordinate (x,y,z), we give the contact form $\alpha=dy-zdx$. Put

$$v_1=\frac{\partial}{\partial x}+z\frac{\partial}{\partial y},\quad v_2=\frac{\partial}{\partial z},\quad v_3=\frac{\partial}{\partial y},\quad v_4=v_2v_1+v_1v_2.$$

A local diffeomorphism ϕ is a *contact diffeomorphism*, if it satisfies $\phi^*(\alpha)=\rho\alpha$ for some nonvanishing function ρ. For a contact diffeomorphism $\phi:(x,y,z)\mapsto(X,Y,Z)$, we define the contact Schwarzian derivatives as follows: for $i,j,k=1,2$, set

$$s_{[ij,k]}(\phi)=v_iv_j(X)v_k(Z)-v_iv_j(Z)v_k(X),$$

and

$$S_{\{ijk\}}(\phi)=\frac{1}{3\Delta(\phi)}\left(s_{[ij,k]}(\phi)+s_{[jk,i]}(\phi)+s_{[ki,j]}(\phi),\right)$$

where $\Delta(\phi)=v_1(X)v_2(Z)-v_1(Z)v_2(X)$. We call the functions $S_{\{ijk\}}(\phi)$ the *contact Schwarzian derivatives* of the contact diffeomorphism ϕ. We denote the quadruple of functions by

$$S(\phi)=(S_{\{111\}}(\phi),S_{\{112\}}(\phi),S_{\{122\}}(\phi),S_{\{222\}}(\phi)).$$

Proposition 1.1. *The inverse ϕ^{-1} of a contact diffeomorphism $\phi:\mathbb{K}^3\to\mathbb{K}^3$ maps the differential equation $Y'''=0$ to*

$$y'''=S_{\{112\}}(\phi,\mathrm{x})+3S_{\{111\}}(\phi,\mathrm{x})y''+3S_{\{222\}}(\phi,\mathrm{x})(y'')^2+S_{\{122\}}(\phi,\mathrm{x})(y'')^3$$

By [S-Y], the condition that $y''' = f(x, y, y', y'')$ is mapped to $y''' = 0$ by a contact diffeomorphism is the vanishing of two curvatures A and $\mathbf{b}$. We obtain that $\mathbf{b} = 0$ is equivalent to $\partial^4 f/\partial x^4 = 0$. Let us consider

$$y''' = P + 3Qy' + 3R(y'')^2 + S(y'')^3,$$

where $P = P(x, y, y')$, $Q = Q(x, y, y')$, $R = R(x, y, y')$, $S = S(x, y, y')$. Then $\mathbf{b} = 0$ and the condition $A = 0$ is equal to

$$\begin{aligned} v_3(P) &= 2(v_1 - 2Q)(M_{11}) + 4PM_4 \\ 3v_3(Q) &= 2(v_2 - 4R)(M_{11}) + 4(v_1 + Q)(M_4) + 4PM_{22} \\ 3v_3(R) &= 2(v_1 + 4Q)(M_{22}) + 4(v_2 - R)(M_4) - 4SM_{11} \\ v_3(S) &= 2(v_2 + 2R)(M_{22}) - 4SM_4. \end{aligned} \tag{IC}$$

where we put

$$\begin{aligned} M_{11} &= -\frac{1}{4}(v_1(Q) - v_2(P) - 2Q^2 + 2PR) \\ M_4 &= -\frac{1}{4}(v_1(R) - v_2(Q) - QR + PS) \\ M_{22} &= -\frac{1}{4}(v_1(S) - v_2(R) - 2R^2 + 2QS). \end{aligned}$$

Theorem 1.1. *Four function P, Q, R, S on $\mathbb{K}^3$ is the Schwarzian derivarives of a contact diffeomorphism $\phi : \mathbb{K}^3 \to \mathbb{K}^3$;*

$$(P, Q, R, S) = S(\phi),$$

if and only if the system of the nonlinear differential equations (IC) *is satisfied.*

We seek a system of linear differential equations whose integarability equation is equal to (IC) and its solutions give the contact diffeomorphism. We call the linear system the linearization of (IC)

2 Fundamental system

Here is the linear differential syatem:

$$\begin{cases} {v_1}^2(\vartheta) = Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta \\ v_4(\vartheta) = 2(Rv_1(\vartheta) - Qv_2(\vartheta) + M_4\vartheta) \\ {v_2}^2(\vartheta) = Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22}\vartheta \end{cases} \tag{Sp}$$

Theorem 2.1. *The necessary and sufficient condition for the linear PDE system* (Sp) *to have 4-dimensional solution space is equal to the nonlinear PDE system* (IC).

Proposition 2.1. *For any two solutions α and β of the PDE system* (Sp), *the function $I(\alpha, \beta)$ defined by*

$$I(\alpha,\beta) = \frac{1}{2}\alpha v_3(\beta) - \frac{1}{2}v_3(\alpha)\beta + v_1(\alpha)v_2(\beta) - v_2(\alpha)v_1(\beta) \tag{1}$$

is constant on (x, y, z). Moreover this skew product $I(\alpha, \beta)$ is non-degenerate, and thus it defines a symplectic structure on the solution space $\mathcal{S}(P, Q, R, S)$ of (Sp), *provided the dimension of $\mathcal{S}(P, Q, R, S)$ is equal to* 4.

Theorem 2.2. *If a map $\phi : (x, y, z) \mapsto (X, Y, Z)$ is contact, then there exists a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(S(\phi))$ of the PDE system* (Sp) *such that ϕ is given by*

$$(x,y,z) \mapsto (\frac{\xi}{\vartheta}, \frac{1}{2}\left(\frac{\eta}{\vartheta} + \frac{\xi\zeta}{\vartheta^2}\right), \frac{\zeta}{\vartheta}). \tag{2}$$

Conversely, given a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(P, Q, R, S)$ of (Sp), *the map ϕ defined by* (2) *is a contact diffeomorphism whose contact Schwarzian derivatives are equal to*

$$S(\phi) = (P, Q, R, S).$$

References.

[Gun] R. Gunning, *On uniformization of complex manifolds: the role of connections*, Math. Notes No.22, Princeton. Princeton University Press. 1978.

[O-S] T. Ozawa and H.Sato, Contact diffeomorphisms and their Schwarzian derivatives, preprint, 1999.

[Sat] H. Sato, Schwarzian derivatives of contact diffeomorphisms, Lobachevskii J. of Math., 4, pp. 89-98(1999).

[S-Y] H. Sato and A. Y. Yoshikawa, Third order ordinary differential equations and Legendre connections, J. Math. Soc. Japan, 50, pp. 993-1013(1998).

[Yos] M. Yoshida, *Fuchsian differential equations*, Aspects of Mathematics, Vieweg, Baunschweig, 1987.