数理解析研究所講究録
1163 巻 2000 年 126-131

A cone angle condition on strong convergence of hyperbolic 3-cone-manifolds

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§0. Introduction.

By a hyperbolic 3-cone-manifold, we will mean an orientable riemannian 3-manifold C of constant sectional curvature -1 with cone-type singularity along simple closed geodesics Σ. To each component of the singularity Σ, is associated a cone angle. Kojima showed in [4] that for any values of cone angles, a non-singular part $C-\Sigma$ carries a complete hyperbolic structure C_{comp} of finite volume, and moreover that if the cone angles of C all are at most π, then there is an angle decreasing continuous family of deformations of C to the complete hyperbolic 3-manifold C_{comp} homeomorphic to $C-\Sigma$. The complete hyperbolic 3-manifold C_{comp} has torus cusps at the parts which correspond to the singularity Σ of C, and C_{comp} can be regarded as a hyperbolic 3-cone-manifold with cone angles all equal to zero.

Kojima proved the latter claim by using two machineries, the local rigidity by Hodgson-Kerckhoff [3] and the pointed Hausdorff-Gromov topology [2]. These machineries are fundamental when cone angles are $\leq 2\pi$. In particular, the local rigidity implies the practicability of deformations of a hyperbolic 3-cone-manifold with arbitrary small changes in the cone angles, in the case where the initial cone angles all are at most 2π. Then, if the cone angles of C all are at most π, one obtains deformations of C with decreasing the cone angles with arbitrary small amount. In [4], for extending such a small deformation globally, he analyzed phenomena which occur in the two cases, that is, in the case where tubular neighborhoods of the singularity Σ in the deformations are uniformly thick, and in the case where they collapse. For this analysis, he established three relative constants for hyperbolic 3-cone-manifolds which control the local geometry of cone-manifolds away from the singularity. Lemma 3.1.1 of [4] gives one of them, and is a key lemma to derive the other constants and also to analyze the phenomena above.

In this paper, we will show that the assumption "$\leq \pi$" in Lemma 3.1.1 [4] about the cone angles can be improved to "$< 2\pi$" (see Lemma 2), by using fundamental properties on Dirichlet domains of 3-cone-manifolds (see Lemma 1). Then, without changing the proof performed in the sections 3 and 5 of [4], it can be seen that, for each sequence $\{C_i\}_{i=1}^{\infty}$ consisting of deformations of C so that tubular neighborhoods of Σ in deformations C_i ($i \in \mathbf{N}$) are uniformly thick, if the cone angles of C_i ($i \in \mathbf{N}$) all are less than 2π, then there is a subsequence $\{C_{i_k}\}_{k=1}^{\infty}$ which converges strongly to a hyperbolic 3-cone-manifold C_* homeomorphic to C (see Theorem).

§1. Dirichlet polyhedra and a relative constant for hyperbolic 3-cone-manifolds.

Assume that the singular set Σ of any 3-cone-manifold C considered in this paper forms a link

$$\Sigma = \Sigma^1 \cup \ldots \cup \Sigma^n$$

of n components. To each component Σ^j of Σ, associated is a cone angle $\alpha^j \in [0, \infty)$.

If C is hyperbolic and $\Sigma \neq \phi$, then $N := C - \Sigma$ has a non-singular but incomplete hyperbolic structure and C inherits a metric induced from a riemannian metric on N. We assume that C is complete with this metric. In particular, the metric completion of N is identical to the metric space C. We have a developing map of N from its universal covering space $\tilde{N}$,

$$\mathcal{D}_C : \tilde{N} \to \mathbf{H}^3,$$

and a holonomy representation

$$\rho_C : \pi_1(N) \to \mathrm{PSL}_2(\mathbf{C}).$$

They are called a developing map and a holonomy representation of a cone-manifold C.

Let L be a number with $L \leq -1$. Let $\mathcal{C}_{[L,0]}^{<\theta}$ be the set of pointed compact orientable 3-cone-manifolds of constant sectional curvature $K \in [L, 0]$ so that the cone angles all are less than θ. Let $\mathcal{C}_K^{<\theta}$ be a subset of $\mathcal{C}_{[L,0]}^{<\theta}$ consisting of cone-manifolds with a particular curvature constant K.

Now take a cone-manifold $C \in \mathcal{C}_K^{<2\pi}$ and a point $x \in C - \Sigma$. Then define the following subset of C,

$$P_x := \{y \in C \mid y \text{ admits the unique shortest path to } x\},$$

and call it a Dirichlet fundamental domain of C about x.

Lemma 1. *The Dirichlet fundamental domain P_x of $C \in \mathcal{C}_K^{<2\pi}$ about x has the following properties.*

(1) *P_x is isometrically realized as an interior of a star-shaped geodesic polyhedron in the simply connected 3-dimensional space $\mathbf{H}_K$ of constant curvature K. The closure is star-shaped geodesic polyhedron. We call this embedded compactified polyhedron a Dirichlet polyhedron of C about x, and denote it again by P_x.*

(2) *Let y be a singular point, then there are two boundary faces of P_x both of which include y and whose dihedral angle equals to the cone angle at y. Moreover, the bisecting surface of these two faces contains x.*

Proof. See Cooper-Hodgson-Kerckhoff [1]. □

If $x \notin C - \Sigma$, the injectivity radius of C at x is to be the injectivity radius of $C - \Sigma$ at x. Denote it by $\mathrm{inj}_x C$. The key lemma in this paper is the following:

Lemma 2. *Given positive numbers D, I, $R > 0$, and a curvature bound $L \leq -1$, there is a constant $U := U(D, I, R, L) > 0$ so that if $C \in \mathcal{C}_{[L,0]}^{<2\pi}$, $x \in C$ with $d(x, \Sigma) \geq D$ and $\mathrm{inj}_x C \geq I$, then*

$$\mathrm{inj}_y C \geq U$$

for any $y \in C$ with $d(y, \Sigma) \geq D$ and $d(y, x) \leq R$.

Proof. Suppose that there is not such a uniform bound U. Then, for some numbers D,I,$R > 0$ and $L \leq -1$, there exists a sequence of cone-manifolds $\{C_i\}_{i=1}^{\infty} \subset \mathcal{C}_{[L,0]}^{<2\pi}$ and points x_i, $y_i \in C_i$ such that

(i) $d(x_i, \Sigma_i) \geq D, d(y_i, \Sigma_i) \geq D$,

(ii) $\mathrm{inj}_{x_i} C_i \geq I$,

(iii) $d(y_i, x_i) \leq R$ and

(iv) $\mathrm{inj}_{y_i} C_i \leq 1/i$.

Take a Dirichlet polyhedron P_{y_i} of C_i about y_i in $\mathbf{H}_{K_i}$, where K_i is a curvature of C_i. There are points p_i, q_i on ∂P_{y_i}, which are identified in C_i and attain the shortest distance to y_i from ∂P_{y_i}. The union of these shortest paths $\overline{p_i y_i}, \overline{q_i y_i}$ forms a homotopically nontrivial shortest loop l_i in C_i based at y_i.

If i is large enough, p_i and q_i are on the interior of the faces of P_{y_i} respectively. Then by (i), (iv), and the properties of P_{y_i} described in Lemma 1, it can been seen that P_{y_i} is bounded by the extensions of the two faces.

Let $\phi_i(\leq \pi)$ be the angle between the segments $\overline{p_i y_i}$ and $\overline{q_i y_i}$ at y_i. If $\phi_i \to \pi$ as $i \to \infty$, then $\mathrm{vol}(B_{R+I}(C_i, y_i)) \to 0$ by (iv). This is a contradiction since $B_I(C_i, x_i) \subset B_{R+I}(C_i, y_i)$ by (iii) and $\mathrm{vol}(B_I(C_i, x_i)) > 0$ by (ii). Thus there is a number ϕ so that $\phi_i \leq \phi < \pi$. Therefore the loop l_i bends at y_i with angle uniformly away from π.

Let us lift l_i to a geodesic segment s_i in $\mathbf{H}_{K_i}$, based at y_i so that $p_i(= q_i)$ is its middle point. Let ρ_i be a holonomy representation of C_i; $\rho_i : \pi_1(C_i - \Sigma_i) \to \mathrm{PSL}_2(\mathbf{C})$. Then the action of $\rho_i(l_i)$ on $\mathbf{H}_{K_i}$ is either parabolic, loxodromic or elliptic. In any cases, the orbit of s_i by the action of a group generated by $\rho_i(l_i)$ forms a piecewise geodesic which bends with angle uniformly away from π, and the length of s_i goes to 0 when $i \to \infty$.

If there is a subsequence $\{k\} \subset \{i\}$ so that $\rho_k(l_k)$ all are parabolic, then the orbit of s_k goes to the ideal boundary of $\mathbf{H}_{K_k}$. This a contradiction, since the bending angle of the orbit of s_k should approaches π as $k \to \infty$ in the case where the orbit of s_k goes to ∞ and the length of s_k goes to 0 as $k \to \infty$.

If $\rho_i(l_i)$ is loxodromic, the orbit of s_i squeezes onto the axis of $\rho_i(l_i)$ since the length of s_i approaches 0 when $i \to \infty$. In particular, the axis of $\rho_i(l_i)$ becomes close to y_i when $i \to \infty$.

If there is a subsequence $\{k\} \subset \{i\}$ so that $\rho_k(l_k)$ all are loxodromic, the length of $\rho_k(l_k)$ goes to 0 when $k \to \infty$. If k is large enough, there is a very short simple closed geodesic in C_k near y_k. Then choose a new reference point z_k on this simple closed geodesic, take the Dirichlet polyhedron P_{z_k} about z_k, consider two hypersurfaces of $\mathbf{H}_{K_i}$ which bounds P_{z_k} and perform the same argument as before. This gives a contradiction.

Therefore $\rho_i(l_i)$ all but finitely many exceptions are elliptic. Take a subsequence $\{j\} \subset \{i\}$ so that $\rho_j(l_j)$ all are elliptic. The orbit of s_j rounds around a geodesic which is an extension of a lift of a component of Σ_j. Since the length of s_j goes 0 when $i \to \infty$, y_j approaches the geodesic. This contradicts (i). □

§2. Strong convergence of hyperbolic 3-cone-manifolds.

Let C be a compact orientable hyperbolic 3-cone-manifold with singularity Σ. The singular set Σ has been assumed to form a link

$$\Sigma = \Sigma^1 \cup \ldots \cup \Sigma^n$$

of n components. Let $\mathcal{T}$ be the maximal tube about Σ, that is, a union of open tubular neighborhoods $\mathcal{T}^j$'s which has the following properties,

(a) each component $\mathcal{T}^j$ is an equidistant tubular neighborhood to the j-th component Σ^j of Σ,

(b) among ones having the property (a), the set of radii arranged in order of magnitude from the smallest one is maximal in lexicographical order.

Let us demote by $\partial\mathcal{T}^j$ an abstract boundary of $\mathcal{T}^j$. The actual boundary $\partial\mathcal{T}$ of $\mathcal{T}$ in C is a union of isometrically embedded tori with a finite number of contact points. The first contact point on $\partial\mathcal{T}$ is the point which admits two shortest paths to Σ from $\partial\mathcal{T}$. The finest point on $\partial\mathcal{T}$ is the point on $\partial\mathcal{T}$ which attains the minimum among $\{\mathrm{inj}_x(C) | x \in \partial\mathcal{T}\}$.

A deformation of a hyperbolic 3-cone-manifold C is a hyperbolic 3-cone-manifold C_a together with a reference homeomorphism $\xi_a : (C, \Sigma) \to (C_a, \Sigma_a)$.

Now take a sequence $\{C_i\}_{i=1}^{\infty}$ of compact orientable hyperbolic 3-cone-manifolds with the following properties,

(1) each C_i is a deformation of C with a reference homeomorphism $\xi_i : C \to C_i$,

(2) $\alpha_i^j < 2\pi$ for all $1 \leq j \leq n$ and any $i \in \mathbf{N}$, where α_i^j is a cone angle along the component Σ_i^j,

(3) $\{\alpha_i^j\}_{i=1}^{\infty}$ converges to a number $\beta^j \in [0, 2\pi]$ for all $1 \leq j \leq n$.

Theorem. *Let $\{C_i\}_{i=1}^{\infty}$ be a sequence of compact orientable hyperbolic 3-cone-manifolds as above. Suppose that there is a constant $D_1 > 0$ such that $D_1 \leq$ radius $\mathcal{T}_i^j$ for any $1 \leq j \leq n$ and any $i \in \mathbf{N}$. Then there is a subsequence $\{C_{i_m}\}_{m=1}^{\infty}$ which converges strongly to a hyperbolic 3-cone-manifold C_* homeomorphic to C, where the notion "converge strongly" is defined as follows; the sequence $\{C_{i_m}\}_{m=1}^{\infty}$ converges geometrically to the cone-manifold C_* homeomorphic to C and a sequence $\{\rho_{i_m}\}_{i_m}^{\infty}$ of their holonomy representations converges algebraically to the holonomy representation ρ_* of C_* with respect to the identification by ξ_{i_m}.*

Remark. The property (2) induces the following one,

(4) there is a constant V_{max} such that $\mathrm{vol}(C_i) \leq V_{max}$.

Remark. By the argument on geometric convergence due to Gromov [2], it can be shown that the following property is satisfied,

(5) the sequence $\{(C_i, c_i)\}_{i=1}^{\infty}$ has a subsequence $\{(C_{i_k}, c_{i_k})\}_{k=1}^{\infty}$ which converges geometrically to a complete metric space.

Proof. Take a subsequence $\{i_k\} \subset \{i\}$ which satisfies the properties (1),…,(5). By choosing a further subsequence, we may assume that the sequence $\{C_{i_k}\}_{k=1}^{\infty}$ satisfies the following properties also,

(6) c_{i_k} lies on a component $\partial\mathcal{T}^c_{i_k}$ with a constant reference number c, and

(7) f_{i_k} lies on a component on a component $\partial\mathcal{T}^f_{i_k}$ with a constant reference number f.

Then the sequence $\{c_{i_k}\}_{k=1}^{\infty}$ has the same property as in Kojima [4,section 4], except for the condition on the range of the cone angles.

By following the arguments described in section 3 and section 5 of [4], we can verify that Corollary 5.1.4 of [4] holds with replacing the cone angle condition "$\alpha_i^j \leq \pi$" with "$\alpha_i^j < 2\pi$", if Lemma 3.1.1 of [4] holds with the cone angle condition "$< 2\pi$". Lemma 2 is exactly such a version of Lemma 3.1.1 of [4]. Then Corollary 5.1.4 of [4] with the cone angle condition "$\alpha_i^j < 2\pi$" holds. This is what we need. □

References

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