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On the Borel summability of divergent solutions of parabolic type equations and Barnes generalized hypergeometric functions

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Introduction

If one considers the Cauchy problem for a non-Kowalevskian equation in the analytic category, the uniquely determined formal solution is divergent by a suitable choice of data, which is often called the inverse theorem of Cauchy–Kowalevski's theorem (cf. Miyake [9] and Mizohata [13]). After the theorem was established by Mizohata, the rate of divergence of formal solutions, which is often called the formal Gevrey order, were studied by Miyake [10] and Miyake–Hashimoto [12] for linear equations and by Gérard–Tahara [6] for nonliear equatons. On the other hand, the existence of asymptotic solutions of divergent solution was studied by Ōuchi [14] for general nonlinear equations which are defined in a sector domain of small opening angle in the plane of time variable. But if we ask the existence of such an asymptotic solution in a sector of large opening angle, we can not expect the existence of asymptotic solutions without any condition for the Cauchy data. From such a point of view, Lutz–Miyake–Schäfke [8] studied divergent solutions of the Cauchy problem of the (complex) heat equation, and gave a necessary and sufficient condition for the Borel summability of divergent solutions in terms of the analytic continuation property and its growth condition of the Cauchy data. The definition of Borel summability will be given in Section 1. This result was extended for non-Kowalevskian equations of constant coefficients in Balser–Miyake [2] where a sufficient condition for the Borel summability of divergent solutions was obtained. In this lecture we shall study a simple equation of non-Kowalevski type, and show the sufficient condition obtained in [2] is also a necessary condition and give an integral expression of Borel sum by using the kernel function, which is a generalization of results in [8].

We should note here that Theorem 1 for the charcterization of Borel summability was proved by Miyake [11], and the integral representation of Borel sum and the other results on the explicit formula of kernel function were proved by K. Ichinobe which will be published in a forthcoming paper.

1 Statement of Results

We shall study the following Cauchy problem for a partial differential equation of non Kowalevski type.

$$\text{(CP)} \qquad \begin{cases} \partial_t^p u(t,x) = c_0 \partial_x^q u(t,x), & c_0 = p^p/q^q > 0, \\ u(0,x) = \varphi(x), \quad \partial_t^j u(0,x) = 0 & (1 \le j \le p-1) \end{cases}$$

where $t, x \in \mathbb{C}$, $1 \le p < q$ and the Cauchy data $\varphi(x)$ is assumed to be holomorphic in a neighbourhood of the origin, say $\varphi(x) \in \mathcal{O}(r_0)$, the set of holomorphic functions on $D(r_0) := \{x \in \mathbb{C}; |x| \le r_0\}$. We remark that the constant c_0 is only a normalized constant which makes the results more readable. For example, for the equation $\partial_t^p u = \partial_x^q u$, the statement of Theorem 1 is exactly the same.

This Cauchy problem (CP) has a unique formal solution,

$$\hat{u}(t,x) = \sum_{n=0}^{\infty} \frac{\varphi^{(qn)}(x)}{(pn)!} (c_0 t^p)^n . \tag{1.1}$$

By the assumption that $q > p$, this formal power series is divergent in general.

We study this divergent formal solution from the view point of Borel summability which has been established in the study of divergent solutions of ordinary differential equations with irregular singular points, (cf. Balser [1]).

Before stating our results we shall prepare some notations and definitions.

For $d \in \mathbb{R}$, $\alpha > 0$ and ρ $(0 < \rho \le \infty)$, we define

$$S = S(d, \alpha, \rho) = \{t \in \mathbb{C}; |d - \arg(\mathrm{t})| < \alpha/2, \ 0 < |\mathrm{t}| < \rho\},$$

where d, α and ρ are called the direction, the opening angle and the radius of S, respectively.

Let $k > 0$. We define that $\hat{u}(t,x) = \sum_{n \ge 0} u_n(x) t^n \in \mathcal{O}(r_0)[[t]]_{1/k}$, if there exist positive constants C and K such that

$$\max_{|x| \le r_0} |u_n(x)| \le C K^n \Gamma\left(1 + \frac{n}{k}\right), \quad n = 0, 1, 2, \cdots,$$

where Γ denotes the gamma function. By this notation we see that the formal solution $\hat{u}(t,x) \in \mathcal{O}(r_0)[[t]]_{(q-p)/p}$.

Let $\hat{u}(t,x) \in \mathcal{O}(r_0)[[t]]_{1/k}$ and $u(t,x) \in \mathcal{O}(S \times D(r_1))$ where $S = S(d, \alpha, \rho)$ and r_1 is a positive constant such that $0 < r_1 \le r_0$. Then we define that

$$u(t,x) \sim_k \hat{u}(t,x) \quad \text{in S}$$

if for any closed subsector S' of S, there exist positive constants r_2 with $r_2 \le r_1$, C and K such that

$$\max_{|x| \le r_2} \left| u(t,x) - \sum_{n=0}^{N-1} u_n(x) t^n \right| \le C K^N \Gamma\left(1 + \frac{N}{k}\right) |t|^N, \quad t \in S', \ N \ge 1.$$

Let $d \in \mathbb{R}$ and $\hat{u}(t,x) \in \mathcal{O}(r_0)[[t]]_{1/k}$. Then we define that $\hat{u}(t,x) \in \mathcal{O}(r_0)\{t\}_{k,d}$ ($k-$Borel summable in d direction), if there exist $S = S(d,\alpha,\rho)$ with $\alpha > \pi/k$ and r $(0 < r \leq r_0)$ such that $u(t,x) \in \mathcal{O}(S \times D(r))$ with $u(t,x) \sim_k \hat{u}(t,x)$ in S. In this case such a function u is called the $k-$Borel sum of $\hat{u}$ in d direction or the Borel sum for short.

Remark 1. Let $\hat{u} \in \mathcal{O}(r_0)[[t]]_{1/k}$ be given.

(i) If $\alpha < \pi/k$, then there are infitely many u's such that $u \sim_k \hat{u}$ in $S(d,\alpha,\rho)$ for any d and some $\rho > 0$.

(ii) If $\alpha > \pi/k$, then there does not exist such u as above with any condition for $\hat{u}$, but it is unique if it does exist. In this sense the notion of the Borel sum is well defined, and the problem of the Borel summability for a given divergent series has an important meaning. (cf. Balser[1])

Now our first purpose is to characterize the Borel summability of the formal solution $\hat{u}(t,x)$ of the Cauchy problem (CP) given by (1.1), which is stated as follows.

Theorem 1 (Borel summability)

Let $\hat{u}(t,x) \in \mathcal{O}(r_0)[[t]]_{1/k}$ with $k = p/(q-p)$ be the formal solution (1.1) of (CP). *Then it is $k-$Borel summable in d direction if and only if the following two conditions for the Cauchy data $\varphi(x)$ are satisfied:*

(i) *$\varphi(x)$ can be continued analytically in q sectors*

$$\Omega(p,q;d,\varepsilon) = \bigcup_{j=0}^{q-1} S\left(\frac{pd + 2\pi j}{q}, \varepsilon, \infty\right), \quad \exists\varepsilon > 0.$$

(ii) *$\varphi(x)$ has the growth condition of exponential order at most $q/(q-p)$, which means that there exist positive constants C and δ such that*

$$|\varphi(x)| \leq C\exp(\delta|x|^{q/(q-p)}), \quad x \in \Omega(p,q;d,\varepsilon).$$

Related Results

(i) The case of $(p,q) = (1,2)$ (the heat equation) was studied by Lutz–Miyake–Schäfke [8], and the above theorem is a generalization of their result.

(ii) The more general case of parabolic type equation was studied by Balser–Miyake [2] and they give a sufficient condition for the Borel summability. The above theorem shows that their condition is also a necessary condition for a special equation.

The next purpose is to give an integral representation of the Borel sum by using a kernel function. Indeed, in the case of the heat equation, the Borel sum is just given by the integral representation by the heat kernel which was proved in [8].

To state our second result, we need the Barnes generalized hypergeometric series,

$${}_aF_b(\boldsymbol{\alpha};\boldsymbol{\beta};z) = {}_aF_b\left(\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};z\right) := \sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\beta})_n}\frac{z^n}{n!},$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_a) \in \mathbb{C}^a$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_b) \in \mathbb{C}^b$,

$$(\boldsymbol{\alpha})_n = \prod_{\ell=1}^{a} (\alpha_\ell)_n, \quad (\boldsymbol{\beta})_n = \prod_{m=1}^{b} (\beta_m)_n$$

and $(c)_n = \Gamma(c+n)/\Gamma(c)$ $(c \in \mathbb{C})$.

Then, our second result is stated as follows.

Theorem 2 (Integral representaion of Borel sum)
In Theorem 1, the $k-$Borel sum $u(t,x)$ is given by

$$u(t,x) = \int_0^{\infty(dp/q)} k(t,\zeta)\Phi(x,\zeta)d\zeta$$

where the integration is taken from 0 to ∞ along the half line of argument dp/q,

$$\Phi(x,\zeta) = \sum_{j=0}^{q-1} \varphi(x+\zeta\omega^j), \quad \omega = \exp(2\pi i/q)$$

and the kernel function $k(t,\zeta)$ is given by

$$\begin{aligned} k(t,\zeta) &= \frac{1}{\zeta} \times \frac{\Gamma(\boldsymbol{p}/p)}{\Gamma(\boldsymbol{q}/q)} \times \sum_{j=1}^{q-1} \frac{\Gamma((\widehat{\boldsymbol{q}/q})_j - j/q)}{\Gamma(\boldsymbol{p}/p - j/q)} \times \left(\frac{\zeta^q}{t^p}\right)^{j/q} \\ &\quad \times {}_pF_{q-1}\left(\begin{matrix} 1 + j/q - \boldsymbol{p}/p \\ 1 + j/q - (\widehat{\boldsymbol{q}/q})_j \end{matrix}; (-1)^{p-q}\frac{\zeta^q}{t^p}\right) \end{aligned} \tag{1.2}$$

where $\boldsymbol{p} = (1,2,\cdots,p) \in \mathbb{N}^p$, $\quad \boldsymbol{q} = (1,2,\cdots,q) \in \mathbb{N}^q$, $\quad \boldsymbol{p}/p = (1/p, 2/p, \cdots, 1)$,

$$1 + \frac{j}{q} - \frac{\boldsymbol{p}}{p} = \left(1 + \frac{j}{q} - \frac{1}{p}, 1 + \frac{j}{q} - \frac{2}{p}, \cdots, 1 + \frac{j}{q} - 1\right),$$

$\widehat{\boldsymbol{q}}_j \in \mathbb{N}^{q-1}$, which is obtained by omitting the $j-$th component from $\boldsymbol{q}$, and we use the following abbreviations:

$$\Gamma(\boldsymbol{p}/p) = \prod_{\ell=1}^{p} \Gamma(\ell/p), \quad \Gamma((\widehat{\boldsymbol{q}/q})_j - j/q) = \prod_{m=1, m\neq j}^{q} \Gamma(m/q - j/q).$$

The kernel function $k(t,\zeta)$ seems to be complicated, but in special cases it can be written down as follows.

$$(p,q) = (1,2) \quad \Longrightarrow \quad k(t,\zeta) = \frac{1}{\sqrt{\pi t}} e^{-\zeta^2/t} \quad (\textit{heat kernel}),$$

$$(p,q) = (2,3) \quad \Longrightarrow \quad k(t,\zeta) = \frac{\Gamma(1/2)}{\Gamma(1/3)\Gamma(2/3)} \frac{1}{\zeta} \exp\left(\frac{1}{2} \times \frac{\zeta^3}{t^2}\right) W_{1/2,1/6}\left(-\frac{\zeta^3}{t^2}\right),$$

$$\begin{aligned} (p,q) = (1,3) \quad \Longrightarrow \quad k(t,\zeta) &= \left(\frac{\zeta}{-t}\right)^{1/2} \left\{ J_{1/3}\left(2\sqrt{\frac{\zeta^3}{(-1)^3 t}}\right) + J_{-1/3}\left(2\sqrt{\frac{\zeta^3}{(-1)^3 t}}\right) \right\} \\ &= \frac{3^{2/3}}{t^{1/3}} Ai\left(3^{2/3}\frac{\zeta}{t^{1/3}}\right), \end{aligned}$$

where $W_{a,b}$ and J_c denote the Whittaker and the Bessel function, respectively, (cf. [5]) and Ai denotes the Airy function (cf. [4]) which is given by

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + zt\right) dt,$$

$$Ai(-z) = \frac{z^{1/2}}{3}\left(J_{1/3}(\xi) + J_{-1/3}(\xi)\right), \quad \xi = \frac{2}{3}z^{3/2}.$$

2 Proof of Theorem 1

A crucial point in the proof of Theorem 1 is the following lemma for the Borel summability.

Lemma 1 *Let $k > 0$, $d \in \mathbb{R}$ and $\hat{u}(t,x) \in \mathcal{O}(r_0)[[t]]_{1/k}$. Then the following three propositions are equivalent:*

(1) $\hat{u}(t,x) \in \mathcal{O}(r_0)\{t\}_{k,d}$.

(2) *Let $v(s,x)$ be the formal $(k-)$Borel transformation of $\hat{u}$,*

$$v(s,x) = (\hat{\mathcal{B}}_k\hat{u})(s,x) := \sum_{n=0}^{\infty} u_n(x)\frac{s^n}{\Gamma(1+n/k)},$$

which is holomorphic in a neighbourhoood of the origin. Then, $v(s,x)$ can be continued analytically in $S_1 = S(d,\varepsilon_0,\infty)$ $(\exists\varepsilon_0 > 0)$ in complex $s-$plane and v has a growth condition of exponential order at most k, that is, there exist positive numbers C and δ such that the following estimate holds,

$$\max_{|x|\le r_0} |v(s,x)| \le C\exp(\delta|x|^k), \quad s \in S_1.$$

In this case, the Borel sum u is given by

$$u(t,x) = (\mathcal{L}_k v)(t,x) := \frac{1}{t^k}\int_0^{\infty(d)} \exp\left(-(s/t)^k\right) v(s,x)ds^k.$$

where $t \in S(d,\beta,\rho)$ $(\beta < \pi/k, \rho > 0)$.

(3) *Let $j \ge 2$ be taken arbitrarily, and $k_l > 0$ $(l = 1,2,\cdots,j)$ satisfy*

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_j}.$$

We define $w(s,x) = (\hat{\mathcal{B}}_{k_1}\circ\hat{\mathcal{B}}_{k_2}\circ\cdots\circ\hat{\mathcal{B}}_{k_j}\hat{u})(s,x)$ which is holomorphic in a neighbourhood of the origin. Then $w(s,x)$ satisfies the same conditions as $v(s,x)$ in (2).
In this case, the Borel sum u is given by

$$u(t,x) = (\mathcal{L}_{k_1}\circ\mathcal{L}_{k_2}\circ\cdots\circ\mathcal{L}_{k_j} w)(t,x).$$

We omit the proof which can be seen in [1] and [11].

⟨Proof of the sufficient condition⟩

First, we assume that the Cauchy data $\varphi(x)$ can be continued analytically on

$$\Omega(p,q;d,\varepsilon)=\bigcup_{j=0}^{q-1} S\left(\frac{pd+2\pi j}{q},\varepsilon,\infty\right),\quad \exists\varepsilon>0.$$

and satisfy the growth condition of exponential order at most $q/(q-p)$ there. Let $w(s,x)$ be the $(q-p)$ times iterated $p-$Borel transform of the formal solution $\hat{u}$ of (CP),

$$w(s,x)=\left((\hat{\mathcal{B}}_p)^{q-p}\hat{u}\right)(s,x)=\sum_{n=0}^{\infty}\frac{\varphi^{(qn)}(x)}{(pn)!}\left(\frac{p^p}{q^q}\right)^n\frac{s^{pn}}{(n!)^{q-p}}.$$

By the Cauchy integral formula, for sufficiently small s and x we have

$$\begin{aligned} w(s,x) &= \frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{\varphi(x+\zeta)}{\zeta}\sum_{n=0}^{\infty}\frac{(qn)!}{(pn)!(n!)^{q-p}}\left(\frac{p^p s^p}{q^q\zeta^q}\right)^n d\zeta \\ &= \frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{\varphi(x+\zeta)}{\zeta}\,{}_qF_{q-1}\left(\frac{\boldsymbol{q}}{q};\frac{\boldsymbol{p}}{p},1,\cdots,1;\frac{s^p}{\zeta^q}\right)d\zeta. \end{aligned} \tag{2.1}$$

Here we notice that ${}_qF_{q-1}(\boldsymbol{\alpha};\boldsymbol{\beta};z)$ is the solution holomorphic at the origin of the following Fuchsian equation with regular singular points at $\{0,1,\infty\}$,

$$D_{q,q-1}(u):=\left(\theta\prod_{m=1}^{q-1}(\theta+\beta_m-1)-z\prod_{\ell=1}^{q}(\theta+\alpha_\ell)\right)u=0$$

where $\theta=z(d/dz)$. Therefore ${}_qF_{q-1}(\boldsymbol{\alpha};\boldsymbol{\beta};z)$ is singular only at $z=1$ and has a polynomial growth as $z\to\infty$. Hence in the integral ${}_qF_{q-1}$ is singular in $\zeta-$plain at q roots of $\zeta^q=s^p$ which we denote by $\zeta=\sqrt[q]{s^p}$ for short.

From these observations we can see that under the assumption for $\varphi(x)$ we can deform the contour of integration to make it possible to continue $w(s,x)$ analytically to the infinity along the half line of d direction in $s-$plain. In fact, the contour is deformed so that it encloses the q rays $\{\sqrt[q]{s^p};s\in Re^{id},R\in[0,T]\}$ $(T\to\infty)$. Of course, it is possible to shift the argument of s near d. The desired growth estimate for $w(s,x)$ of exponential order at most $p/(q-p)$ as $s\to\infty$ is also obtained by this deformation of the contour.

This proves the sufficient condition.

⟨Proof of the necessary condition⟩

First, we remark that the advantage of the equivalent proposition (3) in Lemma 1 is in the proof of the necessary condition as we shall see below. Indeed, to prove the sufficient condition it is enough to use the proposition (2) and the proof is just the same as above.

We may assume that $w(s,x) = \left((\hat{\mathcal{B}}_p)^{q-p}\hat{u}\right)(s,x)$ can be continues analytically in $S_1 = S(d,\varepsilon_0,\infty)$ $(\exists\varepsilon_0 > 0)$ and satisfies the growth condition of exponential order at most $p/(q-p)$ as $s \to \infty$ in S_1 from the assumption of the necessity. Hence we may assume

$$\max_{|x|\le r_0} |w(s,x)| \le C\exp(\delta|s|^{p/(q-p)}), \quad s \in S_1.$$

We should remark that $w(s,x)$ satisfies the following partial differential equation,

$$E_{p,q}(w) \equiv p^{-q}q^q\partial_s^p(s\partial)^{q-p}w(s,x) - \partial_x^q w(s,x) = 0.$$

In fact, this is deduced from the following commutative diagram,

$$\begin{tikzcd}
t^{pn} \arrow[r, "(\hat{\mathcal{B}}_p)^{q-p}"] \arrow[d, "\partial_t^p"'] & \dfrac{s^{pn}}{(n!)^{q-p}} \arrow[d, "p^{p-q}\partial_s^p(s\partial_s)^{q-p}"] \\
(pn-p+1)_p\, t^{pn-p} \arrow[r, "(\hat{\mathcal{B}}_p)^{q-p}"] & (pn-p+1)_p \dfrac{s^{pn-p}}{\{(n-1)!\}^{q-p}}
\end{tikzcd}$$

Here we recall that what we have to prove is that $w(0,x)(=\varphi(x))$ can be continued analytically in q sectors $\Omega(p,q;d,\varepsilon)$ and satisfies the growth condition of exponential order at most $q/(q-p)$. To do so we regard $w(s,x)$ as a solution of the following Cauchy problem in $x-$ direction,

$$\begin{cases} E_{p,q}(w) = 0 \\ \partial_x^j w(s,x_0) = \psi_j(s)\ (0 \le j \le q-1) \end{cases}$$

where x_0 is a fixed point arbitrary such that $|x_0| < r_0$. By the assumption on $w(s,x)$, we can see that $\psi_j(s)$ are analytic in S_1 and satisfy

$$|\psi_j(s)| \le C'\exp(\delta'|s|^{p/(q-p)}), \quad s \in S_1.$$

To prove our assertion we may assume without loss of generality that $\psi_j(s) \equiv 0$ for $1 \le j \le q-1$, by the principle of superposition of solutions of linear equations. We put $\psi_0(s) = \psi(s)$. We take the Taylor expansion of $w(s,x)$ around $x = x_0$ and by putting $s = 0$ we have the following formula after a careful calculations.

$$\begin{aligned}
w(0,x) &= \sum_{n=0}^{\infty} \psi^{(pn)}(0)(n!)^{q-p}\left(\frac{q^q}{p^p}\right)^n \frac{(x-x_0)^{qn}}{(qn)!} \\
&= \frac{1}{2\pi i}\oint_{|s|=r} \frac{\psi(s)}{s}\sum_{n=0}^{\infty}\frac{(pn)!(n!)^{q-p}}{(qn)!}\left(\frac{q^q(x-x_0)^q}{p^p s^p}\right)^n ds \\
&= \frac{1}{2\pi i}\oint_{|s|=r} \frac{\psi(s)}{s}\, {}_qF_{q-1}\left(\frac{\boldsymbol{p}}{p},1,\cdots,1;\frac{\widehat{\boldsymbol{q}_q}}{q};\frac{(x-x_0)^q}{s^p}\right) ds.
\end{aligned}$$

By this expression we can obtain the desired contiuation property and the growth condition for $w(0,x)=\varphi(x)$ by the same way as in the proof of the sufficient condition.

This proves the necessary condition.

3 Proof of Theorem 2

First, we consider the expression (2.1) in the proof of the sufficient condition.

$$w(s,x)=\frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{\varphi(x+\zeta)}{\zeta}h(s,\zeta)d\zeta,$$

where

$$h(s,\zeta)={}_qF_{q-1}\left(\frac{\boldsymbol{q}}{q};\frac{\boldsymbol{p}}{p},1,\cdots,1;\frac{s^p}{\zeta^q}\right)$$

and $h(s,\zeta)$ has q singular points in $\zeta-$plane at q roots of $\zeta^q=s^p$ (say $\zeta=\sqrt[q]{s^p}$) for a fixed $s\neq 0$ with arg $s=d$. We denote by $a=s^{p/q}$ of the root of argment dp/q, and we denote by $[0,a]$ the segument jointing the origin and a. Then $h(s,\zeta)$ is univalent in $\mathbb{C}\setminus\bigcup_{j=0}^{q-1}e^{2\pi ij/q}[0,a]$ (outside of q seguments). Therefore by deforming the contour of integration on these q seguments, we get the following expression for $w(s,x)$,

$$w(s,x)=\frac{1}{2\pi i}\int_0^a\frac{\Phi(x,\zeta)}{\zeta}\left\{h(s,\zeta)-h(s,\zeta\omega^{-1})\right\}d\zeta$$

where

$$\Phi(x,\zeta)=\sum_{j=0}^{q-1}\varphi(x+\zeta\omega^j),\quad \omega=\exp(2\pi i/q).$$

Since $h(s,\zeta)$ is univalent out side of q seguments, we have

$$w(s,x)=\frac{1}{2\pi i}\int_0^{\infty(pd/q)}\frac{\Phi(x,\zeta)}{\zeta}\left\{h(s,\zeta)-h(s,\zeta\omega^{-1})\right\}d\zeta.$$

Hence the Borel sum $u(t,x)$ is given by the following iterated Laplace transform,

$$\begin{aligned} u(t,x) &= \left((\mathcal{L}_p)^{q-p}w\right)(t,x)\\ &= \frac{1}{2\pi i}\int_0^{\infty(pd/q)}\frac{\Phi(x,\zeta)}{\zeta}\left\{(\mathcal{L}_p)^{q-p}\left(h(\cdot,\zeta)-h(\cdot,\zeta\omega^{-1})\right)\right\}(t,\zeta)d\zeta. \qquad (3.1)\end{aligned}$$

This observation shows that the kernel function $k(t,\zeta)$ is given by

$$k(t,\zeta)=\frac{1}{2\pi i}\frac{1}{\zeta}\left\{\left((\mathcal{L}_p)^{q-p}h\right)(\cdot,\zeta)-\left((\mathcal{L}_p)^{q-p}h\right)(\cdot,\zeta\omega^{-1})\right\}(t,\zeta).$$

Now, we shall prove that function $h(s,\zeta)/\zeta$ is an iterated formal Borel transform of the formal solution of the following Cauchy problem for the adjoint equation,

$$(\widetilde{\mathrm{CP}})\qquad \begin{cases} \partial_t^p u(t,\zeta)=c_0(-\partial_\zeta)^q u(t,\zeta), & c_0=p^p/q^q\\ u(0,\zeta)=1/\zeta,\ \partial_t^j u(0,\zeta)=0\ (1\le j\le p-1). \end{cases}$$

This Cauchy problem $(\widetilde{\mathrm{CP}})$ has a unique formal solution,

$$\begin{aligned}\hat{e}(t,\zeta) &= \frac{1}{\zeta}\sum_{n=0}^{\infty}\frac{(qn)!}{(pn)!}\left(\frac{(pt)^p}{(q\zeta)^q}\right)^n \\ &= \frac{1}{\zeta}\,{}_qF_{p-1}\left(\frac{\boldsymbol{q}}{q};\frac{\widehat{\boldsymbol{p}_p}}{p};\frac{t^p}{\zeta^q}\right) \underset{\text{put}}{=} \frac{1}{\zeta}\hat{f}(z),\end{aligned}$$

where $z = t^p/\zeta^q$ and $\hat{f} \in \mathbb{C}[[z]]_{q-p}$. Let $g(\xi) = \left((\hat{\mathcal{B}}_1)^{q-p}\hat{f}\right)(\xi)$. Then $g(s^p/\zeta^q) = h(s,\zeta)$.

Hence, let $f(z)$ be the Borel sum of $\hat{f}(z)$. Then $f(z) = ((\mathcal{L}_1)^{q-p}g)(z)$ and the Borel sum $e(t,\zeta)$ of $\hat{e}(t,\zeta)$ is given by

$$e(t,\zeta) = \frac{1}{\zeta}f(z).$$

Thus we see that the kernel function $k(t,\zeta)$ is given by

$$\begin{aligned}k(t,\zeta) &= \frac{1}{2\pi i}\{e(t,\zeta) - e(te^{\frac{2\pi i}{p}},\zeta)\} \\ &= \frac{1}{2\pi i}\times\frac{1}{\zeta}\{f(z) - f(ze^{2\pi i})\}.\end{aligned} \tag{3.2}$$

In order to complete the proof, we have to study the explicit formula for $f(z)$ which will be given in the following sections from a general point view. In conclusion, we shall prove the following formula, which prives Theorem 2.

$$\begin{aligned}\frac{1}{2\pi i}\{f(z) - f(ze^{2\pi i})\} &= \frac{\Gamma(\boldsymbol{p}/p)}{\Gamma(\boldsymbol{q}/q)}\times\sum_{j=1}^{q-1}\frac{\Gamma(\widehat{(\boldsymbol{q}/q)}_j - j/q)}{\Gamma(\boldsymbol{p}/p - j/q)}\times\left(\frac{\zeta^q}{t^p}\right)^{j/q} \\ &\quad \times{}_pF_{q-1}\left(\begin{matrix}1 + j/q - \boldsymbol{p}/p \\ 1 + j/q - \widehat{(\boldsymbol{q}/q)}_j\end{matrix};(-1)^{p-q}\frac{\zeta^q}{t^p}\right).\end{aligned}$$

4 The Borel sum and its analytic continuation of Barnes generalized hypergeimetric series

We shall deal with the following Barnes hypergeometric series ${}_qF_{p-1}$ of divergent type, that is, the case $q > p$.

$$f(z) = {}_qF_{p-1}\left(\begin{matrix}\boldsymbol{\alpha} \\ \boldsymbol{\gamma}\end{matrix};z\right) := \sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\gamma})_n}\frac{z^n}{n!},$$

where $z \in \mathbb{C}$, $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_q) \in \mathbb{C}^q$ and $\boldsymbol{\gamma} = (\gamma_1,\cdots,\gamma_{p-1}) \in \mathbb{C}^{p-1}$.

In order to make sense of this series we assume $\gamma_j \notin \mathbb{Z}_{\leq 0}$ and to avoid the trivial case we assume $\alpha_j \notin \mathbb{Z}_{\leq 0}$ where $\mathbb{Z}_{\leq 0} := \{0,-1,-2,\cdots\}$.

It is known that f is a formal solution of the following differential equation,

$$D_{q,p-1}(u) := \left(\theta \prod_{m=1}^{p-1} (\theta + \gamma_m - 1) - z \prod_{\ell=1}^{q} (\theta + \alpha_\ell)\right) u(z) = 0$$

where $\theta = z(d/dz)$.

In the case $q \le p$, the order of this differential equation is p. In the case of any two of $\{1, \gamma_1, \cdots, \gamma_{p-1}\}$ does not differ by an integer, the other $(p-1)$ solutions are given by,

$$u_j(z) = z^{1-\gamma_j}{}_qF_{p-1}\left(\begin{matrix} 1-\gamma_j+\boldsymbol{\alpha} \\ 2-\gamma_j, 1-\gamma_j+\widehat{\boldsymbol{\gamma}_j} \end{matrix}; z\right), \quad j = 1, \cdots, p-1.$$

Now, our interest is in the case $q > p$. In this case, our formal series f is divergent and we rewrite $f(z)$ by $\hat{f}(z)$. Then it is easily seen that $\hat{f}(z) \in \mathbb{C}[[z]]_{q-p}$.

We want to investigate the Borel summability of this divergent series. Then, our result on the Borel summability of $\hat{f}$ is stated as follows.

Theorem 3 (Borel sum)
Assume that $\alpha_i - \alpha_j \notin \mathbb{Z}$ $(i \neq j)$. Then $\hat{f}(z)$ is $1/(q-p)-$summable in any direction d such that $d \neq 0 (\mathrm{mod}\ 2\pi)$ and its Borel sum f is given by

$$f(z) \;=\; C_{\alpha\gamma} \sum_{j=1}^{q} C_{\alpha\gamma}(j) \times (-z)^{-\alpha_j}{}_pF_{q-1}\left(\begin{matrix} \alpha_j, 1+\alpha_j-\boldsymbol{\gamma} \\ 1+\alpha_j-\widehat{\boldsymbol{\alpha}_j} \end{matrix}; \frac{(-1)^{p-q}}{z}\right)$$

where $z \in S(\pi, (q-p+2)\pi, \infty)$ and

$$C_{\alpha\gamma} = \frac{\Gamma(\boldsymbol{\gamma})}{\Gamma(\boldsymbol{\alpha})}, \quad C_{\alpha\gamma}(j) = \frac{\Gamma(\alpha_j)\Gamma(\widehat{\boldsymbol{\alpha}_j} - \alpha_j)}{\Gamma(\boldsymbol{\gamma} - \alpha_j)}.$$

Next, our result for the analytic continuation of the Borel sum f is stated as follows.

Theorem 4 (Continuation of Borel sum)
Under the same assumptions as in Theorem 3, we have

$$\begin{aligned} &\frac{1}{2\pi i}\{f(z) - f(ze^{2\pi i})\} \\ &= C_{\alpha\gamma} \sum_{j=1}^{q} \frac{C_{\alpha\gamma}(j)}{\Gamma(\alpha_j)\Gamma(1-\alpha_j)} z^{-\alpha_j}{}_pF_{q-1}\left(\begin{matrix} \alpha_j, 1+\alpha_j-\boldsymbol{\gamma} \\ 1+\alpha_j-\widehat{\boldsymbol{\alpha}_j} \end{matrix}; \frac{(-1)^{p-q}}{z}\right) \end{aligned}$$

where $z \in S(0, (q-p)\pi, \infty)$ and $C_{\alpha\gamma}, C_{\alpha\gamma}(j)$ are the same constants as above.

We remark that by using this Theorem 4 we can obtain the kernel function $k(t, \zeta)$ which is given by (1.2) in Theorem 2.

Remark 2. In 1907, Barnes [3] gave an asymptotic expansion of linear combinations of generalized hypergeometric function as follows, which is an alternative version of

Theorem 3.

Theorem A (Barnes)

Let $q > p$. Then there are q asymptotic expansions of the type

$$C_{\alpha\gamma}\sum_{j=1}^{q} C_{\alpha\gamma}(j) \times t^{\alpha_j-\alpha_1}{}_pF_{q-1}\left(\begin{matrix}\alpha_j, 1+\alpha_j-\gamma\\ 1+\alpha_j-\widehat{\alpha_j}\end{matrix};(-1)^{p-q-1}t\right)$$
$$\sim t^{-\alpha_1}{}_qF_{p-1}\left(\begin{matrix}\alpha\\ \gamma\end{matrix};\frac{-1}{t}\right) \quad as \quad t\to\infty, \quad t\in S(0,(q-p+2)\pi,\infty).$$

The other $(q-1)$ asymptotic expansions are obtained by interchanging α_1 with $\alpha_2,\cdots,\alpha_q$, respectively.

We remark that q functions on the left hand side are fundamental solutions of the following differential equation.

$$D_{p,q-1}(u) \;=\; \{\prod_{\ell=1}^{q}(\theta+\alpha_1-\alpha_\ell) - z(\theta+\alpha_1)\prod_{m=1}^{p-1}(\theta+1+\alpha_1-\gamma_m)\}u(z)=0$$

with $z=(-1)^{p-q-1}t$.

5 Proof of Theorem 3

⟨First proof of Theorem 3⟩

Let $h(\xi)$ be the iterated formal Borel transformation,

$$h(\xi)=\left((\mathcal{B}_1)^{q-p}\hat{f}\right)(\xi)=\sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(\gamma)_n(n!)^{q-p}}\frac{\xi^n}{n!}={}_qF_{q-1}\left(\begin{matrix}\boldsymbol{\alpha}\\ \gamma,1,\cdots,1\end{matrix};\xi\right).$$

This series is convergent in $|\xi|<1$. Then we can see that $h\in\mathcal{O}(\mathbb{C}\setminus[1,\infty))$ and h has at most polynomial growth as $\xi\to\infty$, because h satisfies $D_{q,(q-1)}(h)=0$ which is a Fuchsian equation with singular points $\{0,1,\infty\}$. Therefore $\hat{f}$ is Borel summable in any direction d such that $d\neq 0(\text{mod } 2\pi)$ and the Borel sum f is given by the following iterated Laplace integrals,

$$f(z)=\left((\mathcal{L}_1)^{q-p}h\right)(z)=\frac{1}{z}\int_0^{\infty(d)}\exp\left(-\frac{s_1}{z}\right)ds_1\frac{1}{s_1}\int_0^{\infty(d)}\exp\left(-\frac{s_2}{s_1}\right)ds_2\times\cdots$$
$$\cdots\times\frac{1}{s_{q-p-1}}\int_0^{\infty(d)}\exp\left(-\frac{\xi}{s_{q-p-1}}\right)h(\xi)d\xi,$$

where $|d-\arg z|<\pi/2$ and $\arg s_j=\arg\xi=d\neq 0,(1\le j\le q-p-1)$. We can regard f is analytic in $S(\pi,(q-p+2)\pi,\infty)$ by taking analytic continuation of f by moving the arguments of halflines of this iterated integrations. By a change of variables, we have

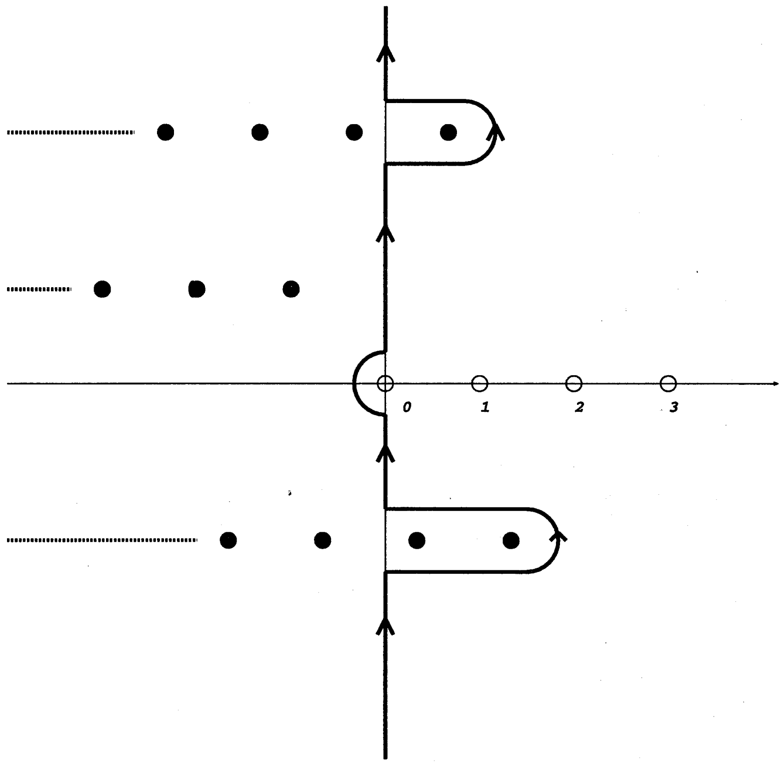
$$f(z) \;=\; \int_0^{\infty(a)}\exp(-u_1)du_1\int_0^{\infty(0)}\exp(-u_2)du_2\cdots\int_0^{\infty(0)}\exp(-u_{q-p})h(\mathbf{u}z)du_{q-p},$$

where $a = d - \arg z$ and $\mathbf{u} = u_1 \cdots u_{q-p}$.

To caluclate this iterated integrals, we employ the Barnes integral representation of $h(\xi)$ (cf. [7]), which is given by,

$$h(\xi) = \frac{C_{\alpha\gamma}}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)\Gamma(-\zeta)}{\Gamma(\gamma+\zeta)\{\Gamma(1+\zeta)\}^{q-p}}(-\xi)^{\zeta}d\zeta, \quad |\arg(-\xi)| < \pi,$$

where $C_{\alpha\gamma}$ is the same constant as in Theorem 3. Here the path of integration I runs from $-i\infty$ to $+i\infty$ on the imaginary axis where the poles of $\Gamma(\boldsymbol{\alpha}+\zeta)$ are ploted by $\bullet$ which are on the left side of I and the poles of $\Gamma(-\zeta)$ are ploted by $\circ$ which are on the right side of I (see the figure below).



By exchanging the order of integrations, we have

$$f(z) = \frac{C_{\alpha\gamma}}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)\Gamma(-\zeta)}{\Gamma(\gamma+\zeta)}(-z)^{\zeta}d\zeta. \tag{5.1}$$

Now, by the residue theorem, we obtain the conclusion. That is, by taking residues of the left side of I, we have

$$f(z) \quad = \quad C_{\alpha\gamma}\sum_{j=1}^{q}\sum_{n=0}^{\infty}\frac{\Gamma(\widehat{\boldsymbol{\alpha}}_j - \alpha_j - n)\Gamma(\alpha_j+n)}{\Gamma(\gamma-\alpha_j-n)}\frac{(-1)^n}{n!}(-z)^{-\alpha_j-n}.$$

By using $\Gamma(c-n) = \Gamma(c)/\{(1-c)_n(-1)^n\}$ ($c \in \mathbb{C}$) which is obtained from the reciprocal formula $\sin(\pi\zeta) = \pi/\Gamma(\zeta)\Gamma(1-\zeta)$ (cf. [5, p.3]), we obtain

$$\begin{aligned} f(z) &= C_{\alpha\gamma}\sum_{j=1}^{q} C_{\alpha\gamma}(j)(-z)^{-\alpha_j}\sum_{n=0}^{\infty}\frac{(\alpha_j)_n(1+\alpha_j-\gamma)_n}{(1+\alpha_j-\widehat{\alpha}_j)_n}\frac{(-1)^{(p-q)n}}{n!}z^{-n} \\ &= C_{\alpha\gamma}\sum_{j=1}^{p} C_{\alpha\gamma}(j)(-z)^{-\alpha_j}{}_pF_{q-1}\left(\begin{matrix}\alpha_j, 1+\alpha_j-\gamma \\ 1+\alpha_j-\widehat{\alpha}_j\end{matrix}; \frac{(-1)^{p-q}}{z}\right). \end{aligned}$$

The proof is complete.

⟨Second proof of Theorem 3⟩

By using the statement (2) in Lemma 1, we shall give an another proof of Theorem 3. Let $g(\xi)$ be the formal $1/(q-p)$–Borel transform of $\hat{f}$,

$$\begin{aligned} g(\xi) &= (\hat{\mathcal{B}}_{1/(q-p)}\hat{f})(\xi) = \sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\gamma})_n\{(q-p)n\}!}\frac{\xi^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\gamma})_n((\boldsymbol{q}-\boldsymbol{p})/(q-p))_n}\frac{1}{n!}\left(\frac{\xi}{(q-p)^{q-p}}\right)^n \\ &= {}_pF_{p-1}\left(\begin{matrix}\boldsymbol{\alpha} \\ \boldsymbol{\gamma}, (\boldsymbol{q}-\boldsymbol{p})/(q-p)\end{matrix}; \frac{\xi}{(q-p)^{q-p}}\right) \end{aligned}$$

where

$$\frac{\boldsymbol{q}-\boldsymbol{p}}{q-p} = \left(\frac{1}{q-p}, \dots, \frac{p-q-1}{q-p}, 1\right) \in \mathbb{C}^{q-p} \text{ and } \left(\frac{\boldsymbol{q}-\boldsymbol{p}}{q-p}\right)_n = \prod_{k=1}^{q-p}\left(\frac{k}{q-p}\right)_n.$$

Therefore, the Borel sum f is given by the following Laplace integral,

$$f(z) = \left(\mathcal{L}_{1/(q-p)}g\right)(\xi) = \frac{1}{z^{1/(q-p)}}\int_0^{\infty(d)}\exp\left[-\left(\frac{\xi}{z}\right)^{1/(q-p)}\right]g(\xi)d\xi^{1/(q-p)},$$

where $d \neq 0(\mathrm{mod}\,2\pi)$, $|d-\arg z| < (q-p)\pi/2$ and $\arg\xi = d$. We can regard that $f(z)$ is analytic in $S(\pi, (q-p+2)\pi, \infty)$ by taking analytic continuation of f by a change of argument of halfline of this integral. By a change of the variable, we have

$$f(z) = \int_0^{\infty(b)} e^{-u}g(zu^{q-p})du,$$

where $b = (d-\arg z)/(q-p)$. We employ the Barnes integral representation of $g(\xi)$ which is given by,

$$g(\xi) = \frac{C_1}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)\Gamma(-\zeta)}{\Gamma(\boldsymbol{\gamma}+\zeta)\Gamma((\boldsymbol{q}-\boldsymbol{p})/(q-p)+\zeta)}\left(-\frac{\xi}{(q-p)^{(q-p)}}\right)^{\zeta}d\zeta,$$

where $|\arg(-\xi)| < \pi$,

$$C_1 = C_{\alpha\gamma} \times \Gamma\left(\frac{\boldsymbol{q}-\boldsymbol{p}}{q-p}\right)$$

and the path of integration I is the same one as in the previous proof, but I is taken so that the point $-1/(q-p)$ is on the left side of I. By exchanging the order of integrations, we have

$$f(z) \;=\; \frac{C_1}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)\Gamma(-\zeta)(-z)^\zeta}{\Gamma(\boldsymbol{\gamma}+\zeta)\Gamma((\boldsymbol{q}-\boldsymbol{p})/(q-p)+\zeta)}\frac{\Gamma(1+(q-p)\zeta)}{(q-p)^{(q-p)\zeta}}d\zeta.$$

Now by using the following multiplication formula (cf. [5, p.4]) with $m = 2, 3, \cdots$,

$$\prod_{k=0}^{m-1}\Gamma\left(\zeta+\frac{k}{m}\right) = (2\pi)^{(m-1)/2}m^{\frac{1}{2}-m\zeta}\Gamma(m\zeta), \quad \zeta+\frac{k}{m}\notin\mathbb{Z}_{\leq 0}\ (k=0,1,\cdots,m-1),$$

we have

$$\frac{\Gamma(1+(q-p)\zeta)}{\Gamma((\boldsymbol{q}-\boldsymbol{p})/(q-p)+\zeta)} = (2\pi)^{-(q-p-1)/2}\cdot(q-p)^{\frac{1}{2}+(q-p)\zeta}$$

and

$$\Gamma\left(\frac{\boldsymbol{q}-\boldsymbol{p}}{q-p}\right) = (2\pi)^{(q-p-1)/2}(q-p)^{-\frac{1}{2}}.$$

Therefore we obtain

$$f(z) = \frac{C_{\alpha\gamma}}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)\Gamma(-\zeta)}{\Gamma(\boldsymbol{\gamma}+\zeta)}(-z)^\zeta d\zeta.$$

This expression is nothing but the formula (5.1) in the previous proof.

⟨Proof of Theorem 4⟩

From (5.1), we get the following formula for the difference of the Borel sum after rotating around the origin.

$$\begin{aligned} f(z)-f(ze^{2\pi i}) &= \frac{C_{\alpha\gamma}}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)\Gamma(-\zeta)}{\Gamma(\boldsymbol{\gamma}+\zeta)}(-z)^\zeta(1-e^{2\pi i\zeta})d\zeta \\ &= \frac{C_{\alpha\gamma}}{2\pi i}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)\Gamma(-\zeta)}{\Gamma(\boldsymbol{\gamma}+\zeta)}z^\zeta(-2i)\sin(\pi\zeta)d\zeta. \end{aligned}$$

By using the reciprocal formula, this expression becomes

$$f(z)-f(ze^{2\pi i}) \;=\; C_{\alpha\gamma}\int_I \frac{\Gamma(\boldsymbol{\alpha}+\zeta)}{\Gamma(\boldsymbol{\gamma}+\zeta)\Gamma(1+\zeta)}z^\zeta d\zeta.$$

Finally by the residue theorem, we get the desired formula.

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