数理解析研究所講究録
1153 巻 2000 年 75-84

Non-landing of stretching rays for the family of real cubic polynomials

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Abstract In this note, the dynamics of real cubic polynomials is investigated. Especially, in the parameter space, landing and non-landing of stretching rays on the parabolic arc is studied. It turns out that stretching rays with irrational Böttcher vectors have non-trivial accumulation sets.

1 Introduction

In this note, we shall investigate the dynamics of the family of real cubic polynomials :

$$P(z) = P_{A,B}(z) = z^3 - 3Az + \sqrt{B}; \quad A, B > 0.$$

Our main concern is the landing of real stretching rays for this family. Since the stretching ray passing through a point in this family stays in this family, we can consider its landing property, especially on $Per_1(1)$, the locus where P has a parabolic fixed point with multiplier 1.

There are only a few works on the landing of stretching rays for degree greater than two. Kiwi [Ki] has considered critical portraits in the visible shift locus of polynomials of arbitrary degree and characterized their impressions in terms of rational laminations. Especially, he showed that the impression of a strictly preperiodic critical portrait consists of a single polynomial, whose critical points are strictly preperiodic. And he conjectured the existence of non-trivial impressions of critical portraits with aperiodic kneading. Willumsen [W] gave necessary conditions for stretching rays to accumulate on some part of $Per_1(1)$ in the family of complex cubic polynomials. This study is much inspired by her work. Quite recently, Buff and Henriksen [BuHe] has announced the existence of stretching rays with non-trivial accumulation sets through the study of the parameter space of the family $f_b(z) = \lambda z + bz^2 + z^3$ where $\lambda = e^{2\pi i\theta}$ and θ is a non-Bruno number.

Here we consider stretching rays only in the family of real cubic polynomials. Especially, in the first quadrant, the boundary of the connectedness locus is very simple. It consists of two real algebraic curves. And stretching rays must accumulate on these curves. This really simplifies things. Our main result is that, most stretching rays in some region of the shift locus of the first quadrant do not land at any point on $Per_1(1)$ (but, of course, they accumulate on it). Hence their accumulation sets must be non-trivial arcs. Although this does not answer the above conjecture in Kiwi [Ki], this gives a feature characteristic to higher degree polynomial dynamics. In fact, the Mandelbrot Local Connectivity Conjecture suggests that this does not happen to quadratic polynomials. The same argument works also in the third quadrant.

The locus $Per_1(1)$ in cubic polynomials has been investigated by several authors. Douady-Hubbard [DH] studied $Per_1(1)$ to show the discontinuity of the straightening map of polynomial-like maps of degree three. Milnor [M] considered the family of real cubic polynomials and conjectured the non local connectivity of the cubic connectedness locus. Lavaurs [L] settled this conjecture by considering the parabolic implosion from $Per_1(1)$. Recently through the study of $Per_1(1)$, Epstein-Yampolsky [EY] showed the conjecture in [M] that the connectedness locus of real cubic polynomials is not locally connected. Thus $Per_1(1)$ reflects the features of the dynamics of cubic polynomials much different from that of quadratic polynomials. And we add one more to it.

The authors would like to express their hearty thanks to Mitsuhiro Shishikura for valuable advice.

2 Stretching rays

Let $\mathcal{P}_d$ be the family of monic centered polynomials of degree $d \geq 2$. For $P \in \mathcal{P}_d$, let $h_P(z) = \lim_{n\to\infty} \frac{1}{d^n} \log_+ |P^n(z)|$ be the Green function for P, which is continued continuously to the whole plane by the functional equation $h_P(P(z)) = d \cdot h_P(z)$ and is harmonic in $\mathbf{C} - K(P)$, the complement of the filled-in Julia set. And let φ_P be the Böttcher coordinate of P defined in a neighborhood of ∞. It satisfies $\varphi_P(P(z)) = \varphi_P(z)^d$ and is tangent to the identity at ∞. Put $G(P) = \max\{h_P(\omega); \omega \text{ is a critical point of } P\}$. Then φ_P can be continued analytically to $U_P = \{z; h_P(z) > G(P)\}$. Actually we have $h_P(z) = \log_+ |\varphi_P(z)|$. For a complex number $u \in H_+ = \{u = s + it \in \mathbf{C}, s > 0\}$, put $f_u(z) = z|z|^{u-1}$ and we define a P-invariant complex structure σ_u by

$$\sigma_u = \begin{cases} (f_u \circ \varphi_P)^* \sigma_0 \text{ on } U_P, \\ \sigma_0 \text{ on } K(P), \end{cases}$$

where σ_0 is the standard complex structure. Then, by the Measurable Riemann Mapping Theorem, there exists a unique qc-map F_u satisfying

$$F_u^* \sigma_0 = \sigma_u, \ \lim_{z\to\infty} \frac{f_u \circ \varphi_P \circ F_u^{-1}(z)}{z} = 1, \ P_u = F_u \circ P \circ F_u^{-1} \in \mathcal{P}_d.$$

Since F_u depends holomorphically on u, so does P_u. Thus we define a holomorphic map $W_P : H_+ \to \mathcal{P}_d$ by $W_P(u) = P_u$. The Böttcher coordinate φ_{P_u} of P_u is equal to $f_u \circ \varphi_P \circ F_u^{-1}$. This operation is called *wringing*. Since P_u is hybrid equivalent to P, it holds $P_u \equiv P$ for $P \in \mathcal{C}_d$, the *connectedness locus*. For $P \in \mathcal{E}_d$, the *escape locus*, we define the *stretching ray* through P by

$$R(P) = W_P(\mathbf{R}_+) = \{P_s; s \in \mathbf{R}_+\}.$$

For example, in case $d = 2$, stretching rays coincide with the external rays for the Mandelbrot set. As for stretching rays, see Branner [Br] or Branner-Hubbard [BH1]. If $P \in \mathcal{P}_d$ is a real polynomial, φ_P and f_s are symmetric with respect to the real axis, hence so are σ_s and F_s, and P_s is also a real polynomial. The following is a direct consequence from the definition.

Lemma 2.1 *Let $\omega_j = \omega_j(P), j = 1, 2$ be two escaping critical points of $P \in \mathcal{E}_d$. Then $\tilde{\eta}(P_s) = \dfrac{h_{P_s}(\omega_1)}{h_{P_s}(\omega_2)}$ is invariant on the stretching ray $R(P)$ through P.*

proof. Since $|\varphi_{P_s}(z)| = |f_s \circ \varphi_P \circ F_s^{-1}(z)| = |\varphi_P \circ F_s^{-1}(z)|^s$, we have $h_{P_s}(z) = s \cdot h_P(F_s^{-1}(z))$ and

$$\tilde{\eta}(P_s) = \frac{h_{P_s}(F_s(\omega_1))}{h_{P_s}(F_s(\omega_2))} = \frac{h_P(\omega_1)}{h_P(\omega_2)} = \tilde{\eta}(P).$$

This completes the proof. □

Generally speaking, in this lemma, we cannot replace $h_P(\omega_j) = \log |\varphi_P(\omega_j)|$ by $\log \varphi_P(\omega_j)$ in the definition of $\tilde{\eta}(P)$. But, in case of real cubic polynomials in the first quadrant, we can do so because both

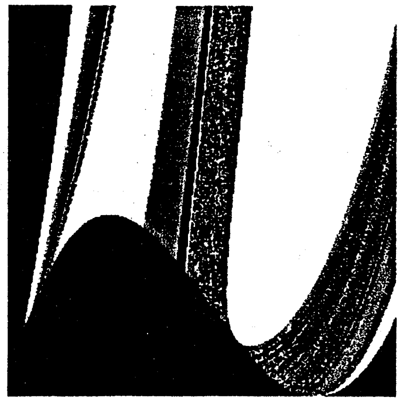

Figure 1: The connectedness locus C_3^R

critical points $\pm\sqrt{A}$ are real and their orbits lie on the positive real axis in the Böttcher coordinate. Here is an advantage of considering the real cubic polynomials. We set $\zeta_P(z) = \dfrac{\log\log\varphi_P(z)}{\log 3}$ and define, for $P \in \mathcal{E}_3^2$ (the real *shift locus*, i.e. the locus where both critical points escape to infinity), the *Böttcher vector* $\eta(P)$ by

$$\eta(P) = \frac{\log h_P(-\sqrt{A}) - \log h_P(\sqrt{A})}{\log 3} = \zeta_P(-\sqrt{A}) - \zeta_P(\sqrt{A}).$$

Note that, since $\varphi_P(\pm\sqrt{A}) > 1$, $\zeta_P(\pm\sqrt{A})$ is well defined. Then Lemma 2.1 implies the following.

Lemma 2.2 *On the stretching ray $R(P)$ through $P \in \mathcal{E}_3^2$, $\eta(P_s)$ is invariant.*

This lemma will play an important role in the following sections. We label each stretching ray the Fatou vector η on it and denote it by $R(\eta)$.

3 The parameter space of real cubic polynomials

We mainly restrict our attention to the first quadrant of the parameter space of real cubic polynomials. In the first quadrant, the connectedness locus C_3^R is bounded by two real algebraic curves :

$$\begin{aligned} Per_1(1) &= \{B = 4(A+1/3)^3; 0 \le A \le 1/9\}, \\ Preper_{(1)1} &= \{B = 4A(A-1)^2; 1/9 \le A \le 1\}. \end{aligned}$$

This terminology is due to Milnor [M]. See Figure 1. C_3^R is the black region and there appear some stretching rays in its complement. The shift locus is colored by the Böttcher vectors. For $Q \in Per_1(1)$, Q has a parabolic fixed point $\beta_Q = \sqrt{A+1/3}$ with multiplier 1. For $Q \in Preper_{(1)1}$, its critical value $Q(-\sqrt{A})$ is the fixed point where the external ray of angle 0 lands.

4 Landing and non-landing of stretching rays on $Per_1(1)$

In this section, we consider the landing and non-landing of stretching rays in the region $\mathcal{D} = \{(A,B) \in \mathbf{R}_+^2; B > 4(A+1/3)^3\}$ on $Per_1(1)$. First we shall show some stretching rays on which certain critical orbit relations hold actually land there. Note that $Per_1(1)$ is parametrized by A and that $Q \in Per_1(1)$ is written by $Q(z) = Q_A(z) = z^3 - 3Az + 2(A+1/3)^{3/2}$. For $Q \in Per_1(1)$ and for $k \ge 1$, put $g_k(A) = Q(-\sqrt{A}) - Q^{k+1}(\sqrt{A})$.

Lemma 4.1 *g_k is a monotone increasing function on* $[0, 1/9]$.

proof. By a direct calculation, we have

$$\begin{aligned} g_k'(A) &= 3(Q^k(\sqrt{A}) + \sqrt{A}) - 3(Q^k(\sqrt{A})^2 - \sqrt{A}) dQ^k(\sqrt{A})/dA \\ &= 3(Q^k(\sqrt{A}) + \sqrt{A})\{1 - (Q^k(\sqrt{A}) - \sqrt{A}) dQ^k(\sqrt{A})/dA\}. \end{aligned}$$

Since $Q^k(\sqrt{A}) > \sqrt{A}$, we have only to show

$$dQ^k(\sqrt{A})/dA < \frac{1}{Q^k(\sqrt{A}) - \sqrt{A}}, \quad 0 \le A \le 1/9.$$

We do this by induction on k. For $k = 1$,

$$\begin{aligned} \frac{1}{dQ(\sqrt{A})/dA} - (Q(\sqrt{A}) - \sqrt{A}) &= \frac{1}{3\sqrt{A+1/3} - 3\sqrt{A}} - \{2(A+1/3)^{3/2} - 2A^{3/2} - \sqrt{A}\} \\ &= \sqrt{A+1/3} + \sqrt{A} - \{2(A+1/3)^{3/2} - 2A^{3/2} - \sqrt{A}\} \\ &= \sqrt{A+1/3}\{1 - 2(A+1/3)\} + 2\sqrt{A}(A+1) > 0, \end{aligned}$$

and the conclusion is true. Next suppose it is true for k. Then by the induction hypothesis,

$$\begin{aligned} dQ^{k+1}(\sqrt{A})/dA &= 3(\sqrt{A+1/3} - Q^k(\sqrt{A})) + 3(Q^k(\sqrt{A})^2 - A) dQ^k(\sqrt{A})/dA \\ &< 3(\sqrt{A+1/3} - Q^k(\sqrt{A})) + 3(Q^k(\sqrt{A}) + \sqrt{A}) \\ &= \frac{1}{\sqrt{A+1/3} - \sqrt{A}} \\ &< \frac{1}{Q^{k+1}(\sqrt{A}) - \sqrt{A}}. \end{aligned}$$

Hence the conclusion holds also for $k + 1$. This completes the proof. □

Lemma 4.2 *There exist a countable set of stretching rays $R_k : P(-\sqrt{A}) - P^{k+1}(\sqrt{A}) = 0, k \ge 1$ landing at $(A_k, B_k) \in Per_1(1)$. R_k is expressed also by $R(k) : \eta(P) = k$.*

proof. Since $g_k(0) = Q_0(0) - Q_0^{k+1}(0) < 0$ and $g_k(1/9) = Q_{1/9}(-1/3) - Q_{1/9}^{k+1}(1/3) = \beta_{Q_{1/9}} - Q_{1/9}^{k+1}(1/3) > 0$, g_k has a unique zero A_k in $(0, 1/9)$. Since $Q^k(\sqrt{A}) < Q^{k+1}(\sqrt{A})$, it follows $A_{k-1} < A_k$. The above estimate holds also for small perturbation $Q_{A,\epsilon}(z) = z^3 - 3Az + 2(A+1/3)^{3/2} + \epsilon$, $\epsilon > 0$ above $Per_1(1)$. Thus we conclude that there exist real algebraic curves $R_k : P(-\sqrt{A}) - P^{k+1}(\sqrt{A}) = 0$ through the point $(A_k, B_k) \in Per_1(1)$. Since this critical orbit relation is preserved under stretching, they form stretching rays and are real algebraic. On the other hand, $Per_1(1)$ is also real algebraic. Hence they must land at some point on $Per_1(1)$. In fact, if their accumulation sets contain an open interval, they must coincide with $Per_1(1)$, which is impossible. This completes the proof. □

Next we consider the stretching rays between $R(k)$. For $Q \in Per_1(1)$, the immediate basin $\mathcal{B}_Q$ of the parabolic fixed point β_Q contains both critical points $\pm\sqrt{A}$ and $J(Q) = \partial\mathcal{B}_Q$ is a Jordan curve. Let $\phi_{Q,-}$ and $\phi_{Q,+}$ be the attracting and repelling Fatou coordinates respectively. Originally, they are defined only on the attracting and repelling petals $\Omega_{Q,-}$ and $\Omega_{Q,+}$ respectively and satisfy the functional equation :

$$\phi_{Q,\pm} \circ Q(z) = \phi_{Q,\pm}(z) + 1.$$

They can be continued analytically by this relation. Especially $\phi_{Q,-}$ is continued to the entire $\mathcal{B}_Q$. Fatou coordinates have ambiguity of additive constants. So, we take real constants $c_\pm \in \mathbf{R}$ satisfying

$c_- < \beta_Q < c_+$ for all $A \in [0, 1/9]$ and normalize them so that they satisfy $\phi_{Q,\pm}(c_\pm) = 0$. Then $\phi_{Q,\pm}$ are determined uniquely and are symmetric with respect to the real axis. We define the *Fatou vector* $\tau(Q)$ of Q by $\tau(Q) = \phi_{Q,-}(-\sqrt{A}) - \phi_{Q,-}(\sqrt{A})$, the difference of the critical points in the attracting Fatou coordinate.

Lemma 4.3 *The Fatou vector gives a real analytic parametrization of $Per_1(1), 0 < A < 1/9$.*

proof. First we show that the Fatou vector map $Q \mapsto \tau(Q)$ has a local inverse in each connected component of $\mathbf{R} - \mathbf{Z}$. Suppose $k < \tau_0 = \tau(Q_0) < k+1$. Take any $\tau \in (k, k+1)$. Consider the piecewise affine map $S_\tau(x + yi) = s_\tau(x) + yi$, where

$$s_\tau(x) = \begin{cases} \frac{\tau}{\tau_0 - k}(x-k) \text{ if } k \le x \le \tau_0, \\ k+1 - \frac{k+1-\tau}{k+1-\tau_0}(k+1-x) \text{ if } \tau_0 \le x \le k+1, \end{cases}$$

It is easy to see that S_τ is a qc-map from $\{k \le \Re w \le k+1\}$ onto itself, identity on its boundary and satisfies $S_\tau(\tau_0) = \tau$. We deform the complex structure by this qc-map in this region and pull it back by the Fatou coordinate $\phi_{Q_0,-}$ and then pull it back to $\mathcal{B}_{Q_0}$ by Q_0. If we take the standard complex structure outside the filled-in Julia set $K(Q_0)$, then we get a complex structure σ_τ. Let ξ_τ be the integrating qc-map of σ_τ so that $Q_\tau = \xi_\tau \circ Q_0 \circ \xi_\tau^{-1} \in Per_1(1)$. Then $\tau(Q_\tau) = \tau$. This gives a local inverse of the Fatou vector map τ.

The above argument does not work when $\tau_0 = k = 1, 2, 3...$ In this case, $Q_0^{k+1}(\sqrt{A}) = Q_0(-\sqrt{A})$ and we do surgery instead of qc-deformation. We normalize the attracting Fatou coordinate by $\phi_{Q_0,-}(Q_0(-\sqrt{A})) = 0$. Then $\phi_{Q_0,-}(Q_0(\sqrt{A})) = -k$. Take a small open neighborhood U of $-k$ in the attracting Fatou coordinate and let $s_\tau : U \to U$ be a qc-map, identity on ∂U and $s_\tau(-k) = \tau - k$. Here $\tau \in (-\epsilon, \epsilon)$ for some small $\epsilon > 0$. Take an open neighborhood V of $Q_0^k(\sqrt{A})$ so that $U \subset T_1(\phi_{Q_0,-}(V))$ and put $U' = T_1^{-1}(U)$, $V' = \phi_{Q_0,-}^{-1}(U') \cap V$. We define a qc-mqp T'_τ by $s_\tau \circ T_1$ on U' and T_1 elsewhere. Then the map R_τ, defined by $\phi_{Q_0,-}^{-1} \circ T'_\tau \circ \phi_{Q_0,-}$ on V' and Q_0 elsewhere is a quasi-regular map on $\mathcal{B}_{Q_0}$ depending real analytically on τ. Let $\sigma_\tau = \phi_{Q_0,-}^* s_\tau^* \sigma_0$ and $\sigma_\tau = \sigma_0$ on the fundamental regions containing V and $R_\tau(V)$ respectively, and then pull it back or push it forward by Q_0. Then we get an R_τ-invariant complex structure σ_τ on $\mathcal{B}_{Q_0}$. Put $\sigma_\tau = \sigma_0$ outside $K(Q_0)$. Let ξ_τ be its integrating qc-map such that $Q'_\tau = \xi_\tau \circ R_\tau \circ \xi_\tau^{-1} \in Per_1(1)$. Then Q'_τ depends real analytically on τ, $Q'_0 = Q_0$ and $\tau(Q'_\tau) = k + \tau$. Thus we obtain a real analytic local parametrization of $Per_1(1)$ at Q_0 with $\tau(Q_0) = k$. This completes the proof. □

The Fatou vector corresponds to $0 < \tau(A) < \infty$. Now Lemma 4.2 can be stated in terms of two vectors.

Lemma 4.4 *The stretching ray $R(k)$ with $k = 1, 2, 3, ..$ lands at a map $Q \in Per_1(1)$ with $\tau(Q) = k$. Conversely, at a map $Q \in Per_1(1)$ with $\tau(Q) = 1, 2, 3, ...,$ a stretching ray $R(\eta)$ with $\eta = \tau(Q)$ lands.*

The "limit" of $R(k)$ is also a stretching ray $R(\infty) : B = 4(A + 1/3)^3$, $A > 1/9$, which consists of a parabolic maps and is contained in the boundary of $\mathcal{D}$. It lands at $(A_\infty, B_\infty) = (1/9, 4^4/9^3)$.

Our main result is the following.

Theorem 4.1 *Suppose η is irrational. Then the stretching ray $R(\eta)$ does not land at any point on $Per_1(1)$. Consequently, its accumulation set $I(\eta) = \overline{R(\eta)} - R(\eta)$ is a non-trivial arc on $Per_1(1)$.*

Figure 2 is an enlargement of Figure 1 and suggests that a stretching ray oscillates like the graph of $\sin(1/x)$ as they approaches $Per_1(1)$.

The proof is an application of the parabolic implosion analysis, for which see Douady [D], Lavaurs [L], Shishikura [Sh] or Willumsen [W]. The following lemma assures the existence of the Fatou coordinates for $Q_{A,\epsilon}$.

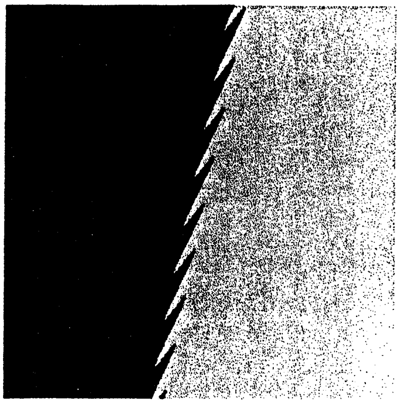

Figure 2: Stretching rays which accumulate but do not land

Lemma 4.5 *Let $\beta^{\pm}_{Q_{A,\epsilon}}$ be the fixed points of $Q_{A,\epsilon}$ bifurcating from β_{Q_A} and let $\rho_{\pm}(\epsilon)$ be their multipliers. Then we have*

$$\begin{aligned}\beta^{\pm}_{Q_{A,\epsilon}} &= \pm i\sqrt{\frac{\epsilon}{3A+1}} + \frac{\epsilon}{18A+6} + O(\epsilon^{3/2}),\\ \rho_{\pm}(\epsilon) &= 1 \pm 2i(A+1/3)^{1/4}\sqrt{3\epsilon} - \frac{2\epsilon}{3\sqrt{A+1/3}} + O(\epsilon^{3/2}).\end{aligned}$$

So, let $\phi_{P,\pm}$ be the Fatou coordinates of $P \in \mathcal{E}^2_3$ above $Per_1(1)$ normalized by $\phi_{P,\pm}(c_{\pm}) = 0$. They are continuous up to $Per_1(1)$. After perturbation, the gate between two fixed points $\beta^{\pm}_{P}$ is open and the incoming Fatou coordinate can be regarded also as an outgoing Fatou coordinate and vice versa. Thus $\phi_{P,+}$ and $\phi_{P,-}$ differ only by an additive constant. We call this difference $\tilde{\sigma}(P) = \phi_{P,+}(z) - \phi_{P,-}(z)$ the *lifted phase* and its class $\sigma(P) = [\tilde{\sigma}(P)]$ in $\mathbf{C}/\mathbf{Z}$ the *phase* of P. Since all mappings are symmetric with respect to the real axis, the lifted phase is always real. Roughly speaking, minus the lifted phase is the time needed for the orbits of P to pass through the gate.

Lemma 4.6 *The lifted phase $\tilde{\sigma}(P_s)$ tends to $-\infty$ as $s \to 0$ on a stretching ray.*

proof. For any s, there exists an $n = n_s$ such that $c_+ \le P_s^n(c_-) < P_s(c_+)$. Then, since

$$0 = \phi_{P_s,+}(c_+) \le \phi_{P_s,+}(P_s^n(c_-)) = \phi_{P_s,+}(c_-) + n < \phi_{P_s,+}(P_s(c_+)) = 1,$$

it follows $-n \le \tilde{\sigma}(P_s) = \phi_{P_s,+}(c_-) < -n+1$. Suppose $\tilde{\sigma}(P_s)$ does not tend to $-\infty$ as $s \to 0$. Then there exists a k and a sequence $P_n \in R(P)$ such that $\tilde{\sigma}(P_n) \ge -k$. This implies $P_n^k(c_-) \ge c_+$. We can assume P_n tends to some $Q \in Per_1(1)$ by taking a subsequence if necessary. Then it follows $Q^k(c_-) \ge c_+$, which is a contradiction. This completes the proof. □

We also define, for $Q \in Per_1(1)$ and for $\tilde{\sigma} \in \mathbf{C}$, the *Lavaurs map* $g_{\tilde{\sigma}} : \mathcal{B}_Q \to \mathbf{C}$ of lifted phase $\tilde{\sigma}$ by $g_{\tilde{\sigma}} = \phi^{-1}_{Q,+} \circ T_{\tilde{\sigma}} \circ \phi_{Q,-}$, where $T_{\tilde{\sigma}}(w) = w + \tilde{\sigma}$. The following is a fundamental fact. (See Douady [D], Prop.18.2, for example.)

Lemma 4.7 *Suppose $P_n \to Q \in Per_1(1)$ and $\sigma(P_n) \to \sigma \in \mathbf{C}/\mathbf{Z}$. Let $\tilde{\sigma}$ be any lift of σ. If we take $N_n \to \infty$ satisfying $N_n + \tilde{\sigma}(P_n) \to \tilde{\sigma}$, then $P_n^{N_n} \to g_{\tilde{\sigma}}$ locally uniformly on $\mathcal{B}_Q$.*

proof. Since we have

$$\begin{aligned}P_n^{N_n} &= \phi^{-1}_{P_n,+} \circ (\phi_{P_n,+} \circ P_n^{N_n} \circ \phi^{-1}_{P_n,-}) \circ \phi_{P_n,-}\\ &= \phi^{-1}_{P_n,+} \circ (T_{N_n} \circ \phi_{P_n,+} \circ \phi^{-1}_{P_n,-}) \circ \phi_{P_n,-}\end{aligned}$$

$$\begin{aligned}
&= \phi_{P_n,+}^{-1} \circ (T_{N_n} \circ T_{\tilde\sigma(P_n)}) \circ \phi_{P_n,-} \\
&= \phi_{P_n,+}^{-1} \circ T_{N_n+\tilde\sigma(P_n)} \circ \phi_{P_n,-} \\
&\to \phi_{Q,+}^{-1} \circ T_{\tilde\sigma} \circ \phi_{Q,-} \\
&= g_{\tilde\sigma},
\end{aligned}$$

this completes the proof. □

Since, in our case, $K(Q)$ is symmetric with respect to the real axis, connected and locally connected, its image in the repelling Fatou coordinate does not intersect the real axis. Then it follows $g_{\tilde\sigma}(\pm\sqrt{A}) \in \mathbf{C} - K(Q)$. Hence we can define the Böttcher vector $\eta(Q,\tilde\sigma)$ with lifted phase $\tilde\sigma$ also for $Q \in Per_1(1)$:

$$\eta(Q,\tilde\sigma) = \zeta_Q(g_{\tilde\sigma}(-\sqrt{A})) - \zeta_Q(g_{\tilde\sigma}(\sqrt{A})).$$

Note that it depends only on the class $\sigma \in \mathbf{C}/\mathbf{Z}$ of $\tilde\sigma$. Generally speaking, $\eta(Q,\tilde\sigma)$ depends on $\tilde\sigma$. But we have

Proposition 4.1 *Suppose $R(\eta)$ lands at $Q \in Per_1(1)$. Then $\eta(Q,\tilde\sigma)$ is equal to η for any $\tilde\sigma$. Especially $\eta(Q,\tilde\sigma)$ is independent of $\tilde\sigma$.*

proof. First we show the following lemma.

Lemma 4.8 *Suppose a sequence P_n in $\mathcal{D}$ converging to $Q \in Per_1(1)$ satisfies $\sigma(P_n) \to \sigma$. Then, $\eta(P_n) \to \eta(Q,\tilde\sigma)$ for any lift $\tilde\sigma$ of σ.*

proof. By Lemma 4.7, for any lift $\tilde\sigma$ of σ, there exists a sequence $N_n \to \infty$ such that $P_n^{N_n} \to g_{\tilde\sigma}$ locally uniformly in $\mathcal{B}_Q$. Then it follows

$$\begin{aligned}
\eta(P_n) &= \zeta_{P_n}(-\sqrt{A(P_n)}) - \zeta_{P_n}(\sqrt{A(P_n)}) \\
&= \zeta_{P_n}(P_n^{N_n}(-\sqrt{A(P_n)})) - \zeta_{P_n}(P_n^{N_n}(\sqrt{A(P_n)})) \\
&\to \zeta_Q(g_{\tilde\sigma}(-\sqrt{A(Q)})) - \zeta_Q(g_{\tilde\sigma}(\sqrt{A(Q)})) \\
&= \eta(Q,\tilde\sigma).
\end{aligned}$$

This completes the proof of Lemma 4.8. □

Now suppose $R(\eta)$ lands at Q. Then Lemma 4.6 says that, for any $\sigma \in \mathbf{R}/\mathbf{Z}$, there exists a sequence $P_n \in R(\eta)$ tending to Q and satisfying $\sigma(P_n) = \sigma$. By Lemma 4.8, it follows $\eta(Q,\tilde\sigma) = \lim_{n\to\infty} \eta(P_n) = \eta$ for any lift $\tilde\sigma$ of σ. Since σ is arbitrary, this completes the proof of Proposition 4.1. □

This proposition is a key to the proof of the main theorem. Let $\tilde{A}(Q)$ be the annulus in the repelling Ecalle cylinder of Q, bounded by the images of the Julia set $J(Q)$. Note that ζ_Q maps $\Omega_{Q,+} - K(Q)$ conformally into the strip region $\Sigma = \{|\Im w| < \pi/(2\log 3)\}$ and satisfies $\zeta_Q \circ Q(z) = \zeta_Q(z) + 1$ there (the same functional equation as the Fatou coordinates). This yields a flat annulus $A'(Q) = \{|\Im w| < \pi/(2\log 3)\}$ in $\mathbf{C}/\mathbf{Z}$ of modulus $\pi/\log 3$. Put $A(Q) = \phi_{Q,+}(\tilde{A}(Q))$. Then the quotient map $\psi_Q : A'(Q) \to A(Q)$ of the map $\phi_{Q,+} \circ \zeta_Q^{-1} : \Sigma \to \Omega_{Q,+} - K(Q)$ gives a conformal equivalence between the annuli $A'(Q)$ and $A(Q)$. In terms of the Lavaurs map, the Fatou vector is also written by

$$\tau(Q) = \phi_{Q,+}(g_{\tilde\sigma}(-\sqrt{A})) - \phi_{Q,+}(g_{\tilde\sigma}(\sqrt{A})),$$

which easily follows from the definition. Now we can see the geometric meanings of $\tau(Q)$ and $\eta(Q,\tilde\sigma)$. That is, $\tau(Q)$ is the difference of $g_{\tilde\sigma}(\pm\sqrt{A})$ in the repelling Fatou coordinate and $\eta(Q,\tilde\sigma)$ is their difference in the ζ_Q-coordinate. $\tau(Q)$ does not depend on $\tilde\sigma$. On the other hand, $\eta(Q,\tilde\sigma)$ generally depends on $\tilde\sigma$. But Proposition 4.1 assures its independence of $\tilde\sigma$ if $R(\eta)$ lands at Q. If we change $\tilde\sigma$, the positions of $g_{\tilde\sigma}(\pm\sqrt{A})$ in the repelling Fatou coordinate are translated according to that change. Nevertheless, their difference in the ζ_Q-coordinate does not change. Since we can take $\tilde\sigma$ arbitrarily, this gives a strong restriction on the property of the map ψ_Q and we get the following lemma.

Lemma 4.9 *Suppose $R(\eta)$ lands at $Q \in Per_1(1)$. Then $\psi_Q(w + [\eta]) = \psi_Q(w) + [\tau(Q)]$. Especially it follows $\tau(Q) = \eta$.*

proof. Since ψ_Q is conformal, we have only to show the relation on the equator $\mathbf{R}/\mathbf{Z}$. The above discussion implies that the difference of the images by $\zeta_Q \circ \phi_{Q,+}^{-1}$ of the two points on the real axis of the repelling Fatou coordinate with difference $\tau(Q)$ is always η. Hence we have $\psi_Q(w + [\eta]) = \psi_Q(w) + [\tau(Q)]$ on the equator. Then ψ_Q gives a real analytic conjugacy of the two rotations with rotàtion numbers $[\tau(Q)]$ and $[\eta]$ on the equator. Hence $[\tau(Q)] = [\eta]$. By Lemma 5.3, this implies $\tau(Q) = \eta$. This completes the proof. □

Now we are in a position to prove the main theorem. Suppose η is irrational and $R(\eta)$ lands at some $Q \in Per_1(1)$. By Lemma 4.9, ψ_Q satisfies $\psi_Q(w + [\eta]) = \psi_Q(w) + [\tau(Q)]$. Then, for any $n \in \mathbf{Z}$, we have $\psi_Q(w + [n\eta]) = \psi_Q(w) + [n\tau(Q)]$. Note that, if η is irrational, the set$\{[n\eta]; n \in \mathbf{Z}\}$ is dense in $\mathbf{R}/\mathbf{Z}$. Then $A(Q)$ must also be a flat annulus. This implies that $J(Q)$ is a real analytic curve, which is a contradiction. In fact, the immediate basin of β_Q contains, locally at β_Q, a sector region with an angle $3\pi/2$. Then $J(Q) = \partial\mathcal{B}_Q$ cannot be smooth at β_Q, consequently at all its preimages densely distributed on $J(Q)$. This completes the proof of the main theorem.

In case of rational η, Lemma 4.9 still holds and we have

Lemma 4.10 *Suppose $\eta = p/q \notin \mathbf{Z}$ is rational and $R(\eta)$ lands at some $Q \in Per_1(1)$. Then $\tau(Q) = \eta$ and the image of $J(Q)$ in the repelling Fatou coordinate is invariant under the translation $w \mapsto w+1/q$.*

Since $\mathbf{Q}$ is dense in $\mathbf{R}$, we have

Lemma 4.11 *There exists a dense subset E of $\mathbf{Q}$ such that, if $\eta \in E$ then $R(\eta)$ does not land at any point on $Per_1(1)$.*

We conjecture that, for any $\eta \in \mathbf{R} - \mathbf{Z}$, $R(\eta)$ does not land at any point on $Per_1(1)$.

5 The third quadrant

The same argument works also in the third quadrant. So we only state the results and omit the details. There, our family is written by

$$P(z) = P_{A,B}(z) = z^3 - 3Az - \sqrt{-B}i; \quad A, B < 0,$$

which is affinely equivalent to the family of real polynomials :

$$p(z) = p_{A,B}(z) = -z^3 - 3Az - \sqrt{-B}.$$

The connectedness locus $\mathcal{C}_3^R$ is bounded by two real algebraic curves :

$$\begin{aligned} Per_2(1) &= \{B = 4(A - 2/3)^3; -1/36 \le A \le 0\}, \\ Preper_{(1)2} &= \{B = -(\sqrt{-A}(2A+1)+1)^2; -1 \le A \le -1/36\}. \end{aligned}$$

We consider the stretching rays in the region $\mathcal{D}' = \{B < 4(A - 2/3)^3\}$. For $q \in Per_2(1)$, q has a parabolic 2-cycle $\{\beta_q, \beta'_q\}$ with multiplier 1. Here β_q, β'_q are the landing points of the external rays of angles $0, 1/2$ respectively. In other words, they are the maximum and minimum real 2-periodic points respectively. Both critical points $\pm\sqrt{-A}$ of q are contained in the immediate basin $\mathcal{B}_q$ of β_q. Let $\phi_{q,\pm}$ be the Fatou coordinates of $q \in Per_2(1)$ at β_q normalized by $\phi_{q,\pm}(c_\pm) = 0$ for some real constants $c_\pm \in \mathbf{R}$ satisfying $c_+ < \beta_q < c_-$ for any $q \in Per_2(1)$. $\phi_{q,\pm}$ satisfy $\phi_{q,\pm}(q^2(z)) = \phi_{q,\pm}(z) + 1$ in their petals. We define the Fatou vector $\tau(q) = \phi_{q,-}(-\sqrt{-A}) - \phi_{q,-}(\sqrt{-A})$ of q in the same way.

Lemma 5.1 *The Fatou vector gives a real analytic parametrization of $Per_2(1)$.*

Put $g_k(A) = q(-\sqrt{-A}) - q^{2k+1}(\sqrt{-A})$ for $k \geq 1$. Then we have

Lemma 5.2 *g_k is monotone increasing on $Per_2(1)$.*

Hence g_k has a unique zero A_k in $(-1/36, 0)$ and the sequence $\{A_k\}$ is monotonely decreasing and converging to $-1/36$. Furthermore, there is a real algebraic curves

$$R(k) : p(-\sqrt{-A}) - p^{2k+1}(\sqrt{-A}) = 0.$$

through the point (A_k, B_k). It is easy to see that $R(k)$ is a stretching ray landing at (A_k, B_k). We also define the Böttcher vector $\eta(p)$ of p in the shift locus by

$$\eta(p) = \frac{\log h_p(-\sqrt{-A}) - \log h_p(\sqrt{-A})}{2 \log 3}.$$

Note that we consider the orbit of p^2 of degree 9 and that $h_{p^2} = h_p$. Then we have

Lemma 5.3 *The stretching ray $R(k)$ with $k = 1, 2, 3, \ldots$ lands at a map $q \in Per_2(1)$ with $\tau(q) = k$. Conversely, at a map $q \in Per_2(1)$ with $\tau(q) = 1, 2, 3, \ldots$ the stretching ray $R(\eta)$ with $\eta = \tau(q)$ lands. $R(k)$ is expressed by $p(-\sqrt{A}) - p^{2k+1}(\sqrt{-A}) = 0$.*

Theorem 5.1 *Suppose η is irrational. Then the stretching ray $R(\eta)$ does not land at any point on $Per_2(1)$. Consequently, its accumulation set $I(\eta)$ is a non-trivial arc on $Per_2(1)$.*

Lemma 5.4 *There exists a dense subset E' of $\mathbf{Q}$ such that, if $\eta \in E'$ then $R(\eta)$ does not land at any point on $Per_2(1)$.*

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