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# Rigidity of infinitely renormalizable polynomials of higher degree

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**Abstract**

The conjecture that hyperbolic rational maps are dense in the space of all rational maps of degree $d$ is one of the central problems in complex dynamics. It is known that no invariant line field conjecture implies the density of hyperbolicity (see [**MS**]).

In the case of quadratic polynomials, McMullen shows that a robust infinitely renormalizable quadratic polynomial carries no invariant line field on its Julia set [**Mc**].

In this paper, we give the extension of renormalization and the above theorem of McMullen to polynomial of any degree.

## 1 Notation and backgrounds

**Notation.** Let $f$ be a polynomial of degree $d$.

- The *Fatou set* $F(f)$ of $f$ is the maximal open set of $\mathbb{C}$ where $\{f^n\}$ is normal.
- The *Julia set* $J(f)$ of $f$ is the complement of $F(f)$.
- The *filled Julia set* $K(f)$ of $f$ is the set of all point in $\mathbb{C}$ whose forward orbit by $f$ does not tend to infinity. Note that $\partial K(f) = J(f)$.
- Let $C(f)$ be the set of critical points of $f$.
- The *postcritical set* $P(f)$ is the closure of the strict forward image of critical points by $f$:

$$P(f) = \overline{\bigcup_{n>1} f^n(C(f))}$$

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**Definition.** A *polynomial-like map* $f : U \to V$ is a proper holomorphic map with $\overline{U} \subset V$.

The *filled Julia set* $K(f)$ of a polynomial-like map $f : U \to V$ is the set of all point $z \in U$ such that $f^n(z) \in U$ for all $n \geq 0$. The *Julia set* $J(f)$ is the boundary of $K(f)$.

Two polynomial-like map $f$ and $g$ are *hybrid equivalent* if there is a quasiconformal map $\phi$ from a neighborhood of $K(f)$ to a neighborhood of $K(g)$, such that $\phi \circ f = g \circ \phi$ and $\overline{\partial}\phi = 0$ on $K(f)$.

**Theorem 1.1.** *Every polynomial-like map $f$ is hybrid equivalent to some polynomial $g$ of the same degree. Furthermore, if $K(f)$ is connected, $g$ is unique up to affine conjugacy.*

See [**DH**, Theorem 1].

**Lemma 1.1.** *Let $f_i : U_i \to V_i$ be polynomial-like maps of degree $d_i$ for $i = 1, 2$. Suppose $f_1 = f_2 = f$ on $U = U_1 \cap U_2$ and let $U'$ be a component of $U$ with $U' \subset f(U') = V'$. Then $f : U' \to V'$ is polynomial-like map of degree $d \leq \min(d_1, d_2)$, and*

$$K(f) = K(f_1) \cap K(f_2) \cap U'.$$

*Moreover, if $d = d_i$, then $K(f) = K(f_i)$.*

See [**Mc**, Theorem 5.11].

**Lemma 1.2.** *Let $f$ be a polynomial with connected Julia set. Let $f^n : U \to V$ be a polynomial-like restriction of degree more than 1 with connected filled Julia set $K$. Then:*

1. *The Julia set of $f^n : U \to V$ is contained in $J(f)$.*

2. *For any closed connected set $L$ contained in $K(f)$, $L \cap K$ is also connected.*

See [**Mc**, Theorem 6.13].

**Definition.** A *line field* supported on $E \subset \mathbb{C}$ is the choice of a real line through the origin of $T_z\mathbb{C}$ at each $z \in E$. It is equivalent to take a Beltrami differential $\mu = \mu(z)d\overline{z}/dz$ supported on $E$ with $|\mu| = 1$.

We say $f$ carries an invariant line field on its Julia set if there exists a measurable Beltrami differential $\mu$ on $\mathbb{C}$ such that $f^*\mu = \mu$ and $|\mu| = 1$ on a set of positive measure contained in $J(f)$ and vanishes elsewhere.

**Conjecture 1.1 (No invariant line fields).** *A polynomial carries no invariant line field on its Julia set.*

If this conjecture is true, the following one is also true. Here the polynomial $f$ is *hyperbolic* if all critical points tend to attracting periodic cycles under iteration.

**Conjecture 1.2 (Density of hyperbolicity).** *Hyperbolic maps are dense in the family of polynomial of degree $d$.*

See [**MS**].

# 2 Renormalization

In this section, we give the definition of renormalization and describe some basic properties.

**Definition.** $f^n$ is called *renormalizable* if there exist open disks $U, V \subset \mathbb{C}$ satisfying the followings:

1. $U \cap C(f) \neq \phi$.
2. $f^n : U \to V$ is a polynomial-like map with connected filled Julia set.
3. For each $c \in C(f)$, there is at most one $i$, $0 < i \leq n$, such that $c \in f^i(U)$.
4. $n > 1$ or $U \not\supset C(f)$.

A *renormalization* is a polynomial-like restriction $f^n : U \to V$ as above.

**Notation.** Let $f^n : U \to V$ be a renormalization.

- The filled Julia set of a renormalization $f^n : U \to V$ is denoted by $K_n(U)$ and the postcritical set by $P_n(U)$.
- For $i = 1, \ldots, n$, the *ith small filled Julia set* is denoted by $K_n(U, i) = f^i(K_n(U))$.
- The *ith small postcritical set* is denoted by $P_n(U, i) = K_n(U, i) \cap P(f)$.
- $C_n(U, i) = K_n(U, i) \cap C(f)$. By definition, $C_n(U, n)$ is nonempty and $C_n(U, i)$ is empty with at most $d - 1$ exceptions.
- $\mathcal{K}_n(U) = \bigcup_{i=1}^n K_n(U, i)$ is the union of the small filled Julia sets.
- $\mathcal{C}_n(U) = \bigcup_{i=1}^n C_n(U, i)$ is the set of critical points appear in the renormalization $f^n : U \to V$.
- Let $V_n(U, i) = f^i(U)$ and $U_n(U, i)$ be the component of $f^{i-n}(U)$ contained in $V_n(U, i)$. Then $f^n : U_n(U, i) \to V_n(U, i)$ is polynomial-like map of the same degree as $f^n : U \to V$.

Now, when it is clear which $U$ we consider, we will simply write $K_n(i)$ instead of $K_n(U,i)$, and so on.

In this paper, we fix a critical point $c_0 \in C(f)$ and consider only renormalizations about $c_0$, i.e. $C_n(U) = C_n(U,n)$ contains $c_0$.

The next proposition implies that two renormalizations are essentially the same if their period and critical points are equal.

**Proposition 2.1.** *Let $f^n : U^k \to V^k$ be renormalizations for $k = 1,2$. If for any $i$, $0 \le i < n$, $C_n(U^1,i) = C_n(U^2,i)$, their filled Julia sets are equal.*

*Proof.* Let $K^k$ be the filled Julia set of $f^n : U^k \to V^k$. By Lemma 1.2, $K = K^1 \cap K^2$ is connected.

Let $U$ be the component of $U^1 \cap U^2$ containing $K$. Let $V = f^n(U)$. Since $V$ contains $f(K) = K$, $V$ contains $U$. By Lemma 1.1, $f^n : U \to V$ is polynomial-like with filled Julia set $K$. Since critical points of these three maps are equal, we have $K = K^1 = K^2$. □

**Proposition 2.2.** *Let $f^a : U_a \to V_a$ and $f^b : U_b \to V_b$ be renormalizations about $c_0$. Then there exists a renormalization $f^c : U \to V$ with filled Julia set $K_c = K_a \cap K_b$ where $c$ is the least common multiple of $a$ and $b$.*

*Proof.* By Lemma 1.2, $K = K_a \cap K_b$ is connected.

Let

$$\tilde{U}_a = \{z \in U_a \mid f^{ja}(z) \in U_a \text{ for } j = 1, \ldots, \frac{c}{a} - 1\}$$
$$\tilde{U}_b = \{z \in U_b \mid f^{jb}(z) \in U_b \text{ for } j = 1, \ldots, \frac{c}{b} - 1\}.$$

Then $f^c : \tilde{U}_a \to V_a$ and $f^c : \tilde{U}_b \to V_b$ are polynomial-like. Let $U_c$ be a component of $\tilde{U}_a \cap \tilde{U}_b$ which contains $K$. Then by Lemma 1.1, $f^c : U_c \to f^c(U_c)$ is polynomial-like map with filled Julia set $K$.

Suppose $c \in C_c(i)$. then $c \in C_c(j)$ is equivalent to $j \equiv i \pmod a$ and $j \equiv i \pmod b$, which means $j = i$. Therefore, $f^c : U_c \to V_c$ is a renormalization with filled Julia set $K_c = K$. □

Define the *intersecting set* of a renormalization $f^n : U \to V$ by

$$I_n(U) = K_n(U) \cap \left( \bigcup_{i=1}^{n-1} K_n(U,i) \right).$$

We say a renormalization is *intersecting* if $I_n(U) \neq \emptyset$.

**Proposition 2.3.** *If a renormalization $f^n : U \to V$ is intersecting, then $I_n(U)$ consists of only one point which is a repelling fixed point of $f^n$.*

*Proof.* Suppose $E = K_n(U) \cap K_n(U, i) \neq \emptyset$ for some $0 < i < n$. By Lemma 1.2, $E$ is connected.

Let $U$ be the component of $U \cap U(i)$ containing $E$. By Lemma 1.1, $f^n : U \to f^n(U)$ is a polynomial-like map of degree 1. By the Schwarz lemma, $E$ consists of a single repelling fixed point $x$ of $f^n$.

Suppose $K_n(U) \cap K_n(U, j) = \{y\}$ with $y \neq x$. Then there is a sequence $\{i_0, i_1, \ldots, i_K\}$ such that $K_n(U, i_k) \cap K_n(U, i_{k+1})$ is nonempty and $K_n(U, i_k) \cap K_n(U, i_{k+1})) \cap K_n(U, i_{k+2})$ is empty (where $K+1$, $K+2$ is interpreted as 0, 1, respectively).

Let

$$L = K_n(U, i_1) \cap \ldots K_n(U, i_K).$$

Then $L$ is a closed connected set in $K(f)$. But $L \cap K_n(U)$ consists of two points and it contradicts Lemma 1.2. □

Since a repelling fixed point separates filled Julia set into a finite number of components, components of $K_n(U) - I_n(U)$ are finite. We say a renormalization is *simple* if $K_n(U) - I_n(U)$ is connected, and *crossed* if it is disconnected.

**Theorem 2.1.** *For $p > 0$, there are finitely many $n > 0$ such that there exists a renormalization $f^n : U_n \to V_n$ such that $K_n(U)$ contains a periodic point of period $p$.*

*Proof.* Let $x$ be a periodic point of period $p$. Assume the filled Julia set of a renormalization $f^n : U \to V$ with $p < n$ contains $x$. Since $x$ is a repelling fixed point of $f^n$ (by Proposition 2.3), $p$ divides $n$ and the number $\rho$ of the components of $K_n(U_n) - \{x\}$ is finite.

Let $E$ be the component of $K_n(U)$ which contains $x$. $E - \{x\}$ has exactly $\rho n/p$ components. Let $q$ be the number of the components of $K(f) - \{x\}$. Since $x$ is a repelling periodic point of $f$, $q < \infty$.

Suppose a component $A$ of $K(f) - \{x\}$ contains two components $B_1, B_2$ of $E - \{x\}$. Then we can take a path in $\overline{A - (B_1 \cup B_2)}$ from $x$ to some point in $B_1$. It contradicts Lemma 1.2.

Therefore each component of $K(f) - \{x\}$ can contain at most one component of $E - \{x\}$. So $q \geq \rho n/p$, it concludes $n \leq pq$.

There are finitely many periodic points of period $p$, the theorem follows. □

**Proposition 2.4.** *Let $f^a : U_a \to V_a$ and $f^b : U_b \to V_b$ be renormalizations about $c_0$. Suppose that $f^b : U_b \to V_b$ is simple. Then either $a$ divides $b$ or $b$ divides $a$.*

*Proof.* Let $c$ be the greatest common devisor of $a$ and $b$. If $c = a$ or $c = b$, the proposition follows. So suppose $c < a, b$.

Since $K_a \cap K_b$ is nonempty (it contains $c_0$), $f^i(K_a) \cap f^i(K_b)$ is nonempty for any $i > 0$. Therefore $K_a(c) \cap K_b(c)$, $K_a(c) \cap K_b$ and $K_a \cap K_b(c)$ are all nonempty. Therefore $L = K_b \cup K_a(c) \cup K_b(c)$ is connected.

By Lemma 1.2, $K_a \cap L$ is connected. Since $K_a \cap K_a(c)$ is at most one point and $L$ is a closed connected set, $K_a \cap (K_b \cup K_b(c))$ is connected. So $K_a \cap K_b \cap K_b(c)$ is nonempty. By Proposition 2.3, $K_b \cap K_b(c) = \{x\}$ where $x$ is a repelling fixed point of $f^b$, so $K_a \ni x$. Since $f^b : U_b \to V_b$ is simple, $x$ does not disconnect $K_b$.

By Proposition 2.2, there exists a renormalization $f^{ab/c} : U \to V$ with Julia set $K_{ab/c} = K_a \cap K_b$. But $K_{ab/c}$ cannot contain $x$ because $K_b - \{x\}$ is connected and $ab/c > b$ (see the proof of Theorem 2.1), it is a contradiction. □

**Example.** Let $f(z) = z^3 - \frac{3}{4}z - \frac{\sqrt{7}}{4}$. Then $C(f) = \{\pm\frac{1}{2}\}$ and $\pm\frac{1}{2}$ are periodic of period 2. Let $W_\pm$ be the Fatou component which contains $\pm\frac{1}{2}$. They are superattracting basin of period 2.

Every renormalization $f^n : U \to V$ must satisfy $U \supset W_-$ or $W_+$. So $n \leq 2$ and by symmetry, we will consider only the case $U \supset W_-$.

**Type I.** Let $K$ be the connected component of the closure of $\bigcup_{n>0} f^{-n}(W_-)$ which contains $W_-$ and let $U_1$ be a small neighborhood of $K$.

Then $f : U_1 \to f(U_1)$ is a renormalization with filled Julia set $K(1, U) = K_1$ which is hybrid equivalent to $z \mapsto z^2 - 1$.

**Type II.** Let $U_2$ be a small neighborhood of $W_-$. Then $f^2 : U_2 \to f^2(U_2)$ is a renormalization with filled Julia set $K(2, U_2) = \overline{W_-}$, which is hybrid equivalent to $z \mapsto z^2$.

**Type III.** Let $K_2'$ be the connected component of $\overline{\bigcup_{n>0} f^{-2n}(W_- \cup W_+)}$ which contains $W_-$ and let $U_2'$ be a small neighborhood of $K_2'$.

Then $f^2 : U_2' \to f^2(U_2')$ is a renormalization with filled Julia set $K_2'$, which is hybrid equivalent to $z \mapsto z^3 - \frac{3}{\sqrt{2}}z$.

**Type IV.** Let $K_2''$ be the connected component of $\overline{\bigcup_{n>0} f^{-2n}(W_- \cup f(W_+))}$ which contains $W_-$ and let $U_2''$ be a small neighborhood of $K_2''$.

Then $f^2 : U_2'' \to f^2(U_2'')$ is a renormalization with filled Julia set $K_2''$ and of degree 4.

Similarly, consider $\overline{\bigcup_{n>0} f^{-2n}(W_- \cup f(W_-) \cup W_+)}$ and then we can construct a polynomial-like map $f^2 : U \to V$ of degree 6. But it is not a renormalization because $-\frac{1}{2}$ is contained in both $U$ and $f(U)$.

# 3 Infinite renormalization

For a subset $C_R \subset C(f)$, let $\mathcal{R}(f, C_R)$ be the set of all $n > 0$ such that there exists a renormalization $f^n : U_n \to V_n$ about $c_0$ with $\mathcal{C}_n(U_n) = C_R$. Let $\mathcal{SR}(f, C_R)$ be the set of such $n \in \mathcal{R}(n, C_R)$ that $f^n : U_n \to V_n$ is simple.

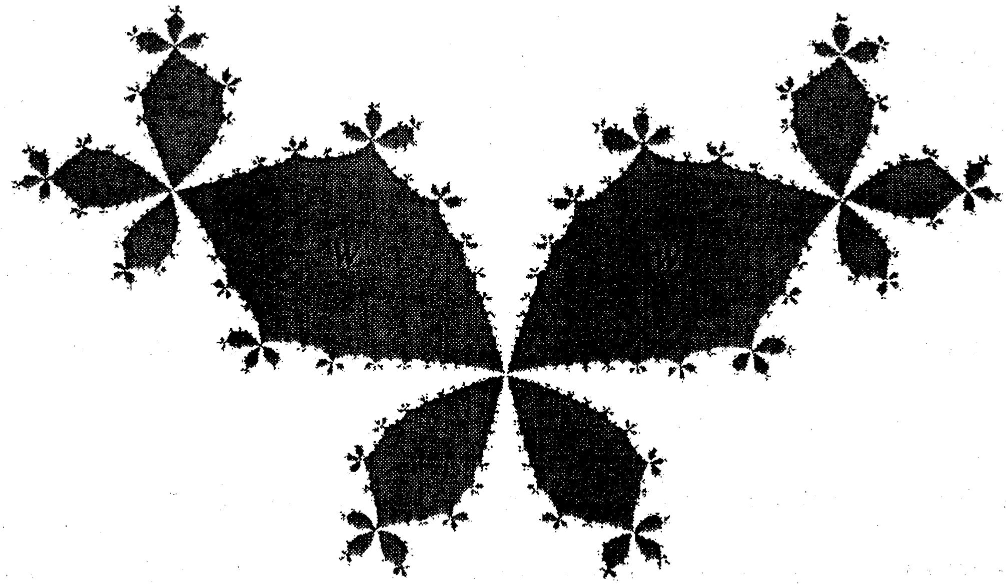

Figure 1: The filled Julia set of $z \mapsto z^3 - \frac{3}{4}z - \frac{\sqrt{7}}{4}$.

Figure 2: Five types of Polynomial-like restrictions.

**Proposition 3.1.** *Let $n_1$, $n_2 \in \mathcal{SR}(f, C_R)$. If $n_1 < n_2$, then $n_1$ divides $n_2$ and $K_{n_1}(U_1) \supset K_{n_2}(U_2)$.*

*Proof.* By Proposition 2.4, $n_1$ divides $n_2$.

Assume $K_{n_1}(U_1) \not\supset K_{n_2}(U_2)$. By Proposition 2.2, there exists a renormalization $f^{n_2} : U'_{n_2} \to V'_{n_2}$ with filled Julia set $K_{n_2}(U'_{n_2}) = K_{n_1}(U_{n_1}) \cap K_{n_2}(U_{n_2})$.

For simplicity, we write $K_{n_1} = K_{n_1}(U_{n_1})$, $K_{n_2} = K_{n_2}(U_{n_2})$ and $K'_{n_2} = K_{n_2}(U'_{n_2})$.

If $\mathcal{C}_{n_2}(U'_{n_2}) = C_R$, then $K'_{n_2} = K_{n_2}$. Therefore there exists a critical point $c_1 \in C_R - \mathcal{C}_{n_2}(U'_{n_2})$. Let $i_k$ be a number which satisfies $K_{n_i}(i_i) \ni c_1$. Then $i_1 \not\equiv i_2$ (mod $n_1$). So there exists $i_0$ such that $K_{n_1}(i_0)$ intersects $K_{n_2}$.

Therefore let a closed connected subset $L$ of $K(f)$ as the following:

$$L = K_{n_1}(i_0) \cup K_{n_2}(i_0) \cup K_{n_1}(2i_0) \cup \cdots \cup K_{n_1}.$$

Then $L \cap K_{n_2}$ is disconnected and it contradicts Lemma 1.2. □

**Proposition 3.2.** *If $f$ can be infinitely renormalizable, $f$ has infinitely many simple renormalizations.*

*More precisely, if $\mathcal{R}(n, C_R)$ is infinite for some $C_R \subset C(f)$, then there exists some $C$, $C_R \subset C \subset C(f)$, such that $\mathcal{SR}(f, C)$ is infinite.*

*Proof.* For $n \in \mathcal{R}(n, C_R)$, Let $\kappa_n$ be the number of components of $\mathcal{K}_n$. Since $\kappa_n$ is equal to the minimum of the period of periodic point of $f$ contained in $K_n$, $\kappa_n \to \infty$ by Theorem 2.1.

Now we show $f^{\kappa_n}$ is simply renormalizable. For sufficiently large $n$, choose a repelling periodic point $x$ of $f$ of period less than $\kappa_n$. Then $x \notin \mathcal{K}_n$. We construct the Yoccoz puzzle from the rays landing at $x$ and some equipotential curve.

For any depth $r \geq 0$, the piece $P_r(c_0)$ containing $c_0$ contains the component $E$ of $\mathcal{K}_n$ containing $c_0$. Thus the tableau $P_r(f^k(c_0))$ for $c_0$ is periodic of period $p$ with $p | \kappa_n$, i.e. for any $r > 0$, $P_r(f^p(c_0)) = P_r(f^p(c_0))$.

Then by slightly thickening the pieces, we can obtain a simple renormalization $f^p : U_p \to V_p$ with $K_p \supset E$ (more precisely, see [**Mi2**, Lemma 2]).

If $p = \kappa_n$, we are done.

Otherwise, let $g$ be the polynomial hybrid equivalent to $f^p : U_p \to V_p$. There exists a renormalization $g^{n/p} : \tilde{U}_{n/p} \to \tilde{V}_{n/p}$ corresponds to $f^n : U_n \to V_n$.

Now apply the argument above to $g$ and the renormalization $g^{n/p} : \tilde{U}_{n/p} \to \tilde{V}_{n/p}$ and eventually we obtain a simple renormalization of $f^{\kappa_n}$. □

Now we assume that $f$ is infinitely renormalizable. By the proposition above, $\#\mathcal{SR}(f, C_R)$ is infinite for some $C_R \subset C(f)$.

Furthermore, suppose $f(C_R) = f(C(f))$, i.e. for any $c' \in C(f) - C_R$, there exists some $c \in C_R$ such that $f(c) = f(c')$.

*Remark.* The above condition is satisfied for a polynomial which is hybrid equivalent to $f^n : U_n \to V_n$ for $n \in \mathcal{SR}(f, C_R)$.

So this assumption is to consider the polynomial hybrid equivalent to some renormalization instead of the original polynomial.

**Definition.** Let $f$ as above. For each $n \in \mathcal{SR}(f, C_R)$, let $\delta_n(i)$ be a closed curve which separates $K_n(i)$ from $P(f) - P_n(i)$ (in our case, such a curve exists and its homotopy class is uniquely determined). Let $\gamma_n(i)$ is the hyperbolic geodesic on $\mathbb{C} - P(f)$ which is homotopic to $\delta_n(i)$ on $\mathbb{C} - P(f)$ and let $\gamma_n = \gamma_n(n)$.

We say $\mathcal{SR}(f, C_R)$ is *robust* if

$$\liminf_{n \to \infty} \ell(\gamma_n) < \infty,$$

where $\ell(\cdot)$ denotes the hyperbolic length on $\mathbb{C} - P(f)$.

Let $\Sigma = \operatorname{proj\,lim}_{n \in \mathcal{SR}(f, C_R)} \mathbb{Z}/n$ and define $\sigma : \Sigma \to \Sigma$ by:

$$\sigma\left((i_n)_{n \in \mathcal{SR}(f,C_R)}\right) = \left(i_n + 1\right).$$

**Theorem 3.1.** *Let $f$ as above. When $\mathcal{SR}(f, C_R)$ is robust, then:*

1. *The postcritical set $P(f)$ is a Cantor set of measure zero.*
2. $\lim_{n \in \mathcal{SR}(f,C_R)} \sup_{0<i\leq n} \operatorname{diam} P_n(i) \to 0.$
3. *$f : P(f) \to P(f)$ is topologically conjugate to $\sigma : \Sigma \to \Sigma$. Especially, $f|_{P(f)}$ is a homeomorphism.*

*Proof.* By the hyperbolic geometry, the geodesics $\gamma_n(i)$ ($n \in \mathcal{SR}(f, C_R)$, $0 < i \leq n$) are simple and mutually disjoint, and their length are comparable with $\ell(\gamma_n)$.

Thus by the collar theorem, there is a standard collar $A_n(i)$ about $\gamma_n(i)$ in $\mathbb{C} - P(f)$ such that they are mutually disjoint and $\operatorname{mod}(A_n(i))$ is a decreasing function of $\ell(\gamma_n(i))$. Note that $A_n(i)$ separates $P_n(i)$ from the rest of the postcritical set.

Let $E_n = \bigcup_{i=1}^{n} A_n(i)$ and $F_n$ be the union of the bounded components of $\mathbb{C} - E_n$. Then $F_n$ contains $P(f)$ and each component of $F_n$ meets $P(f)$.

For any sequence $\{A_n(i_n)\}_{n \in \mathcal{SR}(f,C_R)}$ of nested annuli,

$$\sum_{n \in \mathcal{SR}(f,C_R)} \operatorname{mod} A_n(i_n) = \infty,$$

because $\liminf \ell(\gamma_n) < \infty$.

Therefore $F = \bigcap_{n \in \mathcal{SR}(f,C_R)} F_n$ is a Cantor set of measure zero. Since $F$ contains $P(f)$ and each component of $F_n$ contains $P_n(i)$ for some $i$, $F$ is equal to $P(f)$, so the postcritical set is measure zero and diameter of $P_n(i)$ tends to zero.

For $n \in \mathcal{SR}(f, C_R)$, we define $\phi_n : P(f) \to \mathbb{Z}/n$ by $\phi(z) = i \pmod{n}$ when $z \in P_n(i)$. Then $\phi(f(z)) = \phi(z) + 1 \pmod{n}$.

Therefore, define $\phi : P(f) \to \operatorname{proj\,lim}_{n \in \mathcal{SR}(f,C_R)} \mathbb{Z}/n$ by $\phi(z) = (\phi_n(z))_{n \in \mathcal{SR}(f,C_R)}$ and it gives the conjugacy between $f|_{P(f)}$ and $\sigma$. □

**Corollary 3.1.** *Let $f$ as above. Suppose $\mathcal{SR}(f, C_R)$ is robust. Then for sufficiently large $n \in \mathcal{SR}(f, C_R)$ and any $i$, $0 < i \leq n$, $\#C_n(i) \leq 1$.*

*Proof.* Suppose

$$\#\left(\bigcap_{n \in \mathcal{SR}(f,C_R)} C_n(n)\right) > 1.$$

By Theorem 3.1, $\bigcap P_n(1)$ consists of only one point $x$. Therefore, $f(C_n(n)) = \{x\}$. But it is impossible because there is no other critical point in $U_n$ for sufficiently large $n$. □

# 4 Robust rigidity

In this section, we prove the following theorem:

**Theorem 4.1 (Robust rigidity).** *Let $f$ as above. If $\mathcal{SR}(f, C_R)$ is robust, then $f$ carries no invariant line field on its Julia set.*

The proof depends on the following two lemmas.

**Lemma 4.1.** *Let $f_n : (U_n, u_n) \to (V_n, v_n)$ be a sequence of holomorphic maps between disks and let $\mu_n$ is a sequence of $f_n$-invariant line field on $V_n$. Suppose $f_n$ converge to $f : (U, u) \to (V, v)$ in the Carathéodory topology and $\mu_n$ converge in measure to $\mu$ on $V$. Then $\mu$ is $f$-invariant.*

See [**Mc**, Theorem 5.14].

**Lemma 4.2.** *Let $\mu$ be a measurable line field on $\mathbb{C}$. Assume $\mu$ is almost continuous at $x$ and $|\mu(x)| = 1$. Let $(V_n, v_n) \to (V, v)$ be a convergent sequence of disks, and let $h_n : V_n \to \mathbb{C}$ be a sequence of univalent maps with $h_n'(v_n) \to 0$.*

*Suppose*

$$\sup \frac{|x - h_n(v_n)|}{h_n'(v_n)} < \infty.$$

*Then there exists a subsequence such that $h_n^*(\mu)$ converges in measure to a univalent line field on $V$.*

See [**Mc**, Theorem 5.16].

Now we give the summary of the proof of the theorem. We divide the proof into two cases: whether $\liminf \ell(\gamma_n)$ is zero or not. But outline of these two proof are very similar. We assume there exists a measurable invariant line field $\mu$ supported on $J(f)$ and induce contradiction.

First, we take a point $x \in J(f)$ where $\mu$ is almost continuous, such that $\|(f^k)'(x)\| \to \infty$ with respect to hyperbolic metric on $\mathbb{C} - P(f)$, and such that $f^k(x)$ does not land in but tends to $P(f)$.

Next we construct some critically compact proper map $f^n : X_n \to Y_n$ from $f^n : U_n \to V_n$. By assumption, $f^k(x)$ eventually land in $Y_n$. If we take disks $X_n, Y_n$ properly, we can take a univalent inverse branch $h_n$ of $f^{-k}$ from $Y_n$ to the region near $x$. Note that $h_n^*(\mu) = \mu$ is $f^n$-invariant line field on $Y_n$.

By properly scaling $f^n : X_n \to Y_n$ and taking a subsequence, they converge to a proper map $g : U \to V$. Furthermore, by Lemma 4.2 and Lemma 4.1, $g$ must have an invariant univalent line field $\nu$ on $V$.

But $g$ have a critical point $c \in U \cap V$, then, by invariance, $\nu(c) = 0$, that is a contradiction.

## 4.1 Thin rigidity

**Definition.** A renormalization $f^n : U_n \to V_n$ is *unbranched* if

$$V_n \cap P(f) = P_n.$$

Let $f^n : U_n \to V_n$ be an unbranched renormalization. Let $W$ be a component of $f^{-1}(V_n(i+1))$ which is not $V_n(i)$. Then any inverse branch of $f^{-k}$ on $W$ is univalent because $W$ is disjoint from the postcritical set.

**Lemma 4.3.** *There exists some $M > 0$ such that if $\ell(\gamma_n) < M$, we can choose $U_n$ and $V_n$ such that $f^n : U_n \to V_n$ is unbranched renormalization and*

$$\mathrm{mod}(U_n, V_n) > m(\ell(\gamma_n)) > 0$$

*where $m(\ell) \to \infty$ as $\ell \to 0$.*

*Proof.* Let $A_n$ be the standard collar about $\gamma_n$ with respect to the hyperbolic metric on $\mathbb{C} - P(f)$. Let $B_n$ be the component of $f^{-n}(A_n)$ which is the same homotopy class as $\gamma_n$. Let $D_n$(resp. $E_n$) be the union of $B_n$(resp. $A_n$) and the bounded component of the complement. $f^n : D_n \to E_n$ is a critically compact proper map with postcritical set $P_n$.

When $\ell(\gamma_n)$ is sufficiently small, $\mathrm{mod}(P_n, E_n) \geq \mathrm{mod}(A_n)$ is sufficiently large. Then we can choose $U_n \subset D_n$ and $V_n \subset E_n$ such that $f^n : U_n \to V_n$ is a renormalization and $\mathrm{mod}(U_n, V_n)$ is bounded below in terms of $\mathrm{mod}(P_n, E_n)$.

The modulus of collar $A_n$ depends only on $\ell(\gamma_n)$ and tends to infinity as $\ell(\gamma_n)$ tends to zero. Since $\mathrm{mod}(P_n, E_n) \geq \mathrm{mod}(A_n)$, we are done. □

**Theorem 4.2.** *Let $f$ as above. Suppose for infinitely many $n \in \mathcal{SR}(f, C_R)$ there is a simple unbranched renormalization $f^n : U_n \to V_n$ with $\operatorname{mod}(U_n, V_n) > m$ for a constant $m > 0$.*

*Then $f$ carries no invariant line field on its Julia set.*

By the previous lemma, the following corollary is trivial.

**Corollary 4.1 (Thin rigidity).** *There is $L > 0$ such that if*

$$\liminf_{\mathcal{SR}(f,C_R)} \ell(\gamma_n) < L,$$

*then $f$ carries no invariant line field on its Julia set.*

*Proof of Theorem 4.2.* Let $\mathcal{USR}(f, C_R, m)$ be a set of $n \in \mathcal{SR}(f, C_R)$ such that there is an unbranched simple renormalization $f^n : U_n \to V_n$ with $\operatorname{mod}(U_n, V_n) > m$.

For $n \in \mathcal{USR}(f, C_R, m)$, there is an annulus of definite modulus separating $J_n(i)$ from $P(f) - P_n(i)$. So $\mathcal{SR}(f, C_R)$ is robust and

$$\bigcap_{n \in \mathcal{SR}(f,C_R)} \mathcal{J}_n = P(f).$$

Therefore, by the fact that a forward orbit of almost every point in $J(f)$ tends to $P(f)$, almost every $x$ in $J(f)$ satisfies the followings:

1. The forward orbit of $x$ does not meet the postcritical set.
2. $\|(f^k)'(x)\| \to \infty$ in the hyperbolic metric on $\mathbb{C} - P(f)$.
3. For any $n \in \mathcal{SR}(f, C_R)$, there is a $k > 0$ with $f^k(x) \in \mathcal{J}_n$.
4. For any $k > 0$, there is an $n \in \mathcal{SR}(f, C_R)$ such that $f^k(x) \notin \mathcal{J}_n$.

(Note that the condition 2 is satisfied every point which satisfies the condition 1.)

Suppose that $f$ carries an invariant line field $\mu$ on $J(f)$. Let $x$ be a point in $J(f)$ at which $\mu$ is almost continuous, $|\mu(x)| = 1$ and satisfies the above condition 1-4. For each $n \in \mathcal{SR}(f, C_R)$, let $k(n) \geq 0$ be the least integer such that $f^{k(n+1)}(x) \in \mathcal{J}_n$. By the condition 3, such $k(n)$ exists and tends to infinity by the condition 4. Now $f^{k(n)+1}(x)$ is contained in $J_n(i(n)+1)$ for some $0 \leq i(n) < n$.

For $n$ sufficiently large, $k(n) > 0$ and $f^{k(n)}(x) \notin \mathcal{J}_n$. So $f^{k(n)}(x)$ is contained in some component $W_n$ of $f^{-1}(V_n(i(n)+1))$ which is not $V_n(i(n))$. $W_n$ is disjoint from the postcritical set. Furthermore, $W_n$ contains no critical point for sufficiently large $n$ (actually, it is true if $k(n) > k(n_0)$ where $n_0 = \min(\mathcal{USR}(f, C_R, m))$).

Let $j(n) > i(n)$ be the least number such that $C_n(j(n))$ is nonempty, so that $f^{j(n)-i(n)} : W_n \to V_n(j(n))$ is univalent. Then there exists a univalent branch $h_n$ of $f^{i(n)-j(n)-k(n)}$ defined on $V_n(j(n))$ which maps $f^{j(n)-i(n)+k(n)}(x)$ to $x$.

Let $J_n^* = h_n(J_n(j(n)))$. Since there is an annulus of definite modulus in $\mathbb{C}-P(f)$ enclosing it, the diameter of $f^{k(n)}(J_n^*)$ $(= f^{-1}(J_n(i(n)+1)) \cap W_n)$ is bounded with respect to the hyperbolic metric on $\mathbb{C} - P(f)$. Therefore, by the condition 2, the diameter of $J_n^*$ in the hyperbolic metric on $\mathbb{C} - P(f)$ tends to zero.

Let $c \in C_R$ be a critical point such that for infinitely many $n \in \mathcal{USR}(f, C_R, m)$, $C_n(j(n))$ contains $c$. By taking a subsequence and replacing $f^n : U_n \to V_n$ by $f^n : U_n(j(n)) \to V_n(j(n))$, we may assume $c = c_0$ and $j(n) = n$, so $h_n$ is defined on $V_n$. (Note that $\mathrm{mod}(U_n(j(n)), V_n(j(n))) \geq \frac{1}{d_R}\mathrm{mod}(U_n, V_n) > \frac{m}{d_R}$, where $d_R$ is the degree of renormalization $f^n : U_n \to V_n$. Thus we should replace $m$ by $\frac{m}{d_R}$.)

Let

$$A_n(z) = \frac{z - c_0}{\mathrm{diam}(J_n)},$$
$$g_n = A_n \circ f^n \circ A_n^{-1},$$
$$y_n = A_n(h_n^{-1}(x)).$$

Then

$$g_n : \big(A_n(U_n), 0\big) \to \big(A_n(V_n), A_n(f^n(c_0))\big)$$

is a polynomial-like map with $\mathrm{diam}(J(g_n)) = 1$ and $\mathrm{mod}(A_n(U_n), A_n(V_n)) > m$.

Thus, by taking a subsequence, $g_n$ converges to some polynomial-like map (or polynomial) $g : (U, 0) \to (V, g(0))$ with $\mathrm{mod}(U, V) > m$ in the Carathéodory topology (see [**Mc**, Theorem 5.8]).

Let $k_n = h_n \circ A_n^{-1} : A_n(V_n) \xrightarrow{A_n^{-1}} V_n \xrightarrow{h_n} \mathbb{C}$ and $\nu_n = k_n^*(\mu)$. Then $\nu_n$ is $g_n$-invariant line field on $A_n(V_n)$ because $\mu = h_n^*(\mu)$ is $f$-invariant. Since $\mathrm{diam}(J(g_n)) = 1$ and $\mathrm{diam}(J_n^*) \to 0$, $k_n'(y_n) \to 0$.

Now we take a further subsequence of $n$ so that $(A_n(V_n), y_n) \to (V, y)$. Then by Lemma 4.2, after passing a further subsequence, $\nu_n$ converges to a univalent $g$-invariant line field $\nu$ on $V$.

For $f^n : U_n \to V_n$ have connected Julia set, so does $g$. Thus the critical point and critical value lie in $V$. But it contradicts the fact that $g$ has a univalent invariant line field $\nu$. □

## 4.2 Thick rigidity

**Theorem 4.3 (Thick rigidity).** *Let $f$ as above. Suppose*

$$0 < \liminf_{n \in \mathcal{SR}(f, C_R)} \ell(\gamma_n) < \infty,$$

*Then $f$ carries no invariant line field on its Julia set.*

**Notation.** For $n \in \mathcal{SR}(f, C_R)$,

- Let $\delta_n$ be the component of $f^{-n}(\gamma_n)$ which is homotopic to $\gamma_n$ on $\mathbb{C} - P(f)$.
- Let $X_n$(resp. $Y_n$) be the disk bounded by $\delta_n$(resp. $\gamma_n$). Then $f^n : X_n \to Y_n$ is a proper map whose degree is the same as that of $f^n : U_n \to V_n$.
- $Y_n(i) = f^i(X_n)$ for $0 < i \leq n$. Then $Y_n(i) \cap P(f) = P_n(i)$.
- $\mathcal{Y}_n = \bigcap_{i=1}^{n} Y_n(i)$. Then $\mathcal{Y}_n$ contains $P(f)$.
- Let $B_n$ be the largest Euclidean ball centered at $c_0$ and contained in $X_n \cap Y_n$.

**Lemma 4.4.**

$$\bigcap_{n \in \mathcal{SR}(f, C_R)} \mathcal{Y}_n = P(f).$$

*Proof.* When $n$ is sufficiently large, the diameter of $P_n(i)$ is small. But for $m > n$, $\gamma_m(i)$ separates $P_n(i)$ into two pieces, so $\gamma_m(i)$ passes very close to $P(f)$. Since the hyperbolic length of $\gamma_m(i)$ on $\mathbb{C} - P(f)$ is bounded for infinitely many $m$, the Euclidean diameter of $Y_n(i)$ is also small. □

Thus just as the proof of the thin rigidity, we obtain the following.

**Lemma 4.5.** *Almost every $x$ in $J(f)$ satisfies the followings:*

1. *The forward orbit of $x$ does not meet the postcritical set.*
2. *$\|(f^k)'(x)\| \to \infty$ in the hyperbolic metric on $\mathbb{C} - P(f)$.*
3. *For any $n \in \mathcal{SR}(f, C_R)$, there is a $k > 0$ with $f^k(x) \in \mathcal{Y}_n$.*
4. *For any $k > 0$, there is an $n \in \mathcal{SR}(f, C_R)$ such that $f^k(x) \notin \mathcal{Y}_n$.*

Let

$$\mathcal{SR}(f, C_R, \lambda) = \{n \in \mathcal{SR}(f, C_R) \mid 1/\lambda < \ell(\gamma_n) < \lambda\}.$$

When $0 < \liminf \ell(\gamma_n) < \infty$, $\mathcal{SR}(f, C_R, \lambda)$ is infinite for some $\lambda > 0$.

By using the collar theorem, we obtain the Euclidean diameters of $X_n$, $Y_n$ and $B_n$ are comparable for $n \in \mathcal{SR}(f, C_R, \lambda)$. So let $A_n(z) = \frac{z - c_0}{\mathrm{diam}(B_n)}$ and then after passing a subsequence,

$$(A_n(X_n), 0) \to (X, 0),$$
$$(A_n(Y_n), A_n(f^n(0))) \to (Y, g(0)),$$
$$A_n \circ f^n \circ A_n^{-1} \to g,$$

where $g : (X, 0) \to (Y, g(0))$ is a proper map, $0 \in X \cap Y$ and $g'(0) = 0$.

**Lemma 4.6.** *For each $n \in \mathcal{SR}(f, C_R, \lambda)$, there exists a disk $Z_n \in \mathbb{C} - P(f)$ and an integer $m$, $0 < m < 2n$ such that*

1. $f^m : Z_n \to Y_n(j)$ *is a univalent map for some $j$ with $0 < j \leq n$ and $C_n(j) \neq \emptyset$,*
2. $d(\partial X_n, \partial Z_n)$ *is bounded above in terms of $\lambda$.*
3. $\ell(\partial Z_n) < \lambda$,
4. $\text{area}(Z_n)$ *is bounded below in terms of $\lambda$.*

*in the hyperbolic metric on $\mathbb{C} - P(f)$.*

*Proof.* By the lower bound of $\gamma_n(i)$, there exist $\gamma_n(i)$ and $\gamma_n(j)$ such that $d(\gamma_n(i), \gamma_n(j))$ is bounded above in terms of $\lambda$. Furthermore, $\gamma_n(k)$ and $\partial Y_n(k)$ is uniformly close. So $d(\partial Y_n(i), \partial Y_n(j))$ is bounded above.

Considering backward images of $Y_n(i)$ and $Y_n(j)$, there is a disk $Z_n$ close to $X_n$ and maps to $Y_n(k)$ ($k = i$ or $j$) univalently by $f^{m'}$.

Since $\text{mod}(P_n, Y_n)$ is bounded below and $\|(f^n)'(z)\|$ is not so expanding near $\partial X_n$, $\text{area}(Z_n)$ is bounded below. □

*Proof of Theorem 4.3.* Suppose $\mu$ is an $f$-invariant line field supported on $J(f)$. Let $x$ be a point at which $\mu$ is almost continuous and satisfies the condition 1-4 of Lemma 4.5.

For each $n \in \mathcal{SR}(f, C_R, \lambda)$, let $k(n) \geq 0$ be the least integer such that $f^{k(n)+1}(x) \in \mathcal{Y}_n$. For $k(n) \to \infty$, we consider $n$ sufficiently large so that $k(n) > 0$ (so $f^{k(n)}(x) \notin \mathcal{Y}_n$).

Now we construct univalent maps $h_n : Y_n(j(n)) \to T_n \subset \mathbb{C}$. Let $i(n)$, $0 \leq i(n) < n$, be the number such that $Y_n(i(n) + 1)$ contains $f^{k(n)+1}(x)$.

**Case I.** $i(n) > 0$. Then $f^{k(n)}(x)$ is contained in a component $W_n$ of $f^{-1}(Y_n(i(n) + 1)$, which is not $Y_n(i(n))$. $W_n$ does not meet the postcritical set. Furthermore, for $n$ sufficiently large, $W_n$ contains no critical points.

So let $j(n) \geq i(n)$ be the least integer such that $C_n(j(n)) \neq \emptyset$ and define $h_n$ be the following:

$$Y_n(j(n)) \xrightarrow{f^{i(n)-j(n)}} W_n \xrightarrow{f^{-k(n)}} T_n \subset \mathbb{C}.$$

where the branch of $f^{-k(n)}$ is chosen to maps $f^{k(n)}(x)$ to $x$.

**Case II.** $i(n) = 0$ and $f^{k(n)}(x) \notin X_n - Y_n$. Since $f^{k(n)}(x) \notin X_n$, define $h_n$ just the same as Case I.

**Case III.** $i(n) = 0$ and $f^{k(n)}(x) \in X_n - Y_n$. Since $\partial X_n$ is close to $\partial Y_n$, $f^{k(n)}(x)$ is close to $Z_n$. So let $\zeta_n$ be a path joining $f^{k(n)}(x)$ to $Z_n$ with length bounded above in terms of $\lambda$.

Then by the previous lemma, there is a univalent map $f^m : Z_n \to Y_n(j(n))$. So define $h_n$ by:

$$Y_n(j(n)) \xrightarrow{f^{-m}} Z_n \xrightarrow{f^{-k(n)}} T_n \subset \mathbb{C}.$$

We choose the inverse branch of $f^{-k(n)}$ so that the extension to $Z_n \cap \zeta_n$ maps $f^{k(n)}(x)$ to $x$.

By the estimates for the derivative $\|(f^{k(n)})'(z)\|$ on $\partial T_n$ in terms of $\|(f^{k(n)})'(x)\|$ and $\lambda$, $\mathrm{diam}(T_n) \to 0$ and $d(x, T_n) \leq C_1 \,\mathrm{diam}(T_n)$ where $C_1$ is a constant which depends only on $\lambda$.

Let $k_n = h_n \circ A_n^{-1}$. Then $|k_n'(0)| \to 0$. Therefore,

$$\frac{|x - k_n(0)|}{|k_n'(0)|} \leq C_2 \frac{d(x, T_n) + \mathrm{diam}(T_n)}{\mathrm{diam}(T_n)} \leq C_3,$$

where $C_2$ and $C_3$ depend only on $\lambda$.

Thus we can apply Lemma 4.2 and deduce the contradiction. □

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