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NEW EXTENSIONS OF THE K-K-M TYPE COVERING LEMMA WITH APPLICATIONS TO ECONOMICS*

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Abstract

The existence of equitable allocations of divisible goods is established. The methods used give divisions of a good into geometrically simple sets, such as simplexes or polyhedral convex cones. Market for indivisible goods is modelled, in which a financial intermediary plays the role as an income re-distributer and each consumer can demand as many goods as he wants subject to his budget constraint, and the existence of a competitive equilibrium is proved. These two seemingly unrelated economic problems are solved by applying David Gale's covering lemma and a dual version of its extension. The extension of Gale's lemma and its dual versions are established here; the proofs are based on Ky Fan's fundamental theorem on coincidence of two set-valued functions.

*Extended account of: [1] Tatsuro Ichiishi and Adam Idzik, "Equitable Allocation of Divisible Goods," *Journal of Mathematical Economics* **32** (1999), 389-400; and [2] Tatsuro Ichiishi and Adam Idzik, "Market Allocation of Indivisible Goods," *Journal of Mathematical Economics* **32** (1999), 457-466.

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1 Introduction

We address two types of resource allocation problems: One type is a normative division of a divisible good, and the other type is a descriptive market allocation of indivisible goods.

There has been a substantial amount of the literature that studies allocations of divisible goods, subject to a given welfare criterion such as fairness, equitableness or envy-freeness. While much of the literature concerns divisions of a good into merely *measurable* subsets (see, e.g., Akin (1995) and the references therein), it is desirable from a practical point of view to have divisions into *geometrically simple* subsets, like intervals (see, e.g., Alon (1988)), simplexes or polyhedral convex cones.

The first purpose of our paper is two-fold: (1) to strengthen some of the earlier results on optimal division problems, and (2) to have divisions of a good (a subset of a Euclidean space) into geometrically simple subsets while satisfying a welfare criterion. We establish two theorems on α-equitable divisions, by applying Gale's (1984) extension of the Knaster-Kuratowski-Mazurkiewicz (K-K-M) lemma on closed coverings of a simplex.

We turn to the market allocation of indivisible goods (for another recent work, see van der Laan, Talman and Yang (1997); for normative allocation of indivisible goods, see, e.g., Thomson (1995) and the references therein). Shapley and Scarf (1974) constructed a model of an exchange market, in which each consumer is initially endowed with one unit of an indivisible good. They established the existence of a core allocation, and then in collaboration with David Gale the existence of a competitive equilibrium. Quinzii (1984) introduced a divisible good, called money, into the Shapley-Scarf model, and established the core equivalence (equivalence of a core allocation and a competitive allocation) and the existence of a competitive equilibrium. Gale (1984) provided an extension of the K-K-M lemma, and derived Quinzii's existence result from his extended K-K-M lemma. All three works are on instances of the assignment game broadly defined, in light of their basic postulate that each consumer supplies his indivisible good and demands *one* unit of another. On the other hand, in the price-guided economy of the neoclassical paradigm, a consumer can demand *several* goods as long as these goods are within his budget constraint.

The second purpose of our paper is to consider a modified version of the Shapley-Scarf model in which each consumer can demand *several* goods subject to his budget constraint, thereby bringing together the assignment game and the neoclassical paradigm. Let n be the number of consumers in the economy. While the total demand for each good can be any integer between 0 and n in disequilibrium, it has to be equal to 1 in equilibrium, since the total supply is 1. Therefore, an assignment emerges as a consequence of equilibrium even in our modified Shapley-Scarf model. We introduce to the model a particular role of financial intermediaries, and establish the existence of a competitive equilibrium.

The above two seemingly unrelated economic problems have one thing in common: Gale's (1984) covering lemma. Gale considered n covers of an $(n-1)$-dimensional simplex, each satisfying the K-K-M type boundary condition. In our paper we consider more general theorems on n covers of a simplex. The first result along this line of research is an extension of Gale's lemma, which allows for covers that satisfy Shapley's (1973) boundary condition. The next result is a dual version of our extended Gale lemma, which allows for covers that satisfy the boundary condition studied by Alexandrov and Pasynkov (1957) and Scarf (1967). The third result is yet another dual version of the extended Gale lemma, which allows for covers that satisfy Ichiishi's (1988) boundary condition. It is this second dual version that we apply to establish our aforementioned existence result for the modified Shapley-Scarf model.

The next section presents our extension of Gale's lemma and its two dual versions. Section 3 presents our results on α-equitable divisions. Section 4 presents our result on the modified Shapley-Scarf model.

2 An extension of Gale's covering lemma and its dual versions

This section establishes an extension (Theorem 2.3A) of Gale's (1984) covering lemma and its dual versions (Theorems 2.3B and 2.3C).

Let N be a nonempty finite set. The cardinality of set N is denoted by $\#N$. Denote by $\mathbf{R}^N$ the $(\#N)$-dimensional Euclidean space and by $\mathbf{R}^N_+$ the nonnegative orthant of $\mathbf{R}^N$. Given a subset X of $\mathbf{R}^N$, let co X denote the convex hull of X, int X denote the interior of X, $\mathcal{P}(X)$ denote the family of all nonempty subsets of X, aff X denote the affine hull of X, ri X denote the relative interior of X and ∂X denote the relative boundary of X. Let

$f : X \to \mathcal{P}(\mathbf{R}^N)$ be a function. We say that f is *upper semicontinuous* (*u.s.c.*) on X if the set $\{x \in X \mid f(x) \subset V\}$ is open in X whenever V is an open subset of $\mathbf{R}^N$. Functions $f, g : X \to \mathcal{P}(\mathbf{R}^N)$ have a *coincidence* if there exists $x \in X$ such that $f(x) \cap g(x) \neq \emptyset$. The unit vectors of $\mathbf{R}^N$ are denoted by e_j, $j \in N$; here $e_j^j = 1$ and $e_j^i = 0$ for all $i \in N \setminus \{j\}$. The *unit simplex* is the set $\Delta^N := \text{co}\ \{e_j \mid j \in N\}$ and its *faces* are $\Delta^S := \text{co}\ \{e_j \mid j \in S\}$, $S \subset N$. For a set A and a point x in $\mathbf{R}^N$, define a set $A - x := \{a - x \in \mathbf{R}^N \mid a \in A\}$.

The Euclidean inner product of two vectors x and y in $\mathbf{R}^N$ is denoted by $x \cdot y$. We recall that a *hyperplane* H in $\mathbf{R}^N$ is a set of the form $H = \{x \in \mathbf{R}^N \mid p \cdot x = t\}$, where $p \in \mathbf{R}^N$, $p \neq \mathbf{0}$, and t is a real number. Given a compact convex set X in $\mathbf{R}^N$, a proper subset F of X is called a *proper face* of X, if there exist $p \in \mathbf{R}^N \setminus \{0\}$ and $t \in \mathbf{R}$ such that $F = X \cap \{x \in \mathbf{R}^N \mid p \cdot x = t\}$, and $p \cdot x > t$ for every $x \in X \setminus F$. In this case, the hyperplane $H = \{x \in \mathbf{R}^N \mid p \cdot x = t\}$ is called a *supporting hyperplane* of X. A face G of a compact convex set X is an *opposite* face to face F of X, if $G = X \cap H_1$ and $F = X \cap H_2$ for some parallel supporting hyperplanes H_1 and H_2 of X.

Ky Fan (1972a) proved the following fundamental theorem on coincidence (we formulate its special case here):

Theorem 2.0 (Fan (1972a, Theorem 3)) *Let X be a nonempty compact convex subset of $\mathbf{R}^N$, and let f and g be upper semicontinuous functions from X to $\mathcal{P}(\mathbf{R}^N)$ such that both $f(x)$ and $g(x)$ are nonempty compact convex sets for each $x \in X$, and such that*

$$(\forall\ x \in X) : (\forall\ p \in \mathbf{R}^N : p \cdot x = \min\{p \cdot z \mid z \in X\}) :$$
$$\exists\ u \in f(x) :\ \exists\ v \in g(x) :\ p \cdot u \geq p \cdot v.$$

Then there exists $x \in X$ such that $f(x) \cap g(x) \neq \emptyset$.

We first formulate two special cases (Theorems 2.1A and 2.1B) of the foregoing theorem. Some special cases of these theorems have found applications to unification of theorems on covering of compact convex sets (Ichiishi (1981, 1988), Ichiishi and Idzik (1990, 1991)), to solutions of inequalities (Fan (1968, 1972b)), to the theory of equilibrium (Gale (1984)), to the fair division problem (Akin (1995)), and to multidimensional matrices (Bapat (1982), Bapat and Raghavan (1989)).

Theorem 2.1A *Let X be a nonempty compact convex subset of $\mathbf{R}^N$ and let $f : X \to \mathcal{P}(\mathbf{R}^N)$ and $g : X \to \mathcal{P}(X)$ be upper semicontinuous functions such* *that both $f(x)$ and $g(x)$ are nonempty compact convex sets for each $x \in X$, and such that*

$$(\forall\, x \in X): \;\; (\forall\, p \in \mathbf{R}^N: \; p \cdot x = \min\{p \cdot z \mid z \in X\}): \\ \exists\, u \in f(x): \; p \cdot u = \min\{p \cdot z \mid z \in X\}.$$

Then f and g have a coincidence. In particular: $X \subset f(X)$, and each of f and g has a fixed point.

Theorem 2.1B *Let X be a nonempty compact convex subset of $\mathbf{R}^N$, and let $f : X \to \mathcal{P}(\mathbf{R}^N)$ and $g : X \to \mathcal{P}(X)$ be upper semicontinuous functions such that both $f(x)$ and $g(x)$ are nonempty compact convex sets for each $x \in X$, and such that*

$$(\forall\, x \in X): \;\; (\forall\, p \in \mathbf{R}^N: \; p \cdot x = \min\{p \cdot z \mid z \in X\}): \\ \exists\, u \in f(x): \; p \cdot u = \max\{p \cdot z \mid z \in X\}.$$

Then f and g have a coincidence. In particular: $X \subset f(X)$, and f has a fixed point.

We present yet further special cases (the following Corollaries 2.2A and 2.2B).

Corollary 2.2A *Let X be a nonempty compact convex subset of $\mathbf{R}^N$. Let $f : X \to \mathcal{P}(\mathbf{R}^N)$ and $g : X \to \mathcal{P}(X)$ be upper semicontinuous functions, such that both $f(x)$ and $g(x)$ are nonempty compact convex sets for each $x \in X$. Let f transform every face F of X in such a way that for each $x \in F$, $f(x) \cap \operatorname{aff} F \neq \emptyset$ (which would be the case, e.g., if f transforms every face F of X into aff F). Then f and g have a coincidence. In particular: $X \subset f(X)$, and f and g have fixed points.*

Corollary 2.2A reduces to Akin (1995, Proposition 17), when f transforms every face F of X into F.

Corollary 2.2B *Let X be a nonempty compact convex subset of $\mathbf{R}^N$. Let $f : X \to \mathcal{P}(\mathbf{R}^N)$ and $g : X \to \mathcal{P}(X)$ be upper semicontinuous functions such that both $f(x)$ and $g(x)$ are nonempty compact convex sets for each $x \in X$. Let f transform every face F of X in such a way that $f(F) \cap \operatorname{aff} G \neq \emptyset$ (which* *would be the case, e.g., if f transforms every face F of X into* aff G*), for each face G opposite to F. Then f and g have a coincidence. In particular: $X \subset f(X)$, and f has a fixed point.*

Now, let $n := \# N$, and define $m_S := \sum_{j \in S} e_j / (\# S)$ for each $S \in \mathcal{P}(N)$. Choose a set K in $\mathbf{R}^N$ such that

$$\{e_j \mid j \in N\} \subset K \subset \text{ aff } \Delta^N, \ \# K < \infty;$$

the set K will be fixed throughout this section. A point $v \in$ aff Δ^N is uniquely expressed as an affine combination of the vertices of Δ^N, $v = \sum_{j \in N} b_j^v e_j$, $b_j^v \in \mathbf{R}$, $\sum_{j \in N} b_j^v = 1$. The *support* of v is the set of j for which $b_j^v \neq 0$, and is denoted by supp v.

Theorem 2.3A *For each $i \in N$, let $\{C_i^v\}_{v \in K}$ be a closed cover of Δ^N satisfying*

$$\Delta^T \subset \bigcup \left\{ C_i^v \;\middle|\; v \in K \cap \text{aff } \Delta^T \right\} \quad \textit{for every } T \in \mathcal{P}(N) \setminus \{N\}.$$

Then there exists a function $\pi : N \to K$ such that

$$\bigcap_{i \in N} C_i^{\pi(i)} \neq \emptyset \quad \textit{and} \quad \bigcup_{i \in N} \text{supp } \pi(i) = N.$$

The closed covers considered in Theorem 2.3A were studied by Ichiishi and Idzik (1990, Theorem 2.1). Theorem 2.3A reduces to Gale's (1984) lemma when $K = \{e_j \mid j \in N\}$. In this case each cover $\{C_i^v\}_{v \in K}$ is of the K-K-M type, that is, the boundary condition is: $\Delta^T \subset \cup_{j \in T} C_i^{e_j}$ for every proper subset T of N. Gale's lemma strengthens Svensson's theorem (1983, Theorem 5). Our theorem allows for covers of Shapley's (1973) type ($K = \{m_S \mid S \in \mathcal{P}(N)\}$); in this case the boundary condition becomes: $\Delta^T \subset \cup_{S \subset T} C_i^{m_S}$ for every proper subset T of N.

Proof of Theorem 2.3A For each $i \in N$ and each $S \in \mathcal{P}(N)$, define

$$\tilde{C}_i^S := \bigcup \{C_i^v \mid v \in K, \ \text{ supp } v = S\}.$$

Choose any $x \in \Delta^N$ and define

$$f_i(x) := \text{co}\left\{m_S \;\middle|\; S \in \mathcal{P}(N),\ \tilde{C}_i^S \ni x\right\},$$
$$f := \frac{1}{n}\sum_{i\in N} f_i.$$

Observe that the function $f : \Delta^N \to \mathcal{P}(\Delta^N)$ is u.s.c., and that $v \in$ aff Δ^S iff supp $v \subset S$, so by Corollary 2.2A,

$$\exists\, x^* \in \Delta^N :\ m_N \in f(x^*). \tag{1}$$

Let $nm_N = \sum_{i\in N} x_i^*$, where $x_i^* \in f_i(x^*)$.

By definition of $f_i(x^*)$, there exist $a_i^S \geq 0$, $S \in \mathcal{P}(N)$, such that $[a_i^S > 0$ only if $\tilde{C}_i^S \ni x^*]$, $\sum_{S\in\mathcal{P}(N)} a_i^S = 1$, and $x_i^* = \sum_{S\in\mathcal{P}(N)} a_i^S m_S$.

For each $S \in \mathcal{P}(N)$, define $b_j^S \in \mathbf{R}$, $j \in N$, by: $b_j^S := 1/(\#S)$, if $S \ni j$; and $b_j^S := 0$, if $S \not\ni j$. Then $m_S := \sum_{j\in N} b_j^S e_j$.

We claim that the $n \times n$ matrix whose (i,j)-element is $\sum_{S\in\mathcal{P}(N)} a_i^S b_j^S$ is bistochastic. Indeed, by definition of $\{a_i^S\}_{S\in\mathcal{P}(N)}$ and $\{b_j^S\}_{j\in N}$ as convex coefficients,

$$\forall\, i \in N :\ \sum_{j\in N}\left(\sum_{S\in\mathcal{P}(N)} a_i^S b_j^S\right) = 1.$$

On the other hand, by looking at each component of the vector equality, $nm_N = \sum_{i\in N} x_i^*$, it follows that

$$\forall\, j \in N :\ \sum_{i\in N}\left(\sum_{S\in\mathcal{P}(N)} a_i^S b_j^S\right) = 1.$$

Thus by the Birkhoff - von Neumann theorem, there exists a permutation $\tilde{\pi} : N \to N$ such that

$$\forall\, i \in N :\ \sum_{S\in\mathcal{P}(N)} a_i^S b_{\tilde{\pi}(i)}^S > 0.$$

Consequently,

$$\forall\, i \in N :\ \exists\, S(i) \in \mathcal{P}(N) :\ a_i^{S(i)} b_{\tilde{\pi}(i)}^{S(i)} > 0.$$

For each $i \in N$, $a_i^{S(i)} > 0$, so $\tilde{C}_i^{S(i)} \ni x^*$. In view of the definition of $\tilde{C}_i^S$, we can choose $\pi(i) \in K$ such that $\text{supp}\ \pi(i) = S(i)$ and $C_i^{\pi(i)} \ni x^*$.

We will show that $\pi : N \to K$ is the required function. First, it is clear that $\cap_{i \in N} C_i^{\pi(i)} \ni x^*$. Second, for each i, $b_{\tilde{\pi}(i)}^{S(i)} > 0$, so $\tilde{\pi}(i) \in S(i)$. Since $\tilde{\pi}$ is a permutation,

$$N = \bigcup_{i \in N} \{\tilde{\pi}(i)\} \subset \bigcup_{i \in N} S(i) = \bigcup_{i \in N} \text{supp}\ \pi(i) \subset N,$$

which establishes the second required result. □

Theorem 2.3B *For each $i \in N$, let $\{C_i^v\}_{v \in K}$ be a closed cover of Δ^N satisfying*

$$\Delta^{N \setminus \{j\}} \subset C_i^{e_j} \text{ for every } j \in N.$$

Then there exists a function $\pi : N \to K$ such that

$$\bigcap_{i \in N} C_i^{\pi(i)} \neq \emptyset \quad \text{and} \quad \bigcup_{i \in N} \text{supp}\ \pi(i) = N.$$

For the case $K \subset \Delta^N$, the type of closed covers considered in Theorem 2.3B was studied by Alexandrov and Pasynkov (1957) and by Scarf (1967).

Proof of Theorem 2.3B Define $\tilde{C}_i^S$, f_i and f as in the proof of Theorem 2.3A. Notice that $\tilde{C}_i^{\{j\}} = C_i^{e_j}$. In view of the present boundary condition, Corollary 2.2B is applicable, so condition (1) in the proof of Theorem 2.3A is satisfied. The rest of the proof is the same as the proof of Theorem 2.3A. □

Theorem 2.3C *Suppose $K \subset \Delta^N$ and has the property that for each $v \in K \cap \partial\Delta^N$ there exists $v' \in K$ such that $m_N \in \text{co}\ \{v, v'\}$. For each $i \in N$, let $\{C_i^v\}_{v \in K}$ be a closed cover of Δ^N satisfying*

$$\Delta^T \subset \bigcup \left\{ C_i^{v'} \;\middle|\; v \in K \cap \Delta^T \right\} \text{ for every } T \in \mathcal{P}(N) \setminus \{N\}.$$

Then there exists a function $\pi : N \to K$ such that

$$\bigcap_{i \in N} C_i^{\pi(i)} \neq \emptyset \quad \text{and} \quad \bigcup_{i \in N} \text{supp}\ \pi(i) = N.$$

The closed covers considered in Theorem 2.3C were studied by Ichiishi and Idzik (1990, Theorem 2.5). They reduce to the type considered by Ichiishi (1988) when $K = \{m_S \mid S \in \mathcal{P}(N)\}$ and $(m_S)' = m_{N\setminus S}$; in this case the boundary condition becomes: $\Delta^T \subset \cup_{S\supset N\setminus T} C_i^{m_S}$ for each proper subset T of N.

Proof of Theorem 2.3C Choose any $x \in \Delta^N$ and define

$$\begin{aligned} f_i(x) &:= \text{co}\,\{v \in K \mid C_i^v \ni x\}, \\ f &:= \frac{1}{n}\sum_{i\in N} f_i. \end{aligned}$$

The function $f : \Delta^N \to \mathcal{P}(\Delta^N)$ is u.s.c. Choose any $T \subset N$, $x \in$ ri Δ^T, and $p \in \mathbf{R}^N$ for which $p \cdot x = \min p \cdot \Delta^N$. If $T = N$, then $p \cdot y = \min p \cdot \Delta^N$ for all $y \in \Delta^N$, in particular, $p \cdot m_N = p \cdot y$ for all $y \in f(x)$. If $T \neq N$, then by the present assumption,

$$\exists\, v_i \in K \bigcap \Delta^T : \ (v_i)' \in f_i(x).$$

Since $p \cdot v_i = \min p \cdot \Delta^N$, it follows that $p \cdot v_i \leq p \cdot (v_i)'$. But m_N lies on the segment that joins v_i and $(v_i)'$, so there exists $\alpha \in [0, 1)$ such that

$$p \cdot m_N = \alpha p \cdot v_i + (1-\alpha) p \cdot (v_i)',$$

and consequently

$$p \cdot m_N \leq p \cdot (v_i)'.$$

Set $y := \sum_{i\in N}(v_i)'/n \in f(x)$. Then, $p \cdot m_N \leq p \cdot y$. Define constant function $g : \Delta^N \to \mathcal{P}(\Delta^N)$ by $g(x) \equiv \{m_N\}$.

Ky Fan's coincidence theorem (Theorem 2.0) is now applicable here, so

$$\exists\, x^* \in \Delta^N : \ f(x^*) \bigcap g(x^*) \neq \emptyset,$$

that is, $m_N \in f(x^*)$.

The rest of the proof follows the idea of the proof of Theorem 2.3A. Indeed, let $nm_N = \sum_{i\in N} x_i^*$, where $x_i^* \in f_i(x^*)$. By definition of $f_i(x^*)$, there exist $a_i^v \geq 0$, $v \in K$, such that $[a_i^v > 0$ only if $C_i^v \ni x^*]$, $\sum_{v\in K} a_i^v = 1$, and $x_i^* = \sum_{v \in K} a_i^v v$. Since $K \subset \Delta^N$, each $v \in K$ is a convex combination of the vertices of Δ^N, so there exist uniquely $b_j^v \geq 0$, $j \in N$, such that $\sum_{j \in N} b_j^v = 1$, and $v = \sum_{j \in N} b_j^v e_j$. The $n \times n$ matrix whose (i,j)-element is $\sum_{v \in K} a_i^v b_j^v$ is bistochastic. By the Birkhoff - von Neumann theorem, there exists a permutation $\tilde{\pi} : N \to N$ such that

$$\forall\, i \in N : \sum_{v \in K} a_i^v b_{\tilde{\pi}(i)}^v > 0.$$

Consequently,

$$\forall\, i \in N : \exists\, \pi(i) \in K : a_i^{\pi(i)} b_{\tilde{\pi}(i)}^{\pi(i)} > 0.$$

It is easy to check that $\pi : N \to K$ is the required function. □

Remark 2.4 Theorems 2.3A, 2.3B and 2.3C are also true, if instead of closed covers of Δ^N we consider open covers. Both forms are equivalent. □

A subfamily $\mathcal{B}$ of $\mathcal{P}(N)$ is called *balanced* if there exist $\lambda^S \geq 0$, $S \in \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \lambda^S m_S = m_N$. The following Corollaries 2.5A, 2.5B and 2.5C on n covers of a simplex are consequences of our proofs of Theorems 2.3A, 2.3B and 2.3C, and generalize the theorems on a balanced family due to Shapley (1973), Scarf (1967) and Ichiishi (1988), respectively. (Each corollary reduces to the respective theorem on a balanced family, if the n covers are identical.)

Corollary 2.5A *Suppose $K = \{m_S \mid S \in \mathcal{P}(N)\}$. For each $i \in N$, let $\{C_i^v\}_{v \in K}$ be a closed cover of Δ^N satisfying $\Delta^T \subset \cup_{S \subset T} C_i^{m_S}$ for every $T \in \mathcal{P}(N) \setminus \{N\}$. Then for each $i \in N$ there exist $\mathcal{B}_i \subset \mathcal{P}(N)$, $\mathcal{B}_i \neq \emptyset$, and $\lambda_i^S > 0$, $S \in \mathcal{B}_i$, such that*

$$\bigcap_{i \in N} \bigcap_{S \in \mathcal{B}_i} C_i^{m_S} \neq \emptyset \quad \textit{and} \quad m_N = \frac{1}{n} \sum_{i \in N} \sum_{S \in \mathcal{B}_i} \lambda_i^S m_S.$$

Proof In this case, the set $\tilde{C}_i^S$ defined in the proof of Theorem 2.3A is equal to $C_i^{m_S}$. The required result is a re-statement of condition (1). □

Corollary 2.5B *For each $i \in N$, let $\{C_i^v\}_{v \in K}$ be a closed cover of Δ^N satisfying $\Delta^{N \setminus \{j\}} \subset C_i^{e_j}$ for every $j \in N$. Then for each $i \in N$ there exist $B_i \subset K$, $B_i \neq \emptyset$, and $\lambda_i^v > 0$, $v \in B_i$, such that*

$$\bigcap_{i \in N} \bigcap_{v \in B_i} C_i^v \neq \emptyset \quad \textit{and} \quad m_N = \frac{1}{n} \sum_{i \in N} \sum_{v \in B_i} \lambda_i^v v.$$

Proof Choose any $x \in \Delta^N$ and define

$$f_i(x) := \text{co}\,\{v \in K \mid C_i^v \ni x\},$$
$$f := \frac{1}{n}\sum_{i\in N} f_i.$$

Then, by the same argument as in the proof of Theorem 2.3B, there exists $x^* \in \Delta^N$ such that $m_N \in f(x^*)$. The required result follows. □

Corollary 2.5C *Suppose $K = \{m_S \mid S \in \mathcal{P}(N)\}$. For each $i \in N$, let $\{C_i^v\}_{v\in K}$ be a closed cover of Δ^N satisfying $\Delta^T \subset \cup_{S\supset N\setminus T} C_i^{m_S}$ for every $T \in \mathcal{P}(N) \setminus \{N\}$. Then for each $i \in N$ there exist $\mathcal{B}_i \subset \mathcal{P}(N)$, $\mathcal{B}_i \neq \emptyset$, and $\lambda_i^S > 0$, $S \in \mathcal{B}_i$, such that*

$$\bigcap_{i\in N}\bigcap_{S\in\mathcal{B}_i} C_i^{m_S} \neq \emptyset \quad \textit{and} \quad m_N = \frac{1}{n}\sum_{i\in N}\sum_{S\in\mathcal{B}_i}\lambda_i^S m_S.$$

Proof Define f_i and f as in the proof of Theorem 2.3C. There exists $x^* \in \Delta^N$ such that $m_N \in f(x^*)$. The required result follows. □

3 Equitable allocation of divisible goods

Let $N = \{1,\ldots,n\}$ and let $\alpha = (\alpha_1,\ldots,\alpha_n)$, where α_i $(i \in N)$ is a positive number and $\sum_{i\in N}\alpha_i = 1$. Consider a problem of dividing the unit simplex Δ^N into n subsets for n persons. Each person has a nonatomic signed measure μ_i (the measure of a subset of hyperplanes in aff Δ^N is equal to zero and values can be negative) defined on (Lebesgue) measurable subsets of Δ^N such that $\mu_i(\Delta^N) > 0$, and wants to get a subset $A_i \subset \Delta^N$ which has at least $1/n$ of the value of Δ^N according to his own measure, i.e. $\mu_i(A_i) \geq (1/n)\mu_i(\Delta^N)$ $(i \in N)$.

We call a measurable division (partition) $B = (B_1,\ldots,B_n)$ of Δ^N

- *α-fair*, if there exists a bijection $\pi : N \to N$ such that for every $i \in N$ $\mu_i(B_{\pi(i)}) \geq \alpha_{\pi(i)}\mu_i(\Delta^N)$,

- α–*equitable*, if there exists a bijection $\pi : N \to N$ such that for every $i \in N$ $\alpha_{\pi(i)}^{-1}\mu_i(B_{\pi(i)}) \geq \alpha_j^{-1}\mu_i(B_j)$ for all $j \in N$,
- *envy–free*, if there exists a bijection $\pi : N \to N$ such that for every $i \in N$ $\mu_i(B_{\pi(i)}) \geq \mu_i(B_j)$ for all $j \in N$.

Observe that if a division is α–equitable, then it is α–fair. In the case $\alpha_i = 1/n$ $(i \in N)$ α–equitable divisions coincide with envy–free divisions.

The purpose of this section is to establish the existence of α-equitable divisions. While some of the earlier papers (e.g., Weller (1985), Berliant, Thomson and Dunz (1992) and Akin (1995)) came up with merely measurable divisions, we divide a good into simple subsets, like simplexes or polyhedral convex cones.

Idzik (1995) generalized Woodal's theorem (Woodal (1980, Theorem 3)) and proved the existence of an envy–free division consisting of intervals:

Theorem 3.1 (Idzik, (1995, Theorem 2.2)) *Let $\mu_1, \ldots, \mu_n$ be n nonatomic signed measures defined on the unit interval $I = [0,1]$ such that $\mu_i(I) > 0$ for $i \in N$, and let α_i $(i \in N)$ be a positive number with $\sum_{i\in N} \alpha_i = 1$. Then there exist a partition of I into n subintervals $I_1, \ldots, I_n$ (in order along I) and a bijection $\pi : N \to N$ such that for every $i \in N$*

$$\alpha_{\pi(i)}^{-1}\mu_i(I_{\pi(i)}) \geq \alpha_j^{-1}\mu_i(I_j) \quad \text{for } j \in N,$$

i.e. there exists an α–equitable division of I into intervals.

Now we apply the idea of the proof of Theorem 3.1 and establish the following theorem:

Theorem 3.2 *Let $\mu_1, \ldots, \mu_n$ be n nonatomic signed measures defined on the unit simplex Δ^N such that $\mu_i(\Delta^N) > 0$ for $i \in N$, and let α_i $(i \in N)$ be a positive number with $\sum_{i\in N} \alpha_i = 1$. Then there exist a partition of Δ^N into n subsimplexes $\Delta_1^N, \ldots, \Delta_n^N$ and a bijection $\pi : N \to N$ such that for every $i \in N$*

$$\alpha_{\pi(i)}^{-1}\mu_i(\Delta_{\pi(i)}^N) \geq \alpha_j^{-1}\mu_i(\Delta_j^N) \quad \text{for } j \in N,$$

i.e. there exists an α–equitable division of Δ^N into subsimplexes.

Proof For any point $x \in \Delta^N$ and any $i \in N$, denote by $\Delta_i^N(x)$ the set co $[\{x\} \cup \Delta^{N\setminus\{i\}}]$, and define

$$C_i^j := \{x \in \Delta^N \mid \alpha_j^{-1}\mu_i(\Delta_j^N(x)) \geq \alpha_s^{-1}\mu_i(\Delta_s^N(x)) \text{ for all } s \in N\}.$$

Observe that for each i, $\{C_i^j\}_{j\in N}$ is a cover of Δ^N. The set C_i^j is closed and does not contain subsimplex $\Delta^{N\setminus\{j\}}$. So the assumptions in Sperner's theorem (see Ichiishi and Idzik (1990, Theorem 1.1)), hence the conditions of Theorem 2.3A, are fulfilled. Direct application of Theorem 2.3A establishes the present theorem. □

Remark 3.3 Observe that Theorem 3.2 can describe a more general situation: Instead of the simplex Δ^N we can consider as an object any bounded Lebesgue measurable subset A of aff Δ^N such that $\mu_i(A) > 0$ for $i \in N$; we can assume without loss of generality that $A \subset \Delta^N$. □

We now turn to a problem of the Kuratowski-Steinhaus type. Let $N_0 = \{0, 1, \ldots, n\}$. Choose $p_i \in \mathbf{R}^N$, $i \in N_0$, so that $P :=$ co $\{p_0, p_1, \ldots, p_n\}$ is an n-dimensional simplex and $0 \in$ int P. Define the faces,

$$P_i := \text{ co } \{p_0, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n\}, \; i \in N_0,$$

and the cones

$$M_i = \{\lambda x \in \mathbf{R}^N \mid \lambda \geq 0, \; x \in P_i\}.$$

Let $K_r := \{x \in \mathbf{R}^N \mid \|x\| \leq r\}$, $r > 0$, where $\|\cdot\|$ is the Euclidean norm for $\mathbf{R}^N$.

Theorem 3.4 *Let $A \subset \mathbf{R}^N$ be a bounded Lebesgue measurable set. Let $\mu_0, \ldots, \mu_n$ be $(n+1)$ nonatomic signed measures defined on the Lebesgue measurable subsets of $\mathbf{R}^N$ such that for some $r > 0$, $\mu_i(A - x) > 0$ for all $x \in \mathbf{R}^N \setminus K_r$ and all $i \in N_0$. Let α_i $(i \in N_0)$ be a positive number with $\sum_{i\in N_0} \alpha_i = 1$. Then there exist a point $x \in \mathbf{R}^N$ and a bijection $\pi : N_0 \to N_0$ such that for every $i \in N_0$,*

$$\alpha_{\pi(i)}^{-1}\mu_i((A - x) \cap M_{\pi(i)}) \geq \alpha_j^{-1}\mu_i((A - x) \cap M_j) \text{ for all } j \in N_0.$$

i.e. there exists a point $x \in \mathbf{R}^N$ which generates an α-equitable division of $(A - x)$ into $(n+1)$ sets, each contained in its associated cone M_i, $i \in N_0$.

Proof Since the set A is bounded, we can choose a real number $s > 0$ large enough so that the following properties hold true: Let $d_i := sp_i$, $i \in N_0$. For the simplex D and its faces defined as $D :=$ co $\{d_0, \ldots, d_n\}$, $D_i :=$ co $\{d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n\}$, we have

$$\begin{aligned} &A \subset D, \\ &K_r \subset \text{ int } D, \quad \text{and} \\ &(A - x) \cap M_i = \emptyset \quad \text{for all } x \in D_i. \end{aligned}$$

Define

$$C_i^j := \left\{ x \in D \;\middle|\; \begin{array}{r} \alpha_j^{-1}\mu_i((A-x) \cap M_j) \geq \alpha_s^{-1}\mu_i((A-x) \cap M_s), \\ \text{for each } s \in N_0 \end{array} \right\}.$$

Observe that for each i, $\{C_i^j\}_{j \in N_0}$ is a cover of D. The set C_i^j is closed and does not contain the subsimplex D_j, because $K_r \subset$ int D and consequently $\mu_i(A - x) > 0$ for each $x \in D_j$, but for this x $\mu_i((A - x) \cap M_j) = 0$. So the assumptions in Sperner's theorem (see Ichiishi and Idzik (1990, Theorem 1.1)), hence the conditions of Theorem 2.3A, are fulfilled. Direct application of Theorem 2.3A establishes the present theorem. □

Theorem 3.4 generalizes the result of Kulpa (1994, Corollary, p. 47); Kulpa considered α-fair allocations for the case in which μ_i is the Lebesgue measure for all $i \in N_0$.

4 Market allocation of indivisible goods: The case of segmented housing market with a financial intermediary

This section studies a model of indivisible goods that are traded in a competitive market. Let N be a set of n consumers, $n < \infty$. There is a financial intermediary besides these n consumers. Each consumer j initially holds one unit of an indivisible good (say, a house), called here the jth good. A consumer can obtain a loan from, or make an investment in, a financial intermediary, but his initial balance at the financial intermediary is zero. The loan/investment is a special form of money, so its price is equal to 1. He can buy as many indivisible goods (called henceforth simply goods) as he wishes subject to his budget constraint, but knowing that there is one and only one unit of each good available in the economy, he demands at most one unit of each good. Denote by $\mathcal{P}(N)$ the family of nonempty subsets of N, $2^N \setminus \{\emptyset\}$. Each consumer's *consumption set* is $\mathbf{R} \times \mathcal{P}(N)$; an element $(t, S) \in \mathbf{R} \times \mathcal{P}(N)$ means that he obtains a loan t and holds the set of goods S. A negative loan t means a positive investment $|t|$. In the following, the phrase "to receive a loan t" will be used synonymously with the phrase "to make an investment $-t$". Implicit in our formulation of a consumption set is the postulate that a consumer has to hold at least one good. Consumer j's *initial endowment* is $(0, \{j\}) \in \mathbf{R} \times \mathcal{P}(N)$. His preference relation is summarily represented by a price-dependent *continuous utility function* $u_j : \mathbf{R} \times \mathcal{P}(N) \times \mathbf{R}^N_+ \to \mathbf{R}$. Here, function u_j incorporates both consumer j's taste and the financial intermediary's behavior in the following way: A commodity bundle (t, S) and the lending interest rate on a loan or the deposit interest rate on an investment determine the consumer's utility level, and the financial intermediary is postulated to determine these interest rates as continuous functions of prices $(1, p)$, hence the price-dependent utility function $u_j(t, S, p)$. Each consumer is a price-taker. The financial intermediary also takes prices of goods as given, but has a monopoly power over the loan/investment market.

A *pure exchange economy with indivisible goods and a financial intermediary* (called henceforth simply an *economy*) is a specified list of data $\{\mathbf{R} \times \mathcal{P}(N), u_j, (0, \{j\})\}_{j \in N}$ of consumption set $\mathbf{R} \times \mathcal{P}(N)$, utility function $u_j : \mathbf{R} \times \mathcal{P}(N) \times \mathbf{R}^N_+ \to \mathbf{R}$ and initial endowment $(0, \{j\})$ for every consumer $j \in N$.

When price vector $p \in \mathbf{R}^N_+$ of the goods prevails, consumer j sells his initial endowment in the market, thereby receives the sale value p_j. He may also decide the amount of a loan t_j he receives from the financial intermediary. Let $S \subset N$ be the set of goods he purchases. His total expenditure on goods is then $\sum_{i \in S} p_i$, and he has to satisfy his budget constraint,

$$\sum_{i \in S} p_i \leq p_j + t_j.$$

Consumer j's demand behavior is summarized by his *inverse demand correspondence* from $\mathcal{P}(N)$ to the subsets of $\mathbf{R}^N_+$, $S \mapsto C^S_j$. Here, $p \in C^S_j$ means that j demands goods S if p is the prevailing market price vector of goods; in light of the budget constraint, he is also obtaining a loan of $t_j \geq$

$\sum_{i\in S} p_i - p_j$. His behavior comes from utility-maximization, so $u_j(t_j, S, p) \geq u_j(t'_j, S', p)$ for all (t'_j, S') for which $\sum_{i\in S'} p_i \leq p_j + t'_j$.

In competitive equilibrium, the total demand for good i is equal to its total supply, and the latter is equal to 1, $i \in N$. Each consumer demands at least one good. An equilibrium is achieved, therefore, iff [each consumer demands one and only one good, and each good is demanded by some consumer]. Formally, a *competitive equilibrium* of an economy is a pair (p^*, π^*) of a price vector $p^* \in \mathbf{R}^N_+$ and a bijection $\pi^* : N \to N$ such that $p^* \in \cap_{j\in N} C_j^{\{\pi^*(j)\}}$.

Markets of indivisible goods were considered by a pioneering paper, Shapley and Scarf (1974). They do not introduce any financial intermediaries, but make the postulate that each consumer demands *one* indivisible good. This is contrasted with our setup that a consumer can obtain a loan or make an investment, and can hold *several* indivisible goods at the same time (provided that his budget constraint is satisfied). So, while the consumption set of each consumer is N in the Shapley-Scarf setup, the consumption set of each consumer in our setup is $\mathbf{R} \times \mathcal{P}(N)$; recall that element $j \in N$ is identified with one unit of the jth good, and element $S \in \mathcal{P}(N)$ is identified with set S of goods. We have followed Quinzii (1984) and Gale (1984) in our formulation of an economy, but our model differs from theirs in two important respects: First, while the divisible commodity that Quinzii and Gale introduced is interpreted as money as a store of value, the divisible commodity that we introduce is interpreted as a loan/investment, which essentially functions as a channel for income re-distribution. The price domain both in the Quinzii-Gale setup and in our setup is $\{1\} \times \mathbf{R}^N_+$ (here, the price of money is always equal to 1). Second, while Quinzii and Gale postulate that each consumer can hold a pair of money and *one* indivisible good, we postulate that he can hold a pair of a loan and *several* indivisible goods. Thus, while the consumption set of each consumer is $\mathbf{R}_+ \times N$ in the Quinzii-Gale setup, the consumption set of each consumer in our setup is $\mathbf{R} \times \mathcal{P}(N)$. We point out that although each consumer is allowed to hold *several* indivisible goods in our setup, he ends up holding *one* indivisible good in a competitive equilibrium. In short, an assignment of goods emerges in equilibrium even in our setup.

The purpose of the present section is to establish an equilibrium existence theorem (Theorem 4.3). The first assumption is the following monotonicity condition on a consumer's preference relation, which says that goods affect his utility positively, and a loan affects his utility negatively; the latter assumption is justified because a loan creates commitment to future payments and an investment yields future returns.

Assumption 4.1 *Let $p \in \mathbf{R}_+^N$ be any price vector of goods.*
(i) *For every $t \in \mathbf{R}$, $u_j(t, S, p) > u_j(t, S', p)$ for all $S, S' \in \mathcal{P}(N)$ for which $S \supset S'$ and $S \neq S'$.*
(ii) *For every $S \in \mathcal{P}(N)$, $u_j(t, S, p) > u_j(t', S, p)$ for all $t, t' \in \mathbf{R}$ for which $t < t'$.*

Assumption 4.1 (ii) guarantees that given any $p \in C_j^S$, consumer j demands goods S by obtaining the exact amount of loan $t_j = \sum_{i \in S} p_i - p_j$. Without loss of generality, therefore, his constrained maximization problem becomes:

$$\begin{array}{rl} \text{Maximize} & u_j\left(\sum_{i \in S} p_i - p_j,\ S,\ p\right), \\ \text{subject to} & S \in \mathcal{P}(N), \\ \text{given} & p \in \mathbf{R}_+^N. \end{array}$$

Since $\mathcal{P}(N)$ is a finite set, a solution to this problem always exists. In other words, *for each consumer j, the family $\{C_j^S\}_{S \in \mathcal{P}(N)}$ is a cover of $\mathbf{R}_+^N$.*

We specify behavior of the financial intermediary. By setting the deposit/lending rates, it influences consumers' decisions on the total investment $\sum_{j:t_j<0} |t_j|$ and the total loan $\sum_{j:t_j>0} t_j$. The latter has to be funded from the former,

$$\sum_{j:t_j>0} t_j \leq \sum_{j:t_j<0} |t_j|,$$

that is,

$$\sum_{j \in N} t_j \leq 0,$$

which constitutes the constraint on the intermediary's behavior. The next assumption says that as long as the average price of goods is high, the intermediary can always set the two interest rates so that this constraint is met. This is justified as follows: A loan is typically demanded by relatively low-income consumers, i.e., by those consumers whose initially endowed goods have low prices. Some other consumers must have high incomes, in view of the high average price. A high-income consumer opts to sell his high-priced good, buys low-priced goods and invests the surplus for high future returns. Thus, whenever there is demand for a loan, there is also supply of investment. Recall that $e_j \in \mathbf{R}_+^N$ is a unit vector, $j \in N$. Given a positive number M, define the simplex,

$$\Delta^N(M) := \text{ co } \{Me_j \mid j \in N\}.$$

A price vector p has the average price M/n, iff $p \in \Delta^N(M)$.

Assumption 4.2 *There exists a positive real number M such that for any $p \in \cap_{j\in N} C_j^{S_j} \cap \Delta^N(M)$, it follows that $\sum_{j\in N} t_j \leq 0$, where $t_j := \sum_{i\in S_j} p_i - p_j$.*

The present paper does not specify the intermediary's behavior other than Assumption 4.2 and the continuous dependence of the interest rates on prices. So our analysis is applicable to a broad class of economies. Assumption 4.2 is nothing but Walras' law within the markets for the goods, provided that the average price is M/n. Indeed, when $p \in \cap_{j\in N} C_j^{S_j}$, the total demand for good i is the number of the consumers who demand i, $\#\{j \in N \mid S_j \ni i\}$, so the value of the total excess demand is:

$$\begin{aligned}
&\sum_{i\in N} p_i \left(\#\{j \in N \mid S_j \ni i\} - 1\right) \\
&= \sum_{j\in N} \left(\sum_{i\in S_j} p_i - p_j\right) \\
&= \sum_{j\in N} t_j \\
&\leq 0.
\end{aligned}$$

We will discuss Assumption 4.2 in the example of a segmented housing market later (after the statement of Lemma 4.4).

Theorem 4.3 *Let $\{\mathbf{R} \times \mathcal{P}(N), u_j, (0, \{j\})\}_{j\in N}$ be an economy which satisfies Assumptions 4.1 and 4.2. Then, there exists a competitive equilibrium of the economy.*

In order to prove Theorem 4.3, we need to establish two lemmas; the first says that each consumer demands all the goods whose prices are substantially low.

Lemma 4.4 *For each compact subset C of $\mathbf{R}^N_+$, there exists a positive number δ such that for every $p \in C^S_j \cap C$ it follows that $S \supset \{i \in N \mid p_i \leq \delta\}$.*

In order to see the role of the financial intermediary and Assumption 4.2, consider a housing market. In reality, a housing market is segmented: a buyer usually looks at houses of a particular price range, or rather, he looks at set N of houses of a similar capacity and quality. Suppose several houses are extremely low-priced, say 1 cent each. A buyer will demand them all; this is the content of Lemma 4.4. He does so even when his initially owned house is also 1 cent, so that he has to obtain a loan from the financial intermediary; he is willing to pay interest on the several cents in the future, if he can keep all these low-priced houses. On the other hand, if his initially owned house is extremely high-priced, say 1 billion dollars, he will buy all the 1-cent houses (which are after all of similar quality as his), and sell his house. This way, he can invest in the financial intermediary the large excess of his sale over his purchases and expect high returns in the future. Walras' law for the markets of the goods in the inequality form is satisfied.

A quantitative example of the above paragraph is in order. Let $N = \{1, 2, 3\}$. Let M be the positive number given in Assumption 4.2, let δ be the positive number given in Lemma 4.4 applied to $C = \Delta^N(M)$, and consider price vector $p = (M - 2\delta, \delta, \delta)$. Each consumer demands the second and the third houses (low-priced houses), so his total expenditure is 2δ. Consumer $i \in \{2, 3\}$ receives his sale value δ, so needs to receive a loan of δ. Consumer 1 receives his sale value $(M - 2\delta)$, so he can invest value $(M - 4\delta)$ in the financial intermediary. The value of the total excess demand is, therefore,

$$p_1(0-1) + p_2(3-1) + p_3(3-1) = -M + 6\delta,$$

which is nonpositive because δ is very small, hence Walras' law.

Proof of Lemma 4.4 Suppose the contrary. Then for each positive integer k, there exist $S^k \in \mathcal{P}(N)$, $p^k \in C^{S^k}_j \cap C$ and $i^k \in N$ such that $p^k_{i^k} \leq 1/k$ and $i^k \notin S^k$. By passing through a subsequence if necessary, one may assume without loss of generality that $p^k \to p^* \in C$, and $S^k = S^*$, $i^k = i^*$ for every k. Then, $p^*_{i^*} = 0$, and $i^* \notin S^*$. Define $t^k_j := \sum_{i \in S^*} p^k_i - p^k_j$, $t^*_j := \sum_{i \in S^*} p^*_i - p^*_j$. Clearly, $t^k_j \to t^*_j$. By the monotonicity assumption, $u_j(t^*_j, S^*, p^*) < u_j(t^*_j, S^* \cup \{i^*\}, p^*)$. By continuity of u_j, there exists a neighborhood U of (t^*_j, p^*) in

$\mathbf{R} \times C$ and a positive number τ such that

$$\forall\ (t_j, p) \in U: \qquad u_j(t_j^*, S^*, p^*) + \tau < u_j(t_j, S^* \cup \{i^*\}, p).$$

But for all k sufficiently large, $u_j(t_j^k, S^*, p^k) < u_j(t_j^*, S^*, p^*) + \tau$ and $(t_j^k + p_{i^*}^k, p^k) \in U$. Thus, under price vector p^k, the commodity bundle $(t_j^k + p_{i^*}^k, S^* \cup \{i^*\})$, which satisfies the budget constraint, yields a higher utility than (t_j^k, S^*) for all k sufficiently large; this contradicts the choice of (t_j^k, S^*) as a maximizer of utility $u_j(\cdot, p^k)$ in the budget set given p^k. □

The next lemma says that the demand correspondence is upper semicontinuous and closed-valued: the properties equivalent in the present setup to closedness of the graph of the demand correspondence, $\{(p, \sum_{i \in S} p_i - p_j, S) \in \mathbf{R}_+^N \times \mathbf{R} \times \mathcal{P}(N) \mid p \in C_j^S\}$.

Lemma 4.5 *For each $j \in N$ and each $S \in \mathcal{P}(N)$, the set C_j^S is closed in $\mathbf{R}_+^N$.*

Proof Upper semicontinuity of the demand correspondence follows from the standard argument which uses the maximum theorem. We only need to show that the budget-set correspondence B_j from the price-domain $\mathbf{R}_+^N$ to the subsets of the consumption set $\mathbf{R} \times \mathcal{P}(N)$, defined by

$$B_j(p) := \left\{ (t, S) \in \mathbf{R} \times \mathcal{P}(N) \;\middle|\; \sum_{i \in S} p_i \leq p_j + t \right\},$$

is lower semicontinuous. Let $\{p^k\}_k$ be any sequence in $\mathbf{R}_+^N$ which converges to p^*, and let (t^*, S^*) be any point in $B_j(p^*)$. Choose $t^\circ \in \mathbf{R}$ so that for all k sufficiently large,

$$\sum_{i \in S^*} p_i^k < p_j^k + t^\circ.$$

For each k, define α^k by

$$\alpha^k := \max \left\{ \alpha \in [0, 1] \;\middle|\; \sum_{i \in S^*} p_i^k \leq p_j^k + \alpha t^* + (1 - \alpha) t^\circ \right\}.$$

Clearly, $\alpha^k \to 1$. Define $t^k := \alpha^k t^* + (1 - \alpha^k) t^\circ$. Then, $(t^k, S^*) \in B_j(p^k)$ for all k sufficiently large, and $(t^k, S^*) \to (t^*, S^*)$. □

Proof of Theorem 4.3 Given the positive numbers M of Assumption 4.2 and δ of Lemma 4.4 applied to $C = \Delta^N(M)$, define the trimmed simplex,

$$\Delta_\delta^N(M) := \{p \in \Delta^N(M) \mid \forall\, i \in N: \ p_i \geq \delta\}.$$

Its faces are defined by

$$\Delta_\delta^T(M) := \{p \in \Delta_\delta^N(M) \mid \forall\, i \in N \setminus T: \ p_i = \delta\}, \ \ T \in \mathcal{P}(N).$$

Lemma 4.4 says that for each consumer j,

$$\forall\, T \subset N: \ \Delta_\delta^T(M) \subset \bigcup_{S \supset N\setminus T} C_j^S,$$

and Lemma 4.5 says that each C_j^S is closed. By Theorem 2.3C applied to the covers of $\Delta_\delta^N(M)$, $\{C_j^S \cap \Delta_\delta^N(M)\}_{S\in\mathcal{P}(N)}$, $j \in N$, there exists $\pi : N \to \mathcal{P}(N)$ such that

$$\bigcap_{j\in N} [C_j^{\pi(j)} \cap \Delta_\delta^N(M)] \neq \emptyset,$$
$$\bigcup_{j\in N} \text{supp } \pi(j) = N.$$

This means that

$$\exists\, p^* \in \bigcap_{j\in N} [C_j^{\pi(j)} \cap \Delta_\delta^N(M)], \tag{2}$$
$$\forall\, i \in N: \ \#\{j \in N \mid \pi(j) \ni i\} \geq 1. \tag{3}$$

On the other hand, by Assumption 4.2,

$$\sum_{i\in N} p_i^*(\#\{j \in N \mid \pi(j) \ni i\} - 1) \leq 0. \tag{4}$$

By strict positiveness of p^* and (3), inequality (4) holds true only if $\#\{j \in N \mid \pi(j) \ni i\} - 1 = 0$ for every $i \in N$. This means that each $\pi(j)$ is a singleton, and function π may be regarded as a permutation on N. The theorem is established in view of (2). □

References

Akin, E. (1995). "Vilfredo Pareto cuts the cake," *Journal of Mathematical Economics* **24**, 23-49.

Alexandrov, P. and B. Pasynkov (1957). "Elementary proof of the essentiality of the identical mapping of a simplex" (in Russian), *Uspekhi Matematičeskikh Nauk* **12**, 175-179.

Alon, N. (1988). "Some recent combinatorial applications of Borsuk-type theorems," in: M. M. Deza, P. Frankl and D. G. Rosenberg, eds., *Algebraic, Extremal and Metric Combinatorics*, Cambridge, UK: Cambridge University Press, 1-12.

Bapat, R. B. (1982). "D_1AD_2 theorems for multidimensional matrices," *Linear Algebra and Applications* **48**, 437-442.

Bapat, R. B. and T. E. S. Raghavan (1989). "An extension of a theorem of Darroch and Ratcliff in loglinear models and its application to scaling multidimensional matrices," *Linear Algebra and Applications* **114/115**, 705-715.

Berliant, M., W. Thomson and K. Dunz (1992). "On the fair division of a heterogeneous commodity," *Journal of Mathematical Economics* **21**, 201-216.

Fan, Ky (1968). "A covering property of simplexes," *Mathematica Scandinavica* **22**, 17-20.

Fan, Ky (1972a). "A minimax inequality and applications," in: O. Shisha, ed., *Inequalities III: Proceedings of the Third Symposium on Inequalities held at the University of California, Los Angeles, September 1-9, 1969*, New York: Academic Press, 103-113.

Fan, Ky (1972b). "Covering properties of convex sets and fixed point theorems in topological vector spaces," in: R. D. Anderson, ed., *Symposium on Infinite Dimensional Topology*, Annals of Mathematical Studies 69, Princeton: Princeton University Press, 79-92.

Gale, D. (1984). "Equilibrium in a discrete exchange economy with money," *International Journal of Game Theory* **13**, 61-64.

Ichiishi, T. (1981). "On the Knaster-Kuratowski-Mazurkiewicz-Shapley theorem," *Journal of Mathematical Analysis and Applications* **81**, 297-299.

Ichiishi, T. (1988). "Alternative version of Shapley's theorem on closed coverings of a simplex," *Proceedings of the American Mathematical Society* **104**, 759-763.

Ichiishi, T. and A. Idzik (1990). "Theorems on closed coverings of a simplex and their applications to cooperative game theory," *Journal of Mathematical Analysis and Applications* **146**, 259-270.

Ichiishi, T. and A. Idzik (1991). "Closed covers of compact convex polyhedra," *International Journal of Game Theory* **20**, 161-169.

Idzik, A. (1995). "Optimal divisions of the unit interval," Game Theory and Applications, International Conference in Honor of Robert J. Aumann on his 65th Birthday, Jerusalem.

Kulpa, W. (1994). "Sandwich type theorems," *Acta Universitatis Carolinae - Mathematica et Physica* **35**, 45-50.

Quinzii, M. (1984). "Core and competitive equilibria with indivisibilities," *International Journal of Game Theory* **13**, 41-60.

Scarf, H. (1967). "The approximation of fixed-points of a continuous mappings," *SIAM Journal of Applied Mathematics* **15**, 1328-1342.

Shapley, L. S. (1973). "On balanced games without side payments," in: T. C. Hu and S. M. Robinson, eds., *Mathematical Programming*, New York: Academic Press, 261-290.

Shapley, L. S. and H. Scarf (1974). "On cores and indivisibility," *Journal of Mathematical Economics* **1**, 23-37.

Svensson, L.-G. (1983) "Large indivisibles: an analysis with respect to price equilibrium and fairness," *Econometrica* **51**, 939-954.

Thomson, W. (1995). "The replacement principle in economies with indivisible goods," Rochester Center for Economic Research Working Paper No. 403.

van der Laan, G., D. Talman and Z. Yang (1997). "Existence of an equilibrium in a competitive economy with indivisibilities and money," *Journal of Mathematical Economics* **28**, 101-109.

Weller, D. (1985). "Fair division of a measurable space," *Journal of Mathematical Economics* **14**, 5-17.

Woodal, D. R. (1980). "Dividing a cake fairly," *Journal of Mathematical Analysis and Applications* **78**, 233-247.