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Fredholm determinant of complex Ruelle operator, Ruelle's dynamical zeta-function, and forward/backward Collet-Eckmann condition

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0. Introduction

In this note, we formulate a complex analytic version of the Ruelle's transfer operator applied to a kind of generalized functions related to a complex dynamical system on the Riemann sphere. The Fredholm determinant of this operator factorizes into several factors. One of these factors is a complex version of Ruelle's dynamical ζ-function. We define what we call an η-function derived from this factorization. The condition for this η-function to have poles or zeros leads us to a condition concerning the recurrence of the critical points. Such a condition is formulated and called the forward/backward Collet-Eckmann condition.

1. Prehyperfunctions supported on the Julia set

In this section, we formulate the notion of prehyperfunctions defined in a neighborhood of the Julia set. Complexified version of Ruelle's transfer operator for these functions will be formulated in the next section. Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping of the Riemann sphere to itself. We assume that the infinity is an attractive (or superattractive) fixed point of R. In the case of attractive fixed point, we assume R is of the form

$$R(z) = \sigma z + O(1),$$

near the infinity with $|\sigma| > 1$. The eigenvalue of the infinity is σ^{-1}. Let $F = F(R)$ denote the Fatou set of R, and let $J = J(R)$ denote the Julia set of R. We denote by $C = C(R)$ the set of critical points of R and by $P = P(R)$ the set of postcritical points, *i.e.*, the closure of the set of the forward iterated images of the critical points. Let $C_J = C \cap J$, $C_F = C \cap J$, $P_F = P \cap F$, and $P_J = P \cap J$. We assume that F and J are connected and P_F is compact. Further, we assume that all the critical points are non-degenerate, and the forward orbit of each critical point does not contain other critical points.

Let $\mathcal{O}(J)$ denote the space of functions $g : J \to \mathbb{C}$ which can be extended to a neighborhood of J and holomorphic in the neighborhood of J. The topology is defined as follows : a sequence of functions $\{g_n\}$ in $\mathcal{O}(J)$ converges to some function g_∞ in $\mathcal{O}(J)$ if there exists a neighborhood of J such that $\{g_n\}$ are extendable to this neighborhood and the sequence converges to g_∞ uniformly in this neighborhood.

Let $\mathcal{O}(F)$ denote the space of holomorphic functions $f : F \to \mathbb{C}$ with the topology of local uniform convergence. We denote by $\mathcal{O}_0(F)$ the set of holomorphic functions $f \in \mathcal{O}(F)$ satisfying $f(\infty) = 0$.

The space of prehyperfunctions $\mathcal{H}(J)$ supported on J is defined by a direct sum :

$$\mathcal{H}(J) = \mathcal{O}(J) \oplus \mathcal{O}_0(F).$$

This space is a Fréchet space.

For $\varphi \in \mathcal{H}(J)$, let $\varphi = \varphi_J \oplus \varphi_F$ with $\varphi_J \in \mathcal{O}(J)$ and $\varphi_F \in \mathcal{O}_0(F)$.

DEFINITION 1.1 An open neighborhood U of J with a smooth boundary $\Gamma = \partial U$ is said to be *adapted* to φ_J if φ_J can be extended holomorphically to U, $U \cap (C_F \cup P_F) = \emptyset$, $R^{-1}(\overline{U}) \subset U$, and $R^{-1}(\Gamma)$ is homologous to Γ in $U \setminus J$.

For each $\varphi_J \in \mathcal{O}(J)$, there exists an adapted neighborhood of φ_J. Let U be a neighborhood of J adapted to some $\varphi_J \in \mathcal{O}(J)$. Let $\mathcal{O}(U)$ denote the space of holomorphic functions on U. Let $\mathcal{H}(U) = \mathcal{O}(U) \oplus \mathcal{O}_0(F)$. For $\varphi \in \mathcal{H}(U)$, the decomposition $\varphi = \varphi_J \oplus \varphi_F$ is given by

$$\varphi_J(x) = \frac{1}{2\pi i}\int_{\gamma_J} \frac{\varphi(\tau)}{\tau - x} d\tau, \quad \text{for } x \in U,$$

and

$$\varphi_F(x) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\varphi(\tau)}{\tau - x} d\tau, \quad \text{for } x \in F,$$

where the intagration path $\gamma_J \subset U \setminus J$ turns once aroud the Julia set J in the counterclockwise direction passing near the boudary of U so that x belongs to the inside of the integration path, and the integration path $\gamma_F \subset U \setminus J$ turns once around the Julia set J in the clockwise direction passing near the Julia set J so that x belongs to the outside of the integration path. The integration path depends on x and z. But this defines functions $\varphi_J \in \mathcal{O}(U)$ and $\varphi_F \in \mathcal{O}_0(F)$. Moreover, we have $\varphi = \varphi_J + \varphi_F$ in $U \setminus J$. Here, $\varphi_J + \varphi_F$ means the usual sum of functions, and we don't distinguish the prehyperfunction and the function defined by φ in $U \setminus J$. Note that the decomposition is unique, since a function belonging to $\mathcal{O}(U) \cap \mathcal{O}_0(F)$ is holomorphic on the Riemann shere and vanishes at the infinity, hence it is identically zero.

2. Complex version of Ruelle's transfer operator

In this section, we define the complexified version of Ruelle's transfer operator (with weight functon $(R'(z))^2$) for the prehyperfunctions. Take an open and simply connected neighborfood U_0 of J, with a smooth boundary $\Gamma_0 = \partial U_0$, such that $R^{-1}(\overline{U_0}) \subset U_0$, $U_0 \cap F \cap (P \cup C) = \emptyset$, and that $R^{-1}(\Gamma)$ is homologous to Γ in $\overline{U_0} \setminus J$.

DEFINITION 2.1 Complex Ruelle operator $L : \mathcal{H}(U) \to \mathcal{H}(U)$ is defined by

$$(L\varphi)(x) = \sum_{y \in R^{-1}(x)} \frac{\varphi(y)}{(R'(y))^2}, \quad \varphi \in \mathcal{H}(U), \quad x \in U.$$

This operator can be rewritten in an "integral operator form" as follows.

$$(L\varphi)(x) = \frac{1}{2\pi i} \int_{\gamma_J + \gamma_F} \frac{\varphi(\tau)}{R'(\tau)(R(\tau) - x)} d\tau,$$

where the integration path γ_J and γ_F are taken as in the previous section. For each $x \in U$, this formula defines the value $(L\varphi)(x)$ by choosing the integration path γ_J running sufficiently near the boudary ∂U, and by choosing the integration path γ_F running sufficiently near J. Note that if we fix the integraton path, then this formula defines the value of $(L\varphi)(x)$ for only in some subset of $U \setminus J$. This fact can be verified immediately by applying the residue formula.

Also, note that the obtained function $L\varphi$ can be extended to a prehyperfunction in a larger domain V, if $R^{-1}(V) \subset U$. Hence, if $U \subset V$ and $R^{-1}(V) \subset U$, then L defines a complex linear mapping

$$L : \mathcal{H}(U) \to \mathcal{H}(V),$$

which is compatible with the natural inclusion $\mathcal{H}(V) \subset \mathcal{H}(U)$. Here, $V \setminus J$ may contain critical points of R.

The space of prehyperfunctions $\mathcal{H}(U)$ has a natural decomposition $\mathcal{H}(U) = \mathcal{O}(U) \oplus \mathcal{O}_0(F)$. This natural decomposition induces a natural decomposition of the complex Ruelle operator $L : \mathcal{O}(U) \oplus \mathcal{O}_0(F) \to \mathcal{O}(U) \oplus \mathcal{O}_0(F)$ as

$$L = \begin{pmatrix} L_{JJ} & L_{JF} \\ L_{FJ} & L_{FF} \end{pmatrix}.$$

These components are given by the "integral operator form", or in an explicit form as follows.

$$L_{JJ} : \mathcal{O}(U) \to \mathcal{O}(U),$$

$$\begin{aligned}
(L_{JJ}\varphi_J)(x) &= \frac{1}{2\pi i}\int_{\gamma_J} \frac{\varphi_J(\tau)}{R'(\tau)(R(\tau)-x)}d\tau \\
&= \sum_{y\in R^{-1}(x)} \frac{\varphi_J(y)}{(R'(y))^2} + \sum_{c\in C_J} \frac{\varphi_J(c)}{R''(c)(R(c)-x)}, \quad \varphi_J \in \mathcal{O}(U), \quad x \in U.
\end{aligned}$$

$$L_{JF} : \mathcal{O}_0(F) \to \mathcal{O}(U),$$

$$\begin{aligned}
(L_{JF}\varphi_F)(x) &= \frac{1}{2\pi i}\int_{\gamma_J} \frac{\varphi_F(\tau)}{R'(\tau)(R(\tau)-x)}d\tau \\
&= -\sum_{c\in C_F} \frac{\varphi_F(c)}{R''(c)(R(c)-x)}, \quad \varphi_F \in \mathcal{O}_0(F), \quad x \in U.
\end{aligned}$$

$$L_{FJ} : \mathcal{O}(U) \to \mathcal{O}_0(F),$$

$$\begin{aligned}
(L_{FJ}\varphi_J)(x) &= \frac{1}{2\pi i}\int_{\gamma_F} \frac{\varphi_J(\tau)}{R'(\tau)(R(\tau)-x)}d\tau \\
&= -\sum_{c\in C_J} \frac{\varphi_J(c)}{R''(c)(R(c)-x)}, \quad \varphi_J \in \mathcal{O}(U), \quad x \in F.
\end{aligned}$$

$$L_{FF} : \mathcal{O}_0(F) \to \mathcal{O}_0(F),$$

$$(L_{FF}\varphi_F)(x) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\varphi_F(\tau)}{R'(\tau)(R(\tau)-x)}d\tau$$

$$= \sum_{y\in R^{-1}(x)} \frac{\varphi_F(y)}{(R'(y))^2} + \sum_{c\in C_F} \frac{\varphi_F(c)}{R''(c)(R(c)-x)}, \quad \varphi_F \in \mathcal{O}_0(F), \quad x \in F.$$

For a prehyperfunction $\varphi \in \mathcal{H}(U)$ with $\varphi = \varphi_J \oplus \varphi_F$, we have the following :

$$(L\varphi)(x) \;=\; (L_{JJ}\varphi_J + L_{JF}\varphi_F) \oplus (L_{FJ}\varphi_J + L_{FF}\varphi_F).$$

By taking neighborhoods $U_k, k = 0, 1, 2, \cdots$ of the Julia set J by $U_k = R^{-k}(U)$, we see that

$$\mathcal{H}(U_0) \subset \mathcal{H}(U_1) \subset \cdots \subset \mathcal{H}(U_k) \subset \cdots \subset \mathcal{H}(J)$$

and

$$\mathcal{H}(J) = \bigcup_{k=0}^{\infty} \mathcal{H}(U_k).$$

The complex Ruelle operators $L : \mathcal{H}(U_k) \to \mathcal{H}(U_k)$ commute with the natural inclusions $\mathcal{H}(U_k) \subset \mathcal{H}(U_{k+1})$ and define a complex linear operator $L : \mathcal{H}(J) \to \mathcal{H}(J)$. In fact, $L(\mathcal{H}(U_{k+1})) \subset \mathcal{H}(U_k)$. Note that when we look for its eigen values and eigen functions, we have to consider invariant subspaces. We shall consider the space of holomorphic functions $\mathcal{O}(F \setminus P_F)$. We denote by $\mathcal{O}_0(\mathbb{C} \setminus P_F)$ the space of entire meromorphic functions with all poles in the postcritical set P. The complex Ruelle operator operates on the space of prehyperfunctions $\mathcal{H}_0(\mathbb{C} \setminus P_F) = \mathcal{O}_0(\mathbb{C} \setminus P_F) \oplus \mathcal{O}_0(F)$.

3. Dual spaces and adjoint Ruelle operator

In this section, we define the dual operator of the complex Ruelle operator $L : \mathcal{H}(J) \to \mathcal{H}(J)$ defiined in the previous section. The topology of $\mathcal{O}(J)$ is understood as the uniform convergence in some neighborhood of J.

DEFINITION 3.1 A complex linear functional $\Phi : \mathcal{O}(J) \to \mathbb{C}$ is said to be *holomorphic* if the value $\Phi[g_\mu]$ depends holomorphically upon μ for holomorphic family of functions g_μ.

DEFINITION 3.2 The *dual space* $\mathcal{O}^*(J)$ is the space of continuous, complex linear, and holomorphic functionals $\Phi : \mathcal{O}(J) \to \mathbb{C}$.

DEFINITION 3.3 Function $\chi_\zeta(z) = \frac{1}{z-\zeta}$ is called the *unit pole* at ζ. For each $\zeta \in F$, $\chi_\zeta \in \mathcal{O}(J)$, and for each $\zeta \in J$, $\chi_\zeta \in \mathcal{O}_0(F)$.

Unit pole χ_ζ is also called the *Cauchy kernel* when it is used as a kernel of an integral operator. As is well known in the theory of hyperfunctions, the holomorphic functional defined by the Cauchy kernel χ_ζ behaves as the Dirac's delta function,

$$f(\zeta) = \frac{1}{2\pi i}\int_{\gamma_F} \chi_\zeta(\tau) f(\tau) d\tau, \quad \text{for} \quad f \in \mathcal{O}_0(F), \zeta \in F$$

and

$$g(\zeta) = \frac{1}{2\pi i}\int_{\gamma_J} \chi_\zeta(\tau) g(\tau) d\tau, \quad \text{for} \quad g \in \mathcal{O}(J), \zeta \in U.$$

Holomorphic linear functionals can be represented by holomorphic functions as described in the following proposition.

PROPOSITION 3.4 For each $\Phi \in \mathcal{O}^*(J)$, there exists an $f \in \mathcal{O}_0(F)$, such that

$$\Phi[g] = \frac{1}{2\pi i}\int_{\gamma_F} f(\tau) g(\tau) d\tau, \quad \text{for} \quad g \in \mathcal{O}(J),$$

where the integration path γ_F is taken as in the previous section.

PROOF Let $\chi_\zeta(z) = \frac{1}{z-\zeta}$ be the unit pole at ζ. For each $\zeta \in F$, $\chi_\zeta \in \mathcal{O}(J)$. Hence χ_ζ is a holomorphic family of functions in $\mathcal{O}(J)$. Therefore, by setting $f(\zeta) = \Phi[\chi_\zeta]$, $f : F \to \mathbb{C}$ is a holomorphic function. By the continuity of the functional Φ, we see that $\lim_{\zeta\to\infty} f(\zeta) = 0$, hence $f \in \mathcal{O}_0(F)$. For $g \in \mathcal{O}(J)$, take a neighborhoood U of J adapted to g. Then for $z \in U$, by Cauchy's integration formula, we have

$$g(z) = \frac{1}{2\pi i}\int_{\gamma_J} \chi_z(\zeta) g(\zeta) d\zeta.$$

Hence, we can compute the value $\Phi[g]$ as follows.

$$\begin{aligned}
\Phi[g] = \Phi[\frac{1}{2\pi i}\int_{\gamma_J} \chi_z(\zeta) g(\zeta) d\zeta] &= \Phi[\frac{-1}{2\pi i}\int_{\gamma_J} g(\zeta) \chi_\zeta(z) d\zeta] \\
= \frac{-1}{2\pi i}\int_{\gamma_J} g(\zeta) \Phi[\chi_\zeta] d\zeta &= \frac{-1}{2\pi i}\int_{\gamma_J} g(\zeta) f(\zeta) d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_F} g(\zeta) f(\zeta) d\zeta.
\end{aligned}$$

The last eqality holds since g and f are holomorphic in $U \setminus J$.

Note that such a function $f \in \mathcal{O}_0(F)$ is unique since

$$f(\zeta) = \frac{1}{2\pi i}\int_{\gamma_F} f(\tau) \chi_\zeta(\tau) d\tau = \Phi[\chi_\zeta], \quad \text{for} \quad \zeta \in F.$$

This proves the following proposition.

PROPOSITION 3.5 The dual space $\mathcal{O}^*(J)$ is isomorphic to $\mathcal{O}_0(F)$.

Next, as for the dual space $\mathcal{O}_0^*(F)$ of $\mathcal{O}_0(F)$, we have the following.

PROPOSITION 3.6 For each $\Psi \in \mathcal{O}_0^*(F)$, there exist a germ of holomorphic function $g \in \mathcal{O}(J)$ and a neighborhood U of J adapted to g, such that

$$\Psi[f] = \frac{1}{2\pi i}\int_{\gamma_J} g(\tau)f(\tau)d\tau, \quad \text{for} \quad f \in \mathcal{O}_0(F).$$

PROOF Define a function $g \in \mathcal{O}(J)$ by

$$g(x) = \Psi[\chi_x], \quad \text{for} \quad x \in J.$$

As Ψ is holomorphic and J is a perfect set, $g(x)$ defines a germ of holomorphic function on J. This function $\Psi[\chi_x]$ extends holomorphically to an adapted neighbourhood of J. For $f \in \mathcal{O}_0(F)$ and $z \in F$,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_F} \chi_z(\zeta)f(\zeta)d\zeta.$$

Hence,

$$\begin{aligned}
\Psi[f] = \Psi[\frac{1}{2\pi i}\int_{\gamma_F} \chi_z(\zeta)f(\zeta)d\zeta] &= \Psi[\frac{-1}{2\pi i}\int_{\gamma_F} f(\zeta)\chi_\zeta(z)d\zeta] \\
= \frac{-1}{2\pi i}\int_{\gamma_F} f(\zeta)\Psi[\chi_\zeta]d\zeta &= \frac{-1}{2\pi i}\int_{\gamma_F} f(\zeta)g(\zeta)d\zeta \\
&= \frac{1}{2\pi i}\int_{\gamma_J} f(\zeta)g(\zeta)d\zeta
\end{aligned}$$

PROPOSITION 3.7 The dual space $\mathcal{O}_0^*(F)$ is isomorphic to $\mathcal{O}(J)$.

Isomorphisms in Propositions 3.4 and 3.6 are called *Cauchy transformations*, since they are defined by the Chauchy kernel $\chi_\zeta(z)$.

DEFINITION 3.8 The *pairings* $\langle f, g\rangle_F$ and $\langle g, f\rangle_J$ are defined for $g \in \mathcal{O}(J)$ and $f \in \mathcal{O}_0(F)$ by

$$\langle f, g\rangle_F = \frac{1}{2\pi i}\int_{\gamma_F} f(\tau)g(\tau)d\tau,$$

and

$$\langle g, f\rangle_J = \frac{1}{2\pi i}\int_{\gamma_J} g(\tau)f(\tau)d\tau.$$

For $\varphi = \varphi_J \oplus \varphi_F \in \mathcal{H}(J)$ and $\psi = \psi^J \oplus \psi^F \in \mathcal{O}_0(F) \oplus \mathcal{O}(J) \simeq \mathcal{O}^*(J) \oplus \mathcal{O}_0^*(F) = \mathcal{H}^*(J)$, the pairing $\langle \psi, \varphi \rangle$ is defined by

$$\langle \psi, \varphi \rangle = \langle \psi^J, \varphi_J \rangle_F + \langle \psi^F, \varphi_F \rangle_J.$$

Usually, the integration paths γ_F anf γ_J depend upon the choice of the adapted neighborhood U of J. However, when we fix the adapted neighborhood U, we shall also consider the integrations along these integration paths. We shall abuse this notation to represent such integrals, especially, when we consider kernels of integral operators. In such a case, since the integration path is fixed, the abuse of notation will not cause an ambiguity. Using the notations above for pairings of prehyperhunctions, we can reformulate the complex Ruelle operator as follows. Let $K_x(\tau) = \frac{1}{R'(\tau)(R(\tau)-x)}$ denote the kernel of the complex Ruelle operator L. As rational functions can be decomposed into partial fractions, we have the following proposition.

PROPOSITION 3.9 For each $x \in F \setminus P$, K_x can be decomposed into partial fractions :

$$K_x = \sum_{y \in R^{-1}(x)} \frac{\chi_y}{(R'(y))^2} + \sum_{c \in C} \frac{\chi_x(R(c))\chi_c}{R''(c)}.$$

We decompose the kernel K_x into three parts.

$$\begin{aligned} K_x^L &= \sum_{y \in R^{-1}(x)} \frac{\chi_y}{(R'(y))^2}, \\ K_x^J &= \sum_{c \in C_J} \frac{\chi_x(R(c))\chi_c}{R''(c)}, \\ K_x^F &= \sum_{c \in C_F} \frac{\chi_x(R(c))\chi_c}{R''(c)}. \end{aligned}$$

We see that $K_x^J \in \mathcal{O}_0(F)$ and $K_x^F \in \mathcal{O}(J)$. In the following, we shall assume $x \in U \setminus J$ or $x \in R(U \setminus J)$, as x represents the variable of the image function of the Ruelle operator. The singularities $y \in R^{-1}(x)$ of K_x^L are redarded to be in the appropriate adapted neighborhood, so that they belong to the annulus between γ_J and γ_F. The components of L can be rewritten as follows.

$$(L_{JJ}\varphi_J)(x) = \langle K_x^L + K_x^J, \varphi_J \rangle_J,$$

$$(L_{JF}\varphi_F)(x) = \langle K_x^L + K_x^F, \varphi_F\rangle_J = -\langle K_x^F, \varphi_F\rangle_F,$$
$$(L_{FJ}\varphi_J)(x) = \langle K_x^L + K_x^J, \varphi_J\rangle_F = -\langle K_x^J, \varphi_J\rangle_J,$$
$$(L_{FF}\varphi_F)(x) = \langle K_x^L + K_x^F, \varphi_F\rangle_F.$$

The image $L\chi_\zeta$ of the unit pole at $\zeta \in \mathbb{C} \setminus C$ can be computed directly as follows.

PROPOSITION 3.10

$$L\chi_\zeta = \frac{\chi_{R(\zeta)}}{R'(\zeta)} + \sum_{c\in C} \frac{\chi_\zeta(c)\chi_{R(c)}}{R''(c)}.$$

PROOF If $\zeta \in F \setminus C_F$, then $\chi_\zeta \in \mathcal{O}(J)$. For $x \in U$, we regard that x and its backward image are included in the annulus domain between the two integration paths γ_J and γ_F. The resudue formula is applied to the outside domain of the integration path instead of the inside domain. Or equivalently, we use the fact that for rational functions, the sum of residues of all poles in the Riemann sphere vanishes.

$$\begin{aligned}
(L_{JJ}\chi_\zeta)(x) &= \frac{1}{2\pi i}\int_{\gamma_J} \frac{d\tau}{R'(\tau)(R(\tau)-x)(\tau-\zeta)} \\
&= \frac{-1}{R'(\zeta)(R(\zeta)-x)} - \sum_{c\in C_F} \frac{1}{R''(c)(R(c)-x)(c-\zeta)} \\
&= \frac{\chi_{R(\zeta)}(x)}{R'(\zeta)} + \sum_{c\in C_F} \frac{\chi_\zeta(c)\chi_{R(c)}(x)}{R''(c)}, \\
(L_{FJ}\chi_\zeta)(x) &= \frac{1}{2\pi i}\int_{\gamma_F} \frac{d\tau}{R'(\tau)(R(\tau)-x)(\tau-\zeta)} \\
&= -\sum_{c\in C_J} \frac{1}{R''(c)(R(c)-x)(c-\zeta)} = \sum_{c\in C_J} \frac{\chi_\zeta(c)\chi_{R(c)}(x)}{R''(c)}.
\end{aligned}$$

Similarly, if $\zeta \in J \setminus C_J$, then $\chi_\zeta \in \mathcal{O}_0(F)$, and

$$\begin{aligned}
(L_{JF}\chi_\zeta)(x) &= \frac{1}{2\pi i}\int_{\gamma_J} \frac{d\tau}{R'(\tau)(R(\tau)-x)(\tau-\zeta)} \\
&= -\sum_{c\in C_F} \frac{1}{R''(c)(R(c)-x)(c-\zeta)} = \sum_{c\in C_F} \frac{\chi_\zeta(c)\chi_{R(c)}(x)}{R''(c)}. \\
(L_{FF}\chi_\zeta)(x) &= \frac{1}{2\pi i}\int_{\gamma_F} \frac{d\tau}{R'(\tau)(R(\tau)-x)(\tau-\zeta)}
\end{aligned}$$

$$= \frac{-1}{R'(\zeta)(R(\zeta)-x)} - \sum_{c\in C_J} \frac{1}{R''(c)(R(c)-x)(c-\zeta)}$$
$$= \frac{\chi_{R(\zeta)}(x)}{R'(\zeta)} + \sum_{c\in C_J} \frac{\chi_\zeta(c)\chi_{R(c)}(x)}{R''(c)}.$$

This completes the proof since $L = L_{JJ}+L_{FJ}$ on $\mathcal{O}(J)$ and $L = L_{JF}+L_{FF}$ on $\mathcal{O}_0(F)$.

Let $L^* : \mathcal{H}^*(J) \to \mathcal{H}^*(J)$ denote the dual operator of the complex Ruelle operator $L : \mathcal{H}(J) \to \mathcal{H}(J)$. And let $\mathcal{L}^* : \mathcal{O}_0(F) \oplus \mathcal{O}(J) \to \mathcal{O}_0(F) \oplus \mathcal{O}(J)$ denote its representation via the Cauchy transformation. We call this operator $\mathcal{L}^*$ the *adjoint Ruelle operator*. The dual space of $\mathcal{H}(J)$ will be denoted by $\mathcal{H}^*(J)$, and we abuse this notation to denote the "adjoint" space $\mathcal{O}_0(F) \oplus \mathcal{O}(J)$, too. The components of $\mathcal{L}^*$ with respect to the natural decomposition will be denoted as

$$\mathcal{L}^* = \begin{pmatrix} \mathcal{L}^*_{FF} & \mathcal{L}^*_{FJ} \\ \mathcal{L}^*_{JF} & \mathcal{L}^*_{JJ} \end{pmatrix}$$

The explicit formula for the adjoint Ruelle operator can be computed directly as follows.

PROPOSITION 3.11 For $\psi = \psi^J \oplus \psi^F$ with $\psi^J \in \mathcal{O}_0(F) \simeq \mathcal{O}^*(J)$, and $\psi^F \in \mathcal{O}(J) \simeq \mathcal{O}_0^*(F)$,

$$(\mathcal{L}^*\psi)(z) = \left(\frac{\psi^J(R(z))}{R'(z)} - \sum_{c\in C_F} \frac{\psi^J(R(c))}{R''(c)}\chi_c(z) + \sum_{c\in C_J} \frac{\psi^F(R(c))}{R''(c)}\chi_c(z)\right)$$
$$\oplus \left(\sum_{c\in C_F} \frac{\psi^J(R(c))}{R''(c)}\chi_c(z) + \frac{\psi^F(R(z))}{R'(z)} - \sum_{c\in C_J} \frac{\psi^F(R(c))}{R''(c)}\chi_c(z)\right).$$

And in $U \setminus J$, where ψ defines a holomorphic function,

$$\mathcal{L}^*\psi = \frac{\psi \circ R}{R'}.$$

PROOF The proof is straightforward by direct computations applying the residue theorem. Let $\psi^J \in \mathcal{O}_0(F) \simeq \mathcal{O}^*(J)$, and compute $\mathcal{L}^*\psi^J$ as follows.

First, let us compute the component $\mathcal{L}^*_{FF}\psi^J \in \mathcal{O}_0(F)$. For $z \in F$, $\chi_z \in \mathcal{O}(J)$. Hence,

$$(\mathcal{L}^*_{FF}\psi^J)(z) = \langle \psi^J, L_{JJ}\chi_z\rangle_F = \langle \psi^J, \frac{\chi_{R(z)}}{R'(z)} + \sum_{c\in C_F} \frac{\chi_z(c)\chi_{R(c)}}{R''(c)}\rangle_F$$

$$= \frac{\psi^J(R(z))}{R'(z)} + \sum_{c\in C_F} \frac{\chi_z(c)\psi^J(R(c))}{R''(c)} = \frac{\psi^J(R(z))}{R'(z)} - \sum_{c\in C_F} \frac{\psi^J(R(c))}{R''(c)}\chi_c(z).$$

The component $\mathcal{L}^*_{JF}\psi^J \in \mathcal{O}(J)$ is computed as follows. For $z \in J$, $\chi_z \in \mathcal{O}_0(F)$. Hence,

$$\begin{aligned}(\mathcal{L}^*_{JF}\psi^J)(z) &= \langle\psi^J, L_{JF}\chi_z\rangle_J = \langle\psi^J, \sum_{c\in C_F} \frac{\chi_z(c)\chi_{R(c)}}{R''(c)}\rangle_J \\ &= \sum_{c\in C_F} \frac{\chi_z(c)}{R''(c)}\langle\psi^J, \chi_{R(c)}\rangle_J = \sum_{c\in C_F} \frac{\psi^J(R(c))}{R''(c)}\chi_c(z).\end{aligned}$$

Similarly, components $\mathcal{L}^*_{FJ}\psi^F \in \mathcal{O}_0(F)$ and $\mathcal{L}^*_{JJ}\psi^F \in \mathcal{O}(J)$ are computed as follows. For $z \in F$, $\chi_z \in \mathcal{O}(J)$. Hence,

$$\begin{aligned}(\mathcal{L}^*_{FJ}\psi^F)(z) &= \langle\psi^F, L_{FJ}\chi_z\rangle_F = \langle\psi^F, \sum_{c\in C_J} \frac{\chi_z(c)\chi_{R(c)}}{R''(c)}\rangle_F \\ &= \sum_{c\in C_J} \frac{\chi_z(c)}{R''(c)}\langle\psi^F, \chi_{R(c)}\rangle_F = \sum_{c\in C_J} \frac{\psi^F(R(c))}{R''(c)}\chi_c(z).\end{aligned}$$

Finally, for $z \in J$, $\chi_z \in \mathcal{O}_0(F)$, and

$$(\mathcal{L}^*_{JJ}\psi^F)(z) = \langle\psi^F, L_{FF}\chi_z\rangle_J = \langle\psi^F, \frac{\chi_{R(z)}}{R'(z)} + \sum_{c\in C_J} \frac{\chi_z(c)\chi_{R(c)}}{R''(c)}\rangle_J$$

$$= \frac{\psi^F(R(z))}{R'(z)} + \sum_{c\in C_J} \frac{\chi_z(c)\psi^F(R(c))}{R''(c)} = \frac{\psi^F(R(z))}{R'(z)} - \sum_{c\in C_J} \frac{\psi^F(R(c))}{R''(c)}\chi_c(z).$$

4. Trace of Complex Ruelle operator

Let U_0 be a neighborhood of J adapted to some prehyperfunction $\varphi \in \mathcal{H}(U_0)$. Complex Ruelle operator $L : \mathcal{H}(U_0) \to \mathcal{H}(U_0)$ is defined by

$$(L\varphi)(x) = \sum_{y\in R^{-1}(x)} \frac{\varphi(y)}{(R'(y))^2}, \quad \varphi \in \mathcal{H}(U_0), \quad x \in U_0 \setminus J.$$

The image prehyperfunction $L\varphi \in \mathcal{H}(U_0)$ can be extended holomorphically to a prehyperfunction defined in a larger domain $U_{-1} \setminus J$ with

$$U_{-1} = \{z \in \mathbb{C} \setminus P_F \mid R^{-1}(z) \subset U_0\}.$$

By defining U_{-k} inductively for $k = 1, 2, \cdots$, We have a sequence of spaces of prehyperfunctions :

$$\mathcal{H}(\mathbb{C} \setminus P_F) \subset \cdots \subset \mathcal{H}(U_{-1}) \subset \mathcal{H}(U_0) \subset \mathcal{H}(U_1) \subset \cdots \subset \mathcal{H}(J),$$

and

$$\mathcal{H}(\mathbb{C} \setminus P_F) = \cap_{k=0}^{\infty} \mathcal{H}(U_{-k}).$$

Hence, if the complex Ruelle operator L has an eigen prehyperfunction $\varphi \in \mathcal{H}(J)$, then it must be in $\mathcal{H}(\mathbb{C} \setminus P_F)$. This subspace $\mathcal{H}(\mathbb{C} \setminus P_F)$ is mapped into itself by L. In the following, we regard L as a complex linear operator

$$L : \mathcal{H}(\mathbb{C} \setminus P_F) \to \mathcal{H}(\mathbb{C} \setminus P_F).$$

In the following, we denote the m-times composition of the rarinal function R by R_m, for $m = 0, 1, 2, \cdots$, as we shall use derivatives of R_m. In order to emphasize the iterated inverse of maps, we use usual notation R^{-m}, too. We can compute the iterates L^m of the complex Ruelle operator as follows.

PROPOSITION 4.1

$$(L^m \varphi)(x) \;=\; \sum_{y \in R^{-m}(x)} \frac{\varphi(y)}{(R'_m(y))^2} \;=\; \frac{1}{2\pi i} \int_{\gamma_J + \gamma_F} \frac{\varphi(\tau)}{R'_m(\tau)(R_m(\tau) - x)} d\tau.$$

Note that the preimages of the integration paths are homologous in $F \setminus P_F$ to the initial integration path γ_J and γ_F.

Let $\mathrm{Fix}(R_m)$ denote the set of fixed points of R_m, and let $C(R_m)$ denote the set of critical points of R_m. The components of L^m with respect to the natural decomposition $\mathcal{H}(\mathbb{C} \setminus P_F) = \mathcal{O}(\mathbb{C} \setminus P_F) \oplus \mathcal{O}_0(F)$ will be denoted as

$$L^m \;=\; \begin{pmatrix} L_{JJ}^{(m)} & L_{JF}^{(m)} \\ L_{FJ}^{(m)} & L_{FF}^{(m)} \end{pmatrix}.$$

We assume that the infinity is an attractive fixed point of R and R takes the form $R(z) = \sigma z + \cdots$ near the infinity. For the case where the infinity is a superattractive fixed point, set $\sigma = \infty$ in the following propositions.

PROPOSITION 4.2 The traces of L^m, $L_{JJ}^{(m)}$, and $L_{FF}^{(m)}$, $m = 1, 2, \cdots$ are given by

$$\mathrm{trace}[L^m] \;=\; 0,$$

$$\mathrm{trace}[L_{JJ}^{(m)}] \;=\; \frac{1}{\sigma^m - 1} - \sum_{x \in \mathrm{Fix}(R_m) \cap J} \frac{1}{R'_m(x)} + \sum_{y \in C(R_m) \cap J} \frac{1}{R''_m(y)(R_m(y) - y)},$$

$$\mathrm{trace}[L_{FF}^{(m)}] = \frac{-1}{\sigma^m - 1} + \frac{1}{\sigma^m} + \sum_{y \in C(R_m) \cap F} \frac{1}{R''_m(y)(R_m(y) - y)}.$$

PROOF The proof is straightforward by a direct computation using the resudue formula.

$$\mathrm{trace}[L^m] = \frac{1}{2\pi i} \int_{\gamma_J + \gamma_F} \frac{d\tau}{R'_m(\tau)(R_m(\tau) - \tau)} = 0.$$

$$\mathrm{trace}[L_{JJ}^{(m)}] = \frac{1}{2\pi i} \int_{\gamma_J} \frac{d\tau}{R'_m(\tau)(R_m(\tau) - \tau)}$$

$$= \sum_{x \in \mathrm{Fix}(R_m) \cap J} \frac{1}{R'_m(x)(R'_m(x) - 1)} + \sum_{y \in C(R_m) \cap J} \frac{1}{R''_m(y)(R_m(y) - y)}$$

$$= \sum_{x \in \mathrm{Fix}(R_m) \cap J} \frac{1}{R'_m(x) - 1} - \sum_{x \in \mathrm{Fix}(R_m) \cap J} \frac{1}{R'_m(x)} + \sum_{y \in C(R_m) \cap J} \frac{1}{R''_m(y)(R_m(y) - y)}.$$

As

$$\sum_{x \in \mathrm{Fix}(R_m) \cap J} \frac{1}{R'_m(x) - 1}$$

is a sum of resudues of rational function $\frac{1}{R_m(\tau) - \tau}$, by taking the residue at the infinity into considerations, we obtain the formula of the proposition. Similarly, as the integrand function is meromorphic in $F \cup \infty$, and F is the basin of attraction of the attractive fixed point, we have

$$\mathrm{trace}[L_{FF}^{(m)}] = \frac{1}{2\pi i} \int_{\gamma_F} \frac{d\tau}{R'_m(\tau)(R_m(\tau) - \tau)}$$

$$= \mathrm{Res}_{\tau=\infty}\left(\frac{1}{R'_m(\tau)(R_m(\tau) - \tau)}\right) + \sum_{y \in C(R_m) \cap F} \frac{1}{R''_m(y)(R_m(y) - y)}$$

$$= \frac{-1}{\sigma^m - 1} + \frac{1}{\sigma^m} + \sum_{y \in C(R_m) \cap F} \frac{1}{R''_m(y)(R_m(y) - y)}.$$

5. Fredholm determinant and Ruelle's dynamical ζ-function

The Fredholm determinant $D(\lambda)$ of linear operator L is defined formally by

$$D(\lambda) = \det(I - \lambda L) = \exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^m}{m} \mathrm{trace}[L^m]\right).$$

As we computed in the preceeding section, we have $\mathrm{trace}[L^m] = 0$ for $m = 1, 2, \cdots$. Hence $D(\lambda) = 1$ holds for all $\lambda \in \mathbb{C}$.

Definition 5.1

$$D_J(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\mathrm{trace}[L_{JJ}^{(m)}]\right),$$

and

$$D_F(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\mathrm{trace}[L_{FF}^{(m)}]\right).$$

As trace$[L^m]$ =trace$[L_{JJ}^{(m)}]$+trace$[L_{FF}^{(m)}]$, $m = 1, 2, \cdots$, we have

$$D(\lambda) = D_J(\lambda)D_F(\lambda) = 1.$$

Let

$$D_J^{(1)}(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\sum_{x\in\mathrm{Fix}(R_m)\cap J}\frac{1}{R'_m(x)-1}\right),$$

$$D_J^{(2)}(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\sum_{x\in\mathrm{Fix}(R_m)\cap J}\frac{-1}{R'_m(x)}\right),$$

$$D_J^{(3)}(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\sum_{y\in C(R_m)\cap J}\frac{1}{R''_m(y)(R_m(y)-y)}\right)$$

denote the factors of $D_J(\lambda)$, and let

$$D_F^{(1)}(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\frac{-1}{\sigma^m-1}\right),$$

$$D_F^{(2)}(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\frac{1}{\sigma^m}\right),$$

$$D_F^{(3)}(\lambda) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\sum_{y\in C(R_m)\cap F}\frac{1}{R''_m(y)(R_m(y)-y)}\right)$$

denote the factors of $D_F(\lambda)$.

Proposition 5.2 The factor $D_J^{(1)}(\lambda)$ converges for $|\lambda| < |\sigma|$ and extends holomorphically to an entire function

$$D_J^{(1)}(\lambda) = \prod_{k=1}^{\infty}(1-\frac{\lambda}{\sigma^k}).$$

And the factor $D_F^{(1)}(\lambda)$ converges for $|\lambda| < |\sigma|$ and extends analytically to an entire meromorphic function

$$D_F^{(1)}(\lambda) = \frac{1}{D_J^{(1)}(\lambda)} = \prod_{k=1}^{\infty}(\frac{\sigma^k}{\sigma^k - \lambda}).$$

PROOF By a straightforward calculation, we obtain the following. We assumed that $|\sigma| > 1$. For $|\lambda| < |\sigma|$, we have

$$\begin{aligned} D_J^{(1)}(\lambda) &= \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\frac{1}{\sigma^m - 1}\right) = \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\sum_{k=1}^{\infty}\frac{1}{\sigma^{mk}}\right) \\ &= \exp\left(\sum_{k=1}^{\infty}(-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\lambda}{\sigma^k})^m)\right) = \exp\left(\sum_{k=1}^{\infty}\log(1 - \frac{\lambda}{\sigma^k})\right) \\ &= \prod_{k=1}^{\infty}(1 - \frac{\lambda}{\sigma^k}). \end{aligned}$$

The last expression of $D_J^{(1)}(\lambda)$ in an infinite product form shows that it extends holomorphically to an entire function. The rest of the proof is easy. This factor of the Fredholm determinant is same as the Fredholm Determinant of the transfer operator $L_{(1)} : \mathcal{H}(J) \to \mathcal{H}(J)$ defined by

$$(L_{(1)}\varphi)(x) = \sum_{y\in R^{-1}(x)}\frac{\varphi(y)}{R'(y)} = \frac{1}{2\pi i}\int_{\gamma_J+\gamma_F}\frac{\varphi(\tau)}{R(\tau) - x}d\tau.$$

The complex version of Ruelle's dynamical ζ-function is defined as follows.

DEFINITION 5.3 Complex *dynamical ζ-function* for the Julia set J is defined by

$$\zeta_J(\lambda) = \exp\left(\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\sum_{x\in\mathrm{Fix}(R_m)\cap J}\frac{1}{R'_m(x)}\right),$$

and complex *dynamical ζ-function* for the Fatou set F is defined by

$$\zeta_F(\lambda) = \exp\left(\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\frac{1}{\sigma^m}\right) = \frac{\sigma}{\sigma - \lambda}.$$

As is easily seen, we have

$$D_J^{(2)}(\lambda) = \zeta_J(\lambda)$$

and

$$D_F^{(2)}(\lambda) = \frac{1}{\zeta_F(\lambda)}.$$

For a periodic point x of R, let $p\langle x\rangle$ denote its prime period, let $\langle x\rangle = \{x_1, x_2, \cdots, x_{p\langle x\rangle}\}$ denote its cycle, and let $\rho\langle x\rangle$ denote the eigenvalue of the cycle. For each prime cycle $\langle x\rangle$ of R, we define the *ζ-function* $\zeta_{\langle x\rangle}(\lambda)$ of the prime cycle $\langle x\rangle$ by

$$\zeta_{\langle x\rangle}(\lambda) = \left(1 - \frac{\lambda^{p\langle x\rangle}}{\rho\langle x\rangle}\right)^{-1}.$$

The complex dynamical ζ-function has an "Euler decomposition"

$$\zeta_J(\lambda) = \prod_{\langle x\rangle} \zeta_{\langle x\rangle}(\lambda),$$

where $\langle x\rangle$ ranges over all the prime cycles in J.

LEMMA 5.4 Let $s \geq 0$ and $t \geq 1$ be integers. If $c \in C(R)$ and $y \in R^{-s}(c)$, then

$$R''_{t+s}(y) = R''(c)R'_{t-1}(R(c))(R'_s(y))^2.$$

This lemma shows that the second derivatve of a point in the backward image can be described as a product of three terms. The proof is straightforward and left to the reader. In order to decompose the terms $D_J^{(3)}(\lambda)$ and $D_F^{(3)}(\lambda)$, we define η-functions as follows.

DEFINITION 5.5 For each critical point $c \in C(R)$, the *η-function* $\eta_c(\lambda)$ for c is defined by

$$\eta_c(\lambda) = \exp\left(\frac{-1}{R''(c)} \sum_{m=1}^{\infty} \frac{\lambda^m}{m} \sum_{t=1}^{m} \frac{1}{R'_{t-1}(R(c))} \sum_{y \in R^{-(m-t)}(c)} \frac{1}{(R'_{m-t}(y))^2 (R_t(c) - y)}\right).$$

The *η-function* of dynamical system R is defined by

$$\eta(\lambda) = \prod_{c \in C(R)} \eta_c(\lambda).$$

As critical points of R_m are in the backward image of the critical points, we have

$$\eta(\lambda) = \exp\left(-\sum_{m=1}^{\infty} \frac{\lambda^m}{m} \sum_{y \in C(R_m)} \frac{1}{R''_m(y)(R_m(y) - y)}\right).$$

Clearly, we have

$$D_J^{(3)}(\lambda) = \prod_{c \in C(R) \cap J} \eta_c(\lambda),$$

$$D_F^{(3)}(\lambda) = \prod_{c \in C(R) \cap F} \eta_c(\lambda),$$

and

$$D_J^{(3)}(\lambda) D_F^{(3)}(\lambda) = \eta(\lambda).$$

Putting all together, we obtain the following proposition.

PROPOSITION 5.6 Ruelle's dynamical ζ-function can be expressed in terms of η-function. If the infinity is an attractive fixed point of R with eigenvalue σ^{-1}, then

$$\zeta_J(\lambda) = (1 - \frac{\lambda}{\sigma}) \frac{1}{\eta(\lambda)}.$$

If the infinity is a superattactive fixed point of R, then

$$\zeta_J(\lambda) = \frac{1}{\eta(\lambda)}.$$

6. Dynamical η-function and critical recurrence rate

Our expression of dynamical η-function gives some information about the zeros or poles of the dynamical ζ-function.

DEFINITION 6.1 Positive number θ is called a *critical recurrence rate* if there exists a positive number α, such that

$$\left| \sum_{y \in C(R_m)} \frac{1}{R_m''(y)(R_m(y) - y)} \right| \leq \alpha \theta^m, \quad \text{for} \quad m = 1, 2, \cdots.$$

DEFINITION 6.2 Rational function $R : \mathbb{C} \to \mathbb{C}$ is said to satisfy the *forward/backward Collet-Eckmann condition* if there exists a positive critical recurrence rate.

Clearly, we have the following theorem.

THEOREM 6.2 If R satisfies the forward/backward Collet-Eckmann condition with a critical recurrence rate $\theta > 0$, then $\eta(\lambda)$ is holomorphic for $|\lambda| < \theta^{-1}$. And consequently, $\zeta_J(\lambda)/(1 - \frac{\lambda}{\sigma})$ extends holomorphically to the disk $|\lambda| < \theta^{-1}$, and does not vanish there.

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