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LOCAL GEOMETRIC FINITENESS OF KLEINIAN GROUPS

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A Kleinian group is, by definition, a group of orientation preserving isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$ that acts freely and properly discontinuously. We try to extend a criterion for handy finitely generated Kleinian groups, geometric finiteness, to infinitely generated cases and come up with the following concept of local geometric finiteness: A Kleinian group Γ is defined to be *locally geometrically finite* if every finitely generated subgroup of Γ is geometrically finite.

In this note, we consider several conditions from which the local geometric finiteness follows. Especially we regard the following theorem due to Thurston (see [5, Th.3.11]) as a motivation for considering such conditions geometrically and clarify the relationship with analytic conditions given by the Hausdorff dimension of the limit set.

Theorem 1. *Let G be a geometrically finite Kleinian group with the non-empty region of discontinuity (i.e. of the second kind). Then every finitely generated subgroup of G is geometrically finite. Namely, G is locally geometrically finite.*

First of all, we review geometric finiteness of Kleinian groups. The *convex hull* $\tilde{C}_G$ of the limit set $\Lambda(G)$ is the smallest, convex, closed subset in $\mathbb{H}^3$ that contains all geodesic lines with the end points in $\Lambda(G)$. The *convex core* C_G is a convex, closed subset of the hyperbolic 3-manifold $N_G=\mathbb{H}^3/G$ that is the image of $\tilde{C}_G$ under the projection $\mathbb{H}^3\to N_G$. Let $x\in\Lambda(G)$ be a parabolic fixed point of G. We say that a horoball B_x in $\mathbb{H}^3$ tangent at x is a *cusp horoball* if B_x is equivariant under the stabilizer of x in G. The image of a cusp horoball under the projection $\mathbb{H}^3\to N_G$ is called a *cusp neighborhood*. Then one of mutually equivalent characterizations of geometric finiteness for G is that the convex core C_G is compact except for cusp neighborhoods (see [5, Th.3.7]). Another characterization is that $\Lambda(G)$ is coincident with the conical limit set $\Lambda_c(G)$ up to parabolic fixed points.

In this note, we define a Kleinian group G to be *analytically finite* if the relative boundary ∂C_G of the convex core in N_G is compact except for cusp neighborhoods. It is obvious that if G is geometrically finite then it is analytically finite. Moreover, the Ahlfors finiteness theorem (see [5, Th.4.1]) asserts that every finitely generated Kleinian group is analytically finite.

The assumption of Theorem 1 that G has the non-empty region of discontinuity is essential; this is necessary for the proof and there exists a counterexample for the statement if we drop it. This is equivalent to saying that ∂C_G is not empty. However, assuming for G to be geometrically finite is too restricted; in order to prove Theorem 1, we only use a property of the convex core of a geometrically finite Kleinian group, boundedness of the hyperbolic distance from its boundary. We formulate this weaker condition precisely as follows: A Kleinian group G is, by definition, *geometrically bounded* if $\partial C_G \neq \emptyset$ and if

$$\sup\{d(\partial C_G, q) \mid q \in C_G - P_G\} < \infty$$

is satisfied for the union P_G of some cusp neighborhoods, where $d(\cdot,\cdot)$ means the hyperbolic distance.

By the definitions above, we can easily see the following fact:

Proposition 1. *A Kleinian group G is both geometrically bounded and analytically finite if and only if G is geometrically finite with the non-empty region of discontinuity.*

Now we state the extension of Theorem 1 by using the geometric boundedness and exhibit a proof for it.

Theorem 2. *If a Kleinian group G is geometrically bounded then G is locally geometrically finite.*

Proof. We denote $C_G - P_G$ by $(C_G)_0$ and $\tilde{C}_G - \tilde{P}_G$ by $(\tilde{C}_G)_0$ where $\tilde{P}_G$ is the union of cusp horoballs that is the inverse image of P_G. By assumption, $(\tilde{C}_G)_0$ is within a bounded distance of $\partial\tilde{C}_G$.

Let Γ be a finitely generated subgroup of G. We define $(C_\Gamma)_0 = C_\Gamma - P_\Gamma$ and $(\tilde{C}_\Gamma)_0 = \tilde{C}_\Gamma - \tilde{P}_\Gamma$ similarly for Γ, where a cusp horoball $B_x \subset \tilde{P}_\Gamma$ for a parabolic fixed point x of Γ is chosen so that it is coincident with the cusp horoball for G. Then $(\tilde{C}_\Gamma)_0 \cap (\tilde{C}_G)_0$ is within a bounded distance of $\partial\tilde{C}_\Gamma$ because $\tilde{C}_\Gamma \subset \tilde{C}_G$.

Since Γ is analytically finite by the Ahlfors finiteness theorem, we see that

$$(\partial\tilde{C}_\Gamma \cap (\tilde{C}_\Gamma)_0 \cap \tilde{P}_G)/\Gamma$$

is relatively compact. Thus, replacing $\tilde{P}_G$ with smaller cusp horoballs if necessary, we may assume that $(\tilde{C}_\Gamma)_0 \cap \tilde{P}_G = \emptyset$ and hence $(\tilde{C}_\Gamma)_0 \cap (\tilde{C}_G)_0$ is coincident with $(\tilde{C}_\Gamma)_0$. This implies that $(\tilde{C}_\Gamma)_0$ is within a bounded distance of $\partial\tilde{C}_\Gamma$, namely, Γ is geometrically bounded. Hence, by Proposition 1, Γ is geometrically finite. $\square$

Next we move on the Hausdorff dimension of the limit set. The geometric boundedness has a connection with an analytic condition via the following result [4].

Proposition 2. *If a Kleinian group G is geometrically bounded then the Hausdorff dimension* dim $\Lambda(G)$ *of the limit set is strictly less than* 2.

The conclusion of Proposition 2 is still a sufficient condition for local geometric finiteness; it can be easily seen from a famous result due to Bishop and Jones [1].

Theorem 3. *If a Kleinian group G satisfies* dim $\Lambda(G) < 2$ *then G is locally geometrically finite.*

Proof. Let Γ be a finitely generated subgroup of G. Then

$$\dim \Lambda(\Gamma) \leq \dim \Lambda(G) < 2.$$

By the theorem of Bishop and Jones, dim $\Lambda(\Gamma) < 2$ implies that Γ is geometrically finite. □

Actually, we can prove a slightly stronger result than Theorem 3.

Theorem 3′. *If an infinitely generated Kleinian group G satisfies* dim $\Lambda(G) < 2$ *then every finitely generated subgroup Γ of G satisfies the strict inequality*

$$\dim \Lambda(\Gamma) < \dim \Lambda(G).$$

Proof. By Theorem 3, Γ is geometrically finite. Then the critical exponent of the Poincaré series for Γ is equal to dim $\Lambda(\Gamma)$ and the Poincaré series diverges at this critical exponent. As is shown in [3], if $\Lambda(\Gamma)$ is a proper subset of $\Lambda(G)$, which is always the case for finitely generated Γ and infinitely generated G, then the strict inequality on the Hausdorff dimension follows. □

Finally we weaken the assumption of Theorem 3 slightly and prove that local geometric finiteness follows even from this weaker assumption. This is a consequence of the theorem of Bishop and Jones again.

Theorem 4. *If a Kleinian group G satisfies both that the Hausdorff dimension of the conical limit set $\Lambda_c(G)$ is strictly less than 2 and that the 2-dimensional Hausdorff measure μ_2 of $\Lambda(G)$ is zero, then G is locally geometrically finite.*

Proof. Any subgroup Γ of G satisfies dim $\Lambda_c(\Gamma) < 2$ and $\mu_2(\Lambda(\Gamma)) = 0$, too. By the theorem of Bishop and Jones, if Γ is finitely generated but not geometrically finite then either dim $\Lambda_c(\Gamma) = 2$ or $\mu_2(\Lambda(\Gamma)) > 0$. Hence we can see that every finitely generated subgroup Γ is geometrically finite. □

The assumption of Theorem 4 is by no means a sharp condition for local geometric finiteness. In fact, we can construct the following examples:

Examples. Let G be a Kleinian group of the second kind that is exhausted by a sequence of geometrically finite subgroups Γ_n with dim $\Lambda_c(\Gamma_n) \uparrow 2$. For instance, we can take such G as a certain subgroup of a Kleinian group for an infinite cyclic cover of a closed hyperbolic manifold. Then dim $\Lambda_c(G) = 2$, however G is locally geometrically finite. On the other hand, we can construct an infinitely generated Schottky group G of the second kind so that $\mu_2(\Lambda(G)) > 0$ (see [2, Chapter 8]). However, this G is also locally geometrically finite. Moreover, combining these two examples, we can obtain a locally geometrically finite Kleinian group G satisfying both dim $\Lambda_c(G) = 2$ and $\mu_2(\Lambda(G)) > 0$.

Our next problem is to find an interesting necessary condition for local geometric finiteness.

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