Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at $s = 1$, and explicit lower bounds for relative class numbers of CM-fields (Abridged version)

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Abstract

We provide the reader with various useful explicit upper bounds on residues of Dedekind zeta functions of numbers fields and on absolute values of values at $s = 1$ of L-series associated with primitive characters on ray class groups of number fields. To make it quite clear to the reader how useful such bounds are when dealing with class number problems for CM-fields, we deduce an upper bound on the root discriminants of the normal CM-fields with (relative) class number one.

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1   Introduction

Lately, various class number problems and class groups problems for CM-fields have been solved. These problems include the determinations of the imaginary abelian number fields with class number one (see [CK], [Yam]), relative class number one or class numbers equal to their genus class numbers; the determinations of the non quadratic imaginary cyclic fields of 2-power degrees with cyclic ideal class groups of 2-power orders (see [Lou8]) or with ideal class groups of exponents ≤ 2 (see [Lou3]); the determination of the normal CM-fields of relative class number one with dihedral or dicyclic Galois groups (see [Lef], [LOO], [LO2], [Lou13]); the determination of the non-abelian normal CM-fields of degrees $2n \leq 42$ of class number one (see [LLO], [LO1], [Lou7], see also [LP]); the determination of the dihedral or quaternion octic CM-fields with ideal class groups cyclic of 2-power orders (see [Lou6], [YK]) or of exponents ≤ 2 (see [LO3], [LYK]).

For solving such problems, there are three obstacles to overcome.

First, one must be able to construct the fields he is going to deal with. Usually this is done by using class field theory (e. g. [Lef], [LO2], [LPL]).

Second, one must be able to compute efficiently the relative class numbers of the CM-fields he is going to deal with. This is done by computing approximations of their relative class numbers by using the methods developped by the author in [Lou4], [Lou9], [Lou11], [Lou14] and [Lou16].

Finally, one must obtain a reasonable upper bound on the absolute values of the discriminants of the CM-fields of a given degree or of a given Galois group with a given relative class number, class number or ideal class group. Due to the deep results of [Sta], [Odl] and [Hof] one usually knows before hand that there are only finitely many such CM-fields. However, these three papers which aimed at proving finitness results are of little or no practical use when it comes to explicit determinations for they yield huge bounds on the roots discriminants of the CM-fields with small class numbers and small degrees. In [Lou2], [Lou6], [Lou12] and [Lou15] we developped a wealth of techniques for obtaining lower on relative class numbers of CM-fields, and these lower bounds are particularly good for CM-fields of small degree.

The aim of our talk was to provide the audience with a uniform approach for proving these various useful explicit upper bounds on residues of Dedekind zeta functions of numbers fields and on absolute values of values at $s = 1$ of L-series associated with primitive characters on ray class groups of number fields. Not only did we simplify our previous proofs, but we also obtained new useful bounds (e.g. see (2), (3), (5), (9) and (11)).

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2 Upper bounds for $\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}})$ and $|L(1,\chi)|$

To begin with, we set the notation required for understanding the statements of the results given in this Section. Let $\mathbf{L}$ be number field of degree $m = r_1 + 2r_2$. Let $\zeta_{\mathbf{L}}$ denote its Dedekind zeta function. We set

$$A_{\mathbf{L}} = \sqrt{d_{\mathbf{L}}/4^{r_2}\pi^m}, \quad \Gamma_{\mathbf{L}}(s) = \Gamma^{r_1}(s/2)\Gamma^{r_2}(s), \quad F_{\mathbf{L}}(s) = A_{\mathbf{L}}^s\Gamma_{\mathbf{L}}(s)\zeta_{\mathbf{L}}(s),$$

$$\lambda_{\mathbf{L}} = \mathrm{Res}_{s=1}(F_{\mathbf{L}}), \quad \mu_{\mathbf{L}} = \lim_{s\downarrow 1}\frac{1}{\lambda_{\mathbf{L}}}F_{\mathbf{L}} - \frac{1}{s(s-1)}, \quad B_{\mathbf{L}} = \mu_{\mathbf{L}}\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}).$$

Notice that $\mu_{\mathbf{Q}} = (2+\gamma-\log(4\pi))/2 = 0.023\cdots$ where $\gamma = 0.577\cdots$ denotes Euler's constant. During our lecture, we proved the following results.

Theorem 1 *Let* $\mathbf{L}$ *be a number field of degree* $m > 1$.

1. *(See [Lou10, Th. 1] and [Lou15, Th. 1]). It holds*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq \left(\frac{e\log d_{\mathbf{L}}}{2(m-1)}\right)^{m-1}. \tag{1}$$

2. $\frac{1}{2} \leq \beta < 1$ *and* $\zeta_{\mathbf{L}}(\beta) = 0$ *imply* $\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq (1-\beta)B_{\mathbf{L}}$.

3. *It holds*

$$B_{\mathbf{L}} \leq \left(\frac{e\log d_{\mathbf{L}}}{2m}\right)^m. \tag{2}$$

Therefore, $\frac{1}{2} \leq \beta < 1$ *and* $\zeta_{\mathbf{L}}(\beta) = 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq (1-\beta)\left(\frac{e\log d_{\mathbf{L}}}{2m}\right)^m. \tag{3}$$

Theorem 2 *Let* $\mathbf{L}$ *be a number field of degree* $m \geq 1$. *Let* χ *be a primitive character on some ray class group for* $\mathbf{L}$. *Let* f_{χ} *denote the norm of the finite part of the conductor of* χ.

1. *(See [Lou15, Th. 2]). It holds*

$$|L(1,\chi)| \leq 2\left(\frac{e}{2m}\log(d_{\mathbf{L}}f_{\chi})\right)^m. \tag{4}$$

2. $\frac{2}{3} \leq \beta < 1$ *and* $L(\beta,\chi) = 0$ *imply*

$$|L(1,\chi)| \leq 4(1-\beta)\left(\frac{e}{2(m+1)}\log(d_{\mathbf{L}}f_{\chi})\right)^{m+1}. \tag{5}$$

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Theorem 3 *(See [Lou10, Th. 3] and [Lou12, Th. 1]). Let $\mathbf{L}$ be a given number field. Let χ be non-trivial primitive character χ on a ray class group for $\mathbf{L}$ which is unramified at all the infinite (real) places of $\mathbf{L}$. Let f_χ denote the norm of the finite part of the conductor of χ. We have*

$$|L(1,\chi)| \le \frac{1}{2}\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})\log f_\chi \ + \begin{cases} 2B_{\mathbf{L}} & \text{in all cases,} \\ B_{\mathbf{L}} & \text{if } f_\chi = 1 \text{ or if } f_\chi \ge e^{2\mu_{\mathbf{L}}}. \end{cases} \quad (6)$$

See also [Lou5] and [Lou15, Th. 7] for similar but less satisfactory results when we chuck the assumption that χ is unramified at all the infinite (real) places of $\mathbf{L}$. Since both the upper bounds on $|L(1,\chi)|$ given in Theorem 3 and [Lou15, Th. 7] involve the invariant $B_{\mathbf{L}}$ of $\mathbf{L}$, it was reasonable to determine in Theorem 1 a general upper bound on $B_{\mathbf{L}}$.

Theorem 4 *(See [Lou10, Prop. 6] and [Lou12, Th. 5]). Let $\mathbf{L}$ be a real quadratic field. We have the following improvement on (2):*

$$B_{\mathbf{L}} \le \frac{1}{8}\log^2 d_{\mathbf{L}}. \quad (7)$$

Theorem 5 *Let χ be an even primitive Dirichlet character modulo $f_\chi > 1$.*

1. *(Use the second bound in (6) with $\mathbf{L} = \mathbf{Q}$). It holds*

$$|L(1,\chi)| \le (\log f_\chi + 2\mu_{\mathbf{Q}})/2 \le (\log f_\chi + 0.05)/2. \quad (8)$$

2. $\frac{1}{2} \le \beta < 1$ *and* $L(\beta,\chi) = 0$ *imply*

$$|L(1,\chi)| \le \frac{1-\beta}{8}\log^2 f_\chi \quad (9)$$

which improves upon (5).

Notice that for quadratic characters (9) follows from (3) and (7).

Corollary 6 *Let $\mathbf{L}$ be a real abelian number field of degree $m > 1$. and conductor $f_{\mathbf{L}}$. Notice that $d_{\mathbf{L}} \le f_{\mathbf{L}}^{m-1}$.*

1. *We have the following improvement on (1):*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \le \left(\frac{1}{2}\log f_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1}. \quad (10)$$

2. $\frac{1}{2} \le \beta < 1$ *and* $\zeta_{\mathbf{L}}(\beta) = 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \le (1-\beta)\frac{\log f_{\mathbf{L}}}{4}\left(\frac{1}{2}\log f_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1}, \quad (11)$$

which improves upon (3).

3 Lower bounds for relative class numbers

Let **N** be a CM-field of degree $2m$. Let $\mathbf{N}^+$ denote its maximal totally real subfield (the degree of $\mathbf{N}^+$ is therefore equal to m) and let $Q_{\mathbf{N}} \in \{1, 2\}$, $w_{\mathbf{N}}$ and $h_{\mathbf{N}}^-$ denote its Hasse unit index, its number of complex roots of unity and its relative class number, respectively. Then

$$h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^m} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \frac{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})}{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+})}. \tag{12}$$

Proposition 7 *(See [Lou2, Proposition A]). Let* **N** *be a CM-field of degree* $2m > 2$. *Then,* $\frac{1}{2} \leq 1 - (a/\log d_{\mathbf{N}}) \leq s < 1$ *and* $\zeta_{\mathbf{N}}(s) \leq 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) \geq \epsilon_{\mathbf{N}}(1-s)/e^{a/2}$$

where $\epsilon_{\mathbf{N}} = \max(\epsilon'_{\mathbf{N}}, \epsilon''_{\mathbf{N}})$ *with*

$$\epsilon'_{\mathbf{N}} = 1 - \left(2\pi m e^{a/2m}/r_{\mathbf{N}}\right) \quad and \quad \epsilon''_{\mathbf{N}} = \frac{2}{5} \exp\left(-2\pi m/r_{\mathbf{N}}\right)$$

and where $r_{\mathbf{N}} = d_{\mathbf{N}}^{1/2m}$ *denotes the root number of* **N**.

Notice that the residue at its simple pole $s = 1$ of any Dedekind zeta function $\zeta_{\mathbf{N}}$ is positive (use the analytic class number formula for **N**, or notice that from its definition we get $\zeta_{\mathbf{N}}(s) \geq 1$ for $s > 1$). Therefore, we have $\lim_{\substack{s \to 1 \\ s < 1}} \zeta_{\mathbf{N}}(s) = -\infty$ and $\zeta_{\mathbf{N}}(1 - (a/\log d_{\mathbf{N}})) \leq 0$ if $\zeta_{\mathbf{N}}$ does not have any real zero in the range $1 - (a/\log d_{\mathbf{N}}) \leq s < 1$.

Proposition 8

1. *If* **N** *is a normal CM-field which does not contain any imaginary quadratic subfield, then either* $\zeta_{\mathbf{N}^+}$ *has a real zero in the range* $1 - 1/\log d_{\mathbf{N}} \leq s < 1$ *or* $\zeta_{\mathbf{N}}(s) \leq 0$ *in this range* $1 - 1/\log d_{\mathbf{N}} \leq s < 1$.

2. *If* **N** *is an imaginary abelian field which does not contain any imaginary quadratic subfield, then either* $\zeta_{\mathbf{N}^+}$ *has a real zero in the range* $1 - 2/\log d_{\mathbf{N}} \leq s < 1$ *or* $\zeta_{\mathbf{N}}(s) \leq 0$ *in this range* $1 - 2/\log d_{\mathbf{N}} \leq s < 1$.

Theorem 9 *(Compare with [Lou15, Th. 4]). Let* **N** *be a normal CM-field of degree* $2m > 2$ *which does not contain any imaginary quadratic subfield. Set* $r = d_{\mathbf{N}}^{1/2m}$ *(the root discriminant of* **N***). It holds*

$$h_{\mathbf{N}}^- \geq \epsilon_{\mathbf{N}} \frac{Q_{\mathbf{N}} w_{\mathbf{N}} \sqrt{d_{\mathbf{N}}/d_{\mathbf{N}^+}}}{2\pi\sqrt{e}\left(\frac{\pi e}{m-1}\log d_{\mathbf{N}^+}\right)^{m-1}\log d_{\mathbf{N}}} \geq \epsilon_{\mathbf{N}} \frac{\sqrt{e}}{2u_m}\left(\frac{\sqrt{r}}{\pi e \log r}\right)^m$$

with $u_m = (m-1)(m/(m-1))^m$.

In particular, $h_{\mathbf{N}}^{-} > 1$ *for* $r \geq 40000$, *and* $h_{\mathbf{N}}^{-} > 1$ *for* $m \geq 10$ *and* $r \geq 14000$.

Proof. According to Point 1 of Proposition 8, there are two cases to consider.

First, $\zeta_{\mathbf{N}+}$ has no real zero in the range $1 - 1/\log d_{\mathbf{N}} \leq s < 1$. Then $\zeta_{\mathbf{N}}(1 - (1/\log d_{\mathbf{N}})) \leq 0$ and using Proposition 7 with $a = 1$ we obtain

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) \geq \epsilon_{\mathbf{N}} \frac{1}{\sqrt{e} \log d_{\mathbf{N}}}.$$

Using (1) we obtain

$$\frac{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})}{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}+})} \geq \epsilon_{\mathbf{N}} / \sqrt{e} \left(\frac{e \log d_{\mathbf{N}+}}{2(m-1)} \right)^{m-1} \log d_{\mathbf{N}}. \tag{13}$$

Second, $\zeta_{\mathbf{N}+}$ has a real zero β in the range $1 - 1/\log d_{\mathbf{N}} \leq s < 1$. Then $\zeta_{\mathbf{N}}(\beta) = 0 \leq 0$ and using Proposition 7 with $a = 1$ we obtain

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) \geq \epsilon_{\mathbf{N}} \frac{1-\beta}{\sqrt{e}}.$$

Using (3) we obtain

$$\frac{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})}{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}+})} \geq \epsilon_{\mathbf{N}} / \sqrt{e} \left(\frac{e \log d_{\mathbf{N}+}}{2m} \right)^{m}. \tag{14}$$

Since (14) is always greater than or equal to (13) (for it holds $d_{\mathbf{N}} \geq d_{\mathbf{N}+}^2$), we conclude that (13) is valid in both cases. Using (12) and (13) we get the desired first lower bound. •

Theorem 10 *Let* $\mathbf{N}$ *be an abelian CM-field of degree* $2m > 2$ *which does not contain any imaginary quadratic subfield. Set* $r = d_{\mathbf{N}}^{1/2m}$ *(the root discriminant of* $\mathbf{N}$*). It holds*

$$h_{\mathbf{N}}^{-} \geq \frac{\epsilon_{\mathbf{N}} Q_{\mathbf{N}} w_{\mathbf{N}} \sqrt{d_{\mathbf{N}}/d_{\mathbf{N}+}}}{\pi e \left(\frac{\pi}{(m-1)} \log d_{\mathbf{N}+} + 2\pi\mu_{\mathbf{Q}} \right)^{m-1} \log d_{\mathbf{N}}} \geq \frac{\epsilon_{\mathbf{N}}}{e u_m} \left(\frac{\sqrt{r}}{\pi \log r + 0.146} \right)^m$$

with $u_m = (m-1)(m/(m-1))^m$.

In particular, $h_{\mathbf{N}}^{-} > 1$ *for* $r \geq 10000$, *and* $h_{\mathbf{N}}^{-} > 1$ *for* $m \geq 10$ *and* $r \geq 1200$.

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Proof. The proof of this Theorem 10 is similar to the proof of Theorem 9, apart from the fact that Point 2 of Proposition 8 allows us to use Proposition 7 with $a = 2$ and that we use (10) and (11) (instead of using Point 1 of Proposition 8, (1) and (3)). ●

We refer the reader to [CK] for the solution of the relative class number one problem for the imaginary abelian fields, solution based on refinements of the lower bound given in Theorem 10.

The reader will easily check that our proofs and statements of Theorems 9 and 10 are still valid under the hypothesis that if N contains an imaginary quadratic field k then $\zeta_{\mathbf{k}}(s) < 0$ for $0 < s < 1$. In particular, if we are only interested in solving the relative class number one problem for N, then we assume $h_{\mathbf{N}}^- = 1$ and we would like to use these lower bounds on relative class numbers to obtain an upper bound on the root discriminant $r_{\mathbf{N}}$ of N. We use [Hor, Th. 1] (for the abelian case) or [Oka] (for the normal case) to obtain $h_{\mathbf{k}} = h_{\mathbf{k}}^- = 1$, 2 or 4 for all the imaginary quadratic subfields k of N. Now, according to [Arn] all the imaginary quadratic fields of class numbers 1, 2 and 4 are known and it is only a matter of computation to verify that we have $\zeta_{\mathbf{k}}(s) < 0$ in the range $0 < s < 1$ for all the imaginary quadratic fields of class numbers 1, 2 or 4. Therefore, we are allowed to use our lower bounds and we obtain that the root discriminant $r_{\mathbf{N}}$ of a normal CM-field N (respectively, of an imaginary abelian field N) of degree ≥ 20 with relative class number one is less than or equal to 14000 (respectively, less than or equal to 1200). It may be worth noticing that if N ranges over the CM-fields of degree $2m$ going to infinity, then as we have $r_{\mathbf{N}} \geq r_{\mathbf{N}^+}$ and as $\mathbf{N}^+$ is a totally real field of degree m, Odlyzko's bounds on discriminants yield $\liminf r_{\mathbf{N}} \geq 8\pi e^{\pi \gamma^2} > 215$ under the assumption of the generalized Riemann hypothesis (see [Ser]).

Proposition 11 *Let* **F** *be a real cyclic cubic field and* **K** *be a non-normal CM-sextic field with maximal totally real subfield* **F**. *Let* **N** *denote the normal closure of* **K**. *Then,* **N** *is a CM-field of degree* 24 *with Galois group* $\mathrm{Gal}(\mathbf{N}/\mathbf{Q})$ *isomorphic to the direct product* $\mathcal{A}_4 \times C_2$, $\mathbf{N}^+$ *is a normal subfield of* **N** *of degree* 12 *and Galois group* $\mathrm{Gal}(\mathbf{N}^+/\mathbf{Q})$ *isomorphic to* $\mathcal{A}_4$, *the compositum* $\mathbf{A} = \mathbf{Fk}$ *which is the maximal abelian subfield of* **N** *is an imaginary sextic field and, finally, we have the following factorization of Dedekind zeta functions:*

$$\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+} = (\zeta_{\mathbf{A}}/\zeta_{\mathbf{F}})(\zeta_{\mathbf{K}}/\zeta_{\mathbf{F}})^3. \tag{15}$$

Moreover, $d_{\mathbf{N}}$ *divides* $d_{\mathbf{K}}^{12}$, *hence* $[1 - (1/12 \log d_{\mathbf{K}}), 1[\subseteq [1 - (1/\log d_{\mathbf{N}}), 1[$.

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Lemma 12 *(See [LLO, Lemma 15]). The Dedekind zeta function of a number field* $\mathbf{M}$ *has at most two real zeros in the range* $1 - (1/\log d_{\mathbf{M}}) \le s < 1$.

Theorem 13 *Let* $\mathbf{K}$ *be a non-normal sextic CM-field with maximal totally real subfield a real cyclic cubic field* $\mathbf{F}$ *of conductor* $f_{\mathbf{F}}$. *Set* $r = d_{\mathbf{K}}^{1/6}$ *(the root discriminant of* $\mathbf{K}$*) and* $r_{\mathbf{K}} = 1 - (6\pi e^{1/72}/r)$. *We have*

$$h_{\mathbf{K}}^{-} \ge \frac{\epsilon_{\mathbf{K}}}{12e^{1/24}\pi^3} \frac{\sqrt{d_{\mathbf{K}}/d_{\mathbf{F}}}}{(\log f_{\mathbf{F}} + 0.05)^2 \log d_{\mathbf{K}}} \ge \frac{\epsilon_{\mathbf{K}}}{6e^{1/24}\pi^3} \left(\frac{\sqrt{r}}{3\log r + 0.1}\right)^3. \quad (16)$$

Therefore, $h_{\mathbf{K}}^{-} > 1$ *implies* $r \le 33000$.

Proof. There are two cases to consider. First, assume that $\zeta_{\mathbf{F}}$ has a real zero β in $[1 - (1/12 \log d_{\mathbf{K}}), 1[$. In that case $\zeta_{\mathbf{K}}(\beta) = 0 \le 0$. Second, assume that $\zeta_{\mathbf{F}}$ does not have any real zero in $[1 - (1/12 \log d_{\mathbf{K}}), 1[$. According to (15) and Lemma 12, we conclude that $\zeta_{\mathbf{K}}$ does not have any real zero in $[1 - (1/12 \log d_{\mathbf{K}}), 1[$ and that $\zeta_{\mathbf{N}}(1 - (1/12 \log d_{\mathbf{K}})) \le 0$. •

We refer the reader to [Bou] for the solution of the class number one problem for these non-normal sextic CM-fields, solution based on refinements of the lower bound given in Theorem 13.

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