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On manifolds whose geodesic flows are integrable

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The geodesic flow of an n-dimensional riemannian manifold is said to be (completely) integrable if it admits n first integrals (including the hamiltonian) that mutually commute with respect to the Poisson bracket and that are functionally independent almost everywhere. In this talk I will give an exposition of three kinds of riemannian manifolds whose geodesic flows are integrable, which I studied recently. Two of those are called Liouville manifolds and Kähler-Liouville manifolds respectively. The third one is a certain kind of two-dimensional riemannian manifolds, which are diffeomorphic to the sphere, and whose geodsic flows admit first integrals that are fiberwise homogeneous polynomials of degree greater than two.

1. Liouville manifolds Liouville manifolds are, roughly speaking, riemannian manifolds whose geodesic equations have similar forms as those of ellipsoids. Since such systems were studied by Liouville, and since in two-dimensional case they have been called "Liouville surfaces", we call them Liouville manifolds. The precise definition is as follows: Let M be an n-dimensional complete riemannian manifold, and let E be the associated energy function (the hamiltonian of the geodesic flow). Let $\mathcal{F}$ be an n-dimensional vector space of functions on the cotangent bundle T^*M that are fiberwise homogeneous polynomials of degree two. We say that the pair $(M, \mathcal{F})$ is a Liouville manifold if the following four conditions are satisfied:

(L.1) $E \in \mathcal{F}$;

(L.2) $\{F, H\} = 0$ for any $F, H \in \mathcal{F}$;

(L.3) $F_p \quad (F \in \mathcal{F})$ are simultaneously normalizable for any $p \in M$;

(L.4) $\dim\{F_p \mid F \in \mathcal{F}\} = n$ at some point $p \in M$;

where $F_p = F|_{T_p^*M}$. We assume some non-degeneracy condition, called properness, and obtain the notaion of "rank". It may be said that proper Liouville manifolds of rank one are fairly well understood.

2. Kähler-Liouville manifolds The notion of Kähler-Liouville manifold is a hermitian version (or a complexification) of that of Liouville manifold. The definition is as follows: Let M be a complete Kähler manifold of complex dimension n, I its complex structure, and E its energy function (the hamiltonian of the geodesic flow). Let $\mathcal{F}$ be an n-dimensional vector space of functions on the cotangent bundle T^*M that are fiberwise homogeneous polynomials of degree two. Then we say that $(M, \mathcal{F})$ is a Kähler-Liouville manifold if it satisfies the following conditions:

(KL.1) $E \in \mathcal{F}$;

(KL.2) $\{F, H\} = 0$ for any $F, H \in \mathcal{F}$;

(KL.3) $F_p = F|_{T_p^*M}$ is a hermitian form for every $p \in M$ and $F \in \mathcal{F}$;

(KL.4) F_p $(F \in \mathcal{F})$ are simultaneously normalizable for every $p \in M$;

(KL.5) $\{F_p \mid F \in \mathcal{F}\}$ is n-dimensional at some $p \in M$.

Note that only n first integrals are given in the definition. However, it will turn out that other n first integrals appear automatically if some non-degeneracy condition is assumed. They appear as infinitesimal automorphisms of $(M, \mathcal{F})$. If M is copmact, then those yield a holomorphic $(C^\times)^n$-action on M, and M becomes a toric variety with this action. A typical example is a complex projective space with the standard Kähler metric.

3. Two-dimensional manifolds Let M be a two-dimensional riemannian manifold diffeomorphic to the sphere, and suppose that its geodesic flow has a nontrivial first integral F, which is a homogeneous polynomial of degree k on each cotangent space. Such (M, F) is well-understood when $k = 1$ or 2: If $k = 1$, M is a surface of revolution and F is a Killing vector field generating the revolution (identified with a function on T^*M); and if $k = 2$, then (M, F) is a Liouville surface. Also, two one-parameter families are known for $k = 3$ and 4 by Bolsinov and Fomenko. Here we introduce a family of such (M, F) for every $k \geq 3$, which we found recently. They form a family parametrized by functions in one variable for each k, and each of them is a $C_{2\pi}$ manifold, namely, all geodesics are closed and have the same length 2π.

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