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A note on a paper of Sasaki

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1 Introduction

In his paper [12], Sasaki studied the holomorphic slice $\mathcal{S}$ of the space of punctured torus groups determined by the trace equation $xy=2z$. He found a simply connected domain E contained in $\mathcal{S}$ by using his system of inequalities which characterizes some quasifuchsian punctured torus groups (c.f. [11]). Moreover decomposing the boundary of E into 3 pieces $\partial E=e_1\cup e_2\cup e_3$ he showed that $e_1\cup e_2$ is contained in $\mathcal{S}$ and e_3 (consisting of two points) is in the boundary $\partial\mathcal{S}$. In this paper we consider the slice $\mathcal{S}$ itself more precisely.

Thanks to the recent work by Akiyoshi-Sakuma-Wada-Yamashita (c.f. [1]) to reorganize the work of Jørgensen (c.f. [3]) on the combinatorial pattern of the isometric circles of punctured torus groups, Yamashita made a program which can draw the picture of several slices of the space of punctured torus groups. The picture in this paper is also due to Yamashita. In this picture $\mathcal{S}$ is the complement of the black-coloured regions in $\{\alpha\in\mathbf{C}: Re\ \alpha>1\}$, and E is the white-coloured polygonal subdomain of $\mathcal{S}$. (We remark that the disk-like domain in $\{\alpha\in\mathbf{C}:0<Re\ \alpha<1\}$ is the image of $\mathcal{S}$ under the involution $\alpha\mapsto\frac{1}{\alpha}$.) From this picture it is easy to imagine that $\mathcal{S}$ itself is a simply connected domain.

In this paper we show that $\mathcal{S}$ has a structure of the Teichmüller space of once-punctured tori. More precisely it is so called the (rectangular) Earle slice of puncture torus groups. (For the rhombic Earle slice, see [6].) As a corollary of this result, we can show that $\mathcal{S}$ is connected and simply connected. Moreover $\mathcal{S}$ is a Jordan domain, which is an application of the work of Minsky on the classification of punctured torus groups (c.f. [10] and [7]). The author wishes to thank Yasushi Yamashita for his kind assistance with computer graphics.

2 Punctured torus groups

Let S be an oriented once-punctured torus and $\pi_1(S)$ be its fundamental group. An ordered pair α, β of generators of $\pi_1(S)$ is called *canonical* if the oriented intersection number $i(\alpha, \beta)$ in S with respect to the given orientation of S is equal to $+1$. The commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ represents a loop around the puncture.

Define $\mathcal{R}(\pi_1(S))$ to be the set of $PSL_2(\mathbf{C})$-conjugacy classes of representations from $\pi_1(S)$ to $PSL_2(\mathbf{C})$ which take the commutator of generators to a parabolic element. Let $\mathcal{D}(\pi_1(S))$ denote the subset of $\mathcal{R}(\pi_1(S))$ consisting of conjugacy classes of discrete and faithful representations. Any representative of an element of $\mathcal{D}(\pi_1(S))$ is called a *marked punctured torus group.* Let $\mathcal{QF}$ denote the subset of $\mathcal{D}(\pi_1(S))$ consisting of conjugacy classes of representations ρ such that for the action of $\Gamma = \rho(\pi_1(S))$ on the Riemann sphere $\hat{\mathbf{C}}$ the region of discontinuity Ω has exactly two simply connected invariant components $\Omega^{\pm}$. The quotients $\Omega^{\pm}/\Gamma$ are both homeomorphic to S and inherit an orientation induced from the orientation of $\hat{\mathbf{C}}$. We choose the labelling so that Ω^{+} is the component such that the homotopy basis of Ω^{+}/Γ induced by the ordered pair of marked generators $\rho(\alpha), \rho(\beta)$ of Γ is canonical. Any representative of an element of $\mathcal{QF}$ is called a *marked quasifuchsian punctured torus group.* Considering the algebraic topology $\mathcal{D}(\pi_1(S))$ is closed in $\mathcal{R}(\pi_1(S))$ and $\mathcal{QF}$ is open in $\mathcal{D}(\pi_1(S))$ (see [9]). A quasifuchsian group Γ is called *Fuchsian* if the components $\Omega^{\pm}$ are round discs.

Recall that the set of measured geodesic laminations on a hyperbolic surface is independent of the hyperbolic structure. Denote by $PML(S)$ the set of projective measured laminations on S. Let $\mathcal{C}(S)$ denote the set of free homotopy classes of unoriented simple non-peripheral curves on S. There are in one-to-one correspondence with $\hat{\mathbf{Q}} \equiv \mathbf{Q} \cup \{\infty\}$, after choosing an canonical basis (α, β) for $\pi_1(S)$ as follows; Any element of $H_1(S)$ can be written as $(p, q) = p[\alpha] + q[\beta]$ in the basis $([\alpha], [\beta])$ for $H_1(S)$, and we associate to this the slope $-p/q \in \hat{\mathbf{Q}}$ which describes an element of $\mathcal{C}(S)$. Cosidering projective classes of weighted counting measures, we can identify $\mathcal{C}(S)$ with the set of projective rational raminations. Recall that $PML(S)$ may be identified with $\hat{\mathbf{R}}$, in such a way that rational laminations correspond to $\hat{\mathbf{Q}}$.

We can also embed $\mathcal{D}(\pi_1(S))$ into $\mathbf{C}^3$ by using trace functions on $\mathcal{D}(\pi_1(S))$. Setting $x = \operatorname{Tr} A$, $y = \operatorname{Tr} B$ and $z = \operatorname{Tr} AB$, where A, B are the generator pair of the marked group $\Gamma = \langle A, B \rangle$ in $\mathcal{D}(\pi_1(S))$, gives an embedding of $\mathcal{D}(\pi_1(S))$ into $\{(x, y, z) \in \mathbf{C}^3 : x^2 + y^2 + z^2 = xyz\}$.

3 The slice $\mathcal{S}$ defined by the trace equation $xy = 2z$

Let us consider the following slice $\mathcal{S}$ and the set E

$$\begin{aligned} \mathcal{S} &:= \{(x,y,z) \in \mathbf{C}^3 : xy = 2y\} \cap \mathcal{QF} \\ E &:= \{(x,y,z) \in \mathbf{C}^3 : xy = 2y, x^2 + y^2 + z^2 = xyz, |x| > 2, |y| > 2\}. \end{aligned}$$

Moreover decompose the boundary ∂E of E into $\partial E = e_1 \cup e_2 \cup e_3$ where

$$\begin{aligned} e_1 &:= \{(x,y,z) \in \mathbf{C}^3 : xy = 2y, x^2 + y^2 + z^2 = xyz, |x| = 2, |y| > 2\} \\ e_1 &:= \{(x,y,z) \in \mathbf{C}^3 : xy = 2y, x^2 + y^2 + z^2 = xyz, |x| > 2, |y| = 2\} \\ e_1 &:= \{(x,y,z) \in \mathbf{C}^3 : xy = 2y, x^2 + y^2 + z^2 = xyz, |x| = 2, |y| = 2\}. \end{aligned}$$

In [12] Sasaki proved the next result.

Theorem 3.1 *1.* (theorem 4 in [12]) $E \subset \mathcal{S}$.

2. (theorem 5 in [12]) $e_1 \cup e_2 \subset \mathcal{S}$.

3. (theorem 6 in [12]) $e_3 \in \partial \mathcal{S}$.

By normalizing the generators A, B of $\Gamma = \langle A, B \rangle$ in $\mathcal{S}$, $\mathcal{S}$ can be embedded into the complex plane $\mathbf{C}$ as follows (c.f. [12]); Conjugating by a suitable element of $PSL_2(\mathbf{C})$, we can normalize A, B such that

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, B = \begin{pmatrix} \frac{\alpha^2+1}{\alpha^2-1} & \frac{4\alpha^2}{\alpha^4-1} \\ \frac{\alpha^2+1}{\alpha^2-1} & \frac{\alpha^2+1}{\alpha^2-1} \end{pmatrix}$$

where $\alpha = re^{i\theta}$ satisfying $r > 1$ and $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$. We can take $\alpha \in \mathbf{C}$ as a global holomorphic coordinate of S. The picture in this paper represents $\mathcal{S}$ in this coordinate α.

Generators A, B of $\Gamma = \langle A, B \rangle$ in $\mathcal{S}$ have a following property.

Proposition 3.2 (see theorem 7 in [12])
For $\Gamma = \langle A, B \rangle \in \mathcal{QF}$, Γ is an element of the slice $\mathcal{S}$ if and only if there is an elliptic transformation of order two $I \in PSL_2(\mathbf{C})$ such that $IAI = A, IBI = B^{-1}$.

This proposition is enough for us to show that $\mathcal{S}$ has a nice topological property from the following theorem due to Earle (c.f. [2]). Recall that an isomorphism of Kleinian groups is called *type preserving* if it maps loxodromic elements in $PSL_2(\mathbf{C})$ to loxodromics and parabolics to parabolics.

Theorem 3.3 *Let θ be an involution of $\pi_1(\mathcal{T}_1)$ induced by an orientation reversing diffeomorphism of a Riemann surface $\mathcal{T}_1$ of type $(1,1)$. Let (α, β) be a homotopy basis of $\pi_1(\mathcal{T}_1)$ canonical with respect to the orientation induced by the conformal structure on $\mathcal{T}_1$. Then, up to conjugation in $PSL_2(\mathbf{C})$, there exists a unique marked quasifuchsian group $\rho : \pi_1(\mathcal{T}_1) \to \Gamma = \langle A, B\rangle$, such that:*

1. *There is a conformal map $\mathcal{T}_1 \to \Omega^+/\Gamma$ inducing the representation ρ.*
2. *There is a Möbius transformation $\Theta \in PSL_2(\mathbf{C})$ of order two which induces a conformal homeomorphism $\Omega^+ \to \Omega^-$ such that $\Theta(\gamma z) = \theta(\gamma)\Theta(z)$ for all $\gamma \in \Gamma$ and $z \in \Omega^+$.*

Theorem 3.3 shows that the Earle slice is a holomophic embedding of the Teichmüller space Teich$(\mathcal{T}_1)$ of $\mathcal{T}_1$ into $\mathcal{QF}$. The embedding depends only on the choice of the involution θ of $\pi_1(\mathcal{T}_1)$. We call the image, an *Earle slice* of $\mathcal{QF}$, and denote it $\mathcal{E}_\theta$.

Let $\theta : \pi_1(\mathcal{T}_1) \to \pi_1(\mathcal{T}_1)$ be the involution defined by $\theta(\alpha) = \alpha$ and $\theta(\beta) = \beta^{-1}$. Clearly, θ satisfies the condition of theorem 3.3.

Corollary 3.4 *$\mathcal{S} = \mathcal{E}_\theta$. In particular $\mathcal{S}$ is connected and simply connected.*

4 Properties of $\mathcal{S}$ as the Earle slice

For $A, B \in PSL_2(\mathbf{C})$, put $w = \operatorname{Tr} AB^{-1}$. Then the trace equation $xy = 2z$ is equivalent to $z = w$. Therefore

Proposition 4.1

$$\mathcal{S} = \{(x, y, z) \in \mathbf{C}^3 : z = w\} \cap \mathcal{QF}.$$

We remark that the rhombic Earle slice can be written by $\{(x, y, z) \in \mathbf{C}^3 : x = y\} \cap \mathcal{QF}$ (c.f. remark 3.2 in [6]).

We call a torus a *rectangle* if it admits two anticonformal involutions. In [4] Keen characterized rectangular quasifuchsian puncture torus groups (c.f. theorem 4.2 and 4.3 in [4]). From the normalization of the generators A, B of $\Gamma = \langle A, B\rangle$ in $\mathcal{S}$,

Proposition 4.2 *The Fuchsian locus in $\mathcal{S}$ is equal to $\{\alpha \in \mathbf{R} : \alpha > 1\}$. This Fuchsian locus in $\mathcal{S}$ coincides with the set of rectangular Fuchsian groups in $\mathcal{QF}$.*

From this proposition it seems reasonable to call $\mathcal{S}$ the *rectangular* Earle slice.

We can find anticonformal and conformal symmetries of $\mathcal{S}$ (see proposition 3.4 and 3.6 in [6]).

Proposition 4.3 *1. $\mathcal{S}$ is invariant under complex conjugation.*

2. $\mathcal{S}$ is invariant under the map $\alpha \mapsto \frac{\alpha+1}{\alpha-1}$.

We can see these symmetries from the picture of $\mathcal{S}$ in this paper.

Next we consider the pleating locus of $\mathcal{S}$ (c.f. [5]). Let $\alpha \in \mathcal{S}$ and let $\Gamma_\alpha = \langle A_\alpha, B_\alpha \rangle$ be the corresponding marked quasifuchsian group with regular set and limit set $\Omega_\alpha, \Lambda_\alpha$ respectively. Let $\partial\mathcal{C}_\alpha$ be the boundary in $\mathbf{H}^3$ of the hyperbolic convex hull of Λ_α; it is clearly invariant under the action of Γ_α. The nearest point retraction $\Omega_\alpha \to \partial\mathcal{C}_\alpha$ by mapping $x \in \Omega_\alpha$ to the unique point of contact with $\partial\mathcal{C}_\alpha$ of the largest horoball in $\mathbf{H}^3$ centered at x with interior disjoint from $\partial\mathcal{C}_\alpha$, can easily be modified to a Γ_α-equivariant homeomorphism. We denote two connected components of $\partial\mathcal{C}_\alpha$ corresponding to $\Omega_\alpha^\pm$ by $\partial\mathcal{C}_\alpha^\pm$ respectively. Thus each component $\partial\mathcal{C}_\alpha^\pm/\Gamma_\alpha$ is topologically a punctured torus. $\partial\mathcal{C}_\alpha^\pm/\Gamma_\alpha$ are pleated surfaces in $\mathbf{H}^3/\Gamma_\alpha$. More precisely, there are complete hyperbolic surfaces $S_\alpha^\pm$, each homeomorphic to S, and maps $f^\pm : S_\alpha^\pm \to \mathbf{H}^3/\Gamma_\alpha$, such that every point in $S_\alpha^\pm$ is in the interior of some geodesic arc which is mapped by $f^\pm$ to a geodesic arc in $\mathbf{H}^3/\Gamma_\alpha$, and such that $f^\pm$ induce isomorphisms $\pi_1(S) \to \Gamma_\alpha$. Further, $f^\pm$ are isometries onto their images with the path metric induced from $\mathbf{H}^3$. The *bending* or *pleating locus* of $\partial\mathcal{C}_\alpha^\pm/\Gamma_\alpha$ consists of those points of $S_\alpha^\pm$ contained in the interior of one and only one geodesic arc which is mapped by $f^\pm$ to a geodesic arc in $\mathbf{H}^3/\Gamma_\alpha$. For Γ_α non-Fuchsian, the pleating loci are geodesic laminations, meaning they are unions of pairwise disjoint simple geodesics on $S_\alpha^\pm$. We denote these laminations by $|pl^\pm(\alpha)|$, and usually identify such a lamination with its image under $f^\pm$ in $\mathbf{H}^3/\Gamma_\alpha$. A geodesic lamination is called *rational* if it consists entirely of closed leaves. Since the maximum number of pairwise disjoint simple closed curves on a punctured torus is one, such a lamination consists of a single simple closed geodesic and is therefore of the form $\gamma(p/q)(\alpha)$ for some $p/q \in \hat{\mathbf{Q}}$.

For $p/q, r/s \in \hat{\mathbf{Q}}$, define

$$\mathcal{P}(p/q, r/s) = \{\alpha \in \mathcal{S} : |pl^+(\alpha)| = \gamma(p/q)(\alpha), |pl^-(\alpha)| = \gamma(r/s)(\alpha)\}$$

Then by the similar arguments of [6] (especially, see theorem 5.1 and 5.11), we can show the next result.

Theorem 4.4 *1.* $\mathcal{P}(p/q, r/s) \neq \emptyset$ *if and only if* $r/s = -p/q$ *and* $p/q \neq 0, \infty$. $\mathcal{P}(p/q, -p/q)$ *is an embedded arc from the Fuchsian locus in* $\mathcal{S}$ *to the* $(p/q, -p/q)$*-cusp in* $\partial\mathcal{S}$.

2. The set of rational pleating rays $\mathcal{P}(p/q, -p/q)$ $(p/q \in \mathbf{Q} - \{0\})$ *are dense in* $\mathcal{S}$.

Moreover by using the argument in [7],

Theorem 4.5 $\mathcal{S}$ *is a Jordan domain.*

As a corollary of this theorem, we can determine the end invariants of the boundary groups in $\partial\mathcal{S}$ (c.f. [10]) which are $(x, -x)$ where $x \in \mathbf{R} - \{0\}$. Especially no boundary groups in $\partial\mathcal{S}$ are b-groups, which was also shown by Sasaki (see theorem 8 in [12]).

References

[1] H. Akiyoshi, M. Sakuma, M. Wada and Y. Yamashita, preprint (1999).

[2] C. J. Earle, Some intrinsic coordinates on Teichmüller space, Proc. Amer. Math. Soc. 83 (1981), 527-531.

[3] T. Jørgensen, Once punctured tori, unpublished preprint.

[4] L. Keen, Teichmüller Space of Punctured Tori: I, Complex Variables, Vol.2 (1983), 199-211.

[5] L. Keen and C. Series, Pleating invariants for punctured torus groups, Revised version of Warwick University preprint, 10/1998.

[6] Y. Komori and C. Series, Pleating coordinates for the Earle embedding, Warwick University preprint, 46/1998.

[7] Y. Komori, On the boundary of the Earle slice for punctured torus groups, preprint (1999).

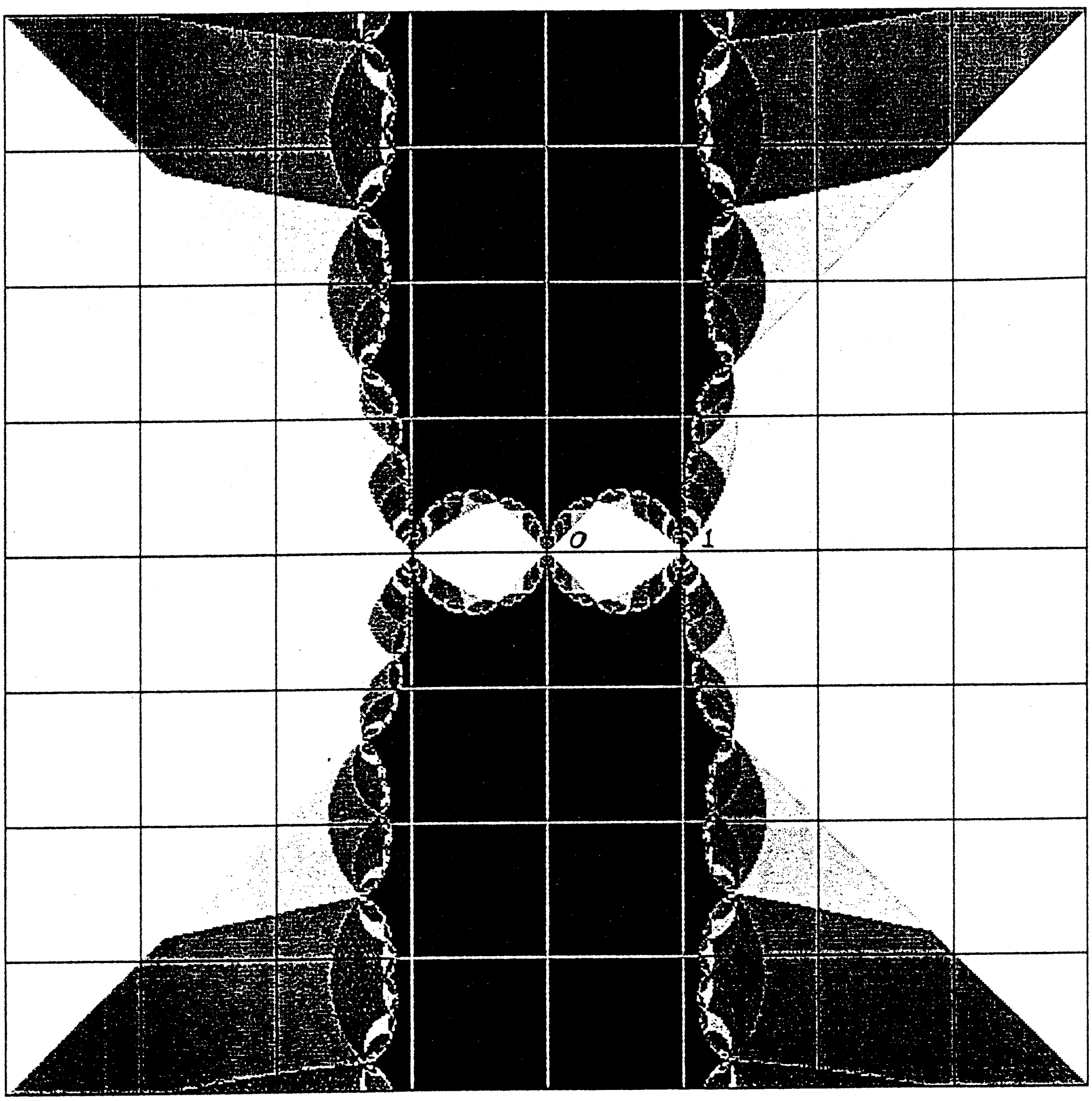
[8] I. Kra, On algebraic curves (of low genus) defined by Kleinian groups, Annales Polonici Math. XLVI (1985), 147-156.

[9] K.Matsuzaki and M.Taniguchi, Hyperbolic manifolds and Kleinian groups, Oxford Mathematical Monograph, 1998.

[10] Y. Minsky, The classification of punctured torus groups, Ann. of Math. 149 (1999), 559-626.

[11] T. Sasaki, A fundamental domain for some quasi-Fuchsian groups, Osaka J. Math. 27 (1990), 67-80.

[12] T. Sasaki, The slice determined by moduli equation $xy = 2z$ in the deformation space of once punctured tori, Osaka J. Math. 33 (1996), 475-484.



The holomorphic slice S. Courtesy of Yasushi Yamashita