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The Area of Figures Representable by Büchi Automata

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Abstract.
Yen Hsu-Chun and Lin Yih-Kai showed that Büchi automata represent various kinds of figures. They proved that if a figure is represented by a deterministic Büchi automaton, then the area of the figure is a rational number. This paper shows the theorem that if a figure is represented by a non-deterministic Büchi automaton, then the area of the closure of the figure is a rational number. as is an extension of their theorem for deterministic Büchi automata.

1 Büchi Automaton

Definition 1.1 (Büchi Automaton) A *Büchi automaton* is defined by the datum which is consists of five components $(\Sigma, S, \delta, s_0, F)$, where each component has the following meaning:

Σ	: alphabet, the set of symbols
S	: the set of states
$\delta \subset S \times \Sigma \times S$	: transition relation
$s_0 \in S$	: the initial state
F	: the set of final states

Actually, final states are not final, but are to be visited infinitely many times.

Let B be a Büchi automaton such as $B = (\Sigma, S, \delta, s_0, F)$. Then $L(B)$ is a subset of Σ^ω which defined as the following. For $(\sigma_1, \sigma_2, ...) \in \Sigma^\omega$,

$$(\sigma_1, \sigma_2, ...) \in L(B)$$

iff there is $(s_1, s_2, ...) \in S^\omega$ such that $(s_{i-1}, \sigma_i, s_i) \in \delta$ for each $i = 1, 2, ...$, and that there are infinitely many i's such that $s_i \in F$. The set $L(B)$ is called the *language* of B.

Definition 1.2 (Determinism) A Büchi automaton $B = (\Sigma, S, \delta, s_0, F)$ is *deterministic* iff for each $s \in S$ and each $\sigma \in \delta$, there exist at most one $s' \in S$ such that $(s, \sigma, s') \in \delta$.

Definition 1.3 (Measure over infinite words) Let Σ be a set which consists of N characters. If μ is written as a measure over the set Σ^ω, then μ denotes the ordinal measure over Σ^ω, which is defined as following: We write $(x_1, x_2, ..., x_n, *)$ for the set $\{(y_1, y_2, ...) \in \Sigma^\omega | y_1 = x_1, y_2 = x_2, ..., y_n = x_n\}$. Then, $\mu(x_1, x_2, ..., x_n, *) = 1/N^n$. Hence $\mu(\Sigma^\omega) = 1$.

Definition 1.4 (Closure) For $E \subset \Sigma^\omega$, we write $\bar{E}$ for the closure of E with respect to the ordinal topology of Σ^ω. That is, for each $(\sigma_1, \sigma_2, ...) \in \Sigma^\omega$, $(\sigma_1, \sigma_2, ...) \in \bar{E}$ iff for any positive integer n, there exists an infinite sequence $(\sigma'_n, \sigma'_{n+1}, \sigma'_{n+2}, ...) \in \Sigma^\omega$ such that $(\sigma_1, \sigma_2, ..., \sigma_{n-1}, \sigma'_n, \sigma'_{n+1}, ...) \in E$

2 Representations of Figures

Definition 2.1 The sets $\mathbf{2}$, $\mathbf{2}^2$, $\mathbf{2}^3$ is written as follows.

$$\mathbf{2} = \{0,1\},\ \mathbf{2}^2 = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbf{2}\},\ \mathbf{2}^3 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \in \mathbf{2}\}.$$

The sets $\mathbf{2}^\omega$, $(\mathbf{2}^2)^\omega$, $(\mathbf{2}^3)^\omega$ is written as follows.

$$\begin{array}{ll} \mathbf{2}^\omega & = \{(x_1, x_2, ...) \mid x_i \in \mathbf{2}\}, \\ (\mathbf{2}^2)^\omega & = \{(\sigma_1, \sigma_2, ...) \mid \sigma_i \in \mathbf{2}^2\}, \\ (\mathbf{2}^3)^\omega & = \{(\sigma_1, \sigma_2, ...) \mid \sigma_i \in \mathbf{2}^3\}. \end{array}$$

The sets $\mathbf{2}^n$ and $(\mathbf{2}^4)^\omega$ for $n = 4, 5, ...$ are defined similarly.

Definition 2.2 The function ϕ maps $\mathbf{2}$ into the unit interval $[0,1]$ such as:

$$\phi : (x_1, x_2, ...) \mapsto \phi(x_1, x_2, ...) = \sum_{i=0}^{\infty} 2^{-i} x_i$$

The function ϕ is continuous and surjective, but not injective. The function ϕ also maps $(\mathbf{2}^2)^\omega$ into the unit square $[0,1]^2$ such as:

$$\phi : (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, ...) \mapsto \phi(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, ...) = \begin{pmatrix} \phi(x_1, x_2, ...) \\ \phi(y_1, y_2, ...) \end{pmatrix}$$

The function ϕ also maps a subset $E \subset (\mathbf{2}^2)^\omega$ into a subset $\phi(E) \subset [0,1]^2$ such as:

$$\phi(E) = \{\phi(\vec{\sigma}) \mid \vec{\sigma} \in E\}.$$

The functions ϕ over elements $\vec{\sigma} \in (\mathbf{2}^n)^\omega$ and over subsets $E \subset (\mathbf{2}^n)^\omega$ are also defined similarly.

Lemma 2.3 (Cascade Product) *Let B and B' be Büchi automata with $\mathbf{2}^2$ as their alphabet. Then there is a Büchi automaton B'' which satisfies the following:*

$$(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, ...) \in L(B'')$$

iff there is $(z_1, z_2, ...) \in \mathbf{2}^\omega$ *such that*

$$(\begin{pmatrix} x_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ z_2 \end{pmatrix}, ...) \in L(B) \quad \textit{and} \quad (\begin{pmatrix} z_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ y_2 \end{pmatrix}, ...) \in L(B').$$

In the case of the previous lemma, we call B'' *a* cascade product *of* B *and* B'.

Remark 2.4 Cascade products are defined not only for automata with $\mathbf{2}^2$ as their alphabet, but also for automata with $\mathbf{2}^3$, or sets of higher dimension, as their alphabets.

Lemma 2.5 *There is a Büchi automaton* B_0 *such that*

$$(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, ...) \in L(B_0) \quad \textit{iff} \quad \phi(x_1, x_2, ...) = \phi(y_1, y_2, ...).$$

Remark 2.6 For each Büchi automaton B with $\mathbf{2}^n$ as its alphabet, there is a Büchi automaton B' such that $\vec{\sigma} \in L(B')$ iff $\phi(\vec{\sigma}) \in \phi(L(B))$. This B' is made as a cascade product of B and n duplications of B_0 of Lemma 2.5.

Put $n = 2$ especially. For this B' above, it holds that if $\phi(x_1, x_2, ...) = \phi(x'_1, x'_2, ...)$ and $\phi(y_1, y_2, ...) = \phi(y'_1, y'_2, ...)$, then

$$(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, ...) \in L(B') \text{ iff } (\begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix}, \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix}, ...) \in L(B').$$

Theorem 2.7 (Affine Transformation) *For each Büchi automaton* B *with* $\mathbf{2}^2$ *as its alphabet, and for each 2×2-matrix* A *over rational numbers, there is a Büchi automaton* B' *such that* $\phi(L(B')) = A(\phi(L(B)))$

Proof. In [JS'99]. ∎

Theorem 2.8 (Non-representability of Circles) *There is no Büchi automaton* B *such that* $\phi(L(B))$ *is a circle.*

Proof. In [JS'99]. ∎

Definition 2.9 (Measure over real numbers) If μ is written as a measure over the interval $[0, 1]$, then μ denotes the ordinal Lebesque measure over $[0, 1]$.

Similarly, if μ is written as a measure over an interval $[0, 1]^n$, then μ denotes the ordinal Lebesque measure over $[0, 1]^n$.

Lemma 2.10 *The function* ϕ *preserves* μ. *That is, for any subset* $E \subset 2^\omega$, $\mu(\phi(E)) = \mu(E)$.

Lemma 2.11 *The function* ϕ *preserves the closure operation. That is, for any subset* $E \subset 2^\omega$, $\phi(\bar{E}) = \overline{\phi(E)}$.

3 Measure of Languages

Theorem 3.1 (Lin & Yen '00) *For a deterministic Büchi automaton B, the measure of the language $\mu(L(B))$ is rational.*

Proof. In [Lin&Yen'00]. ∎

Remark 3.2 Lin and Yen proves the theorem above by the property of Markov chains. A deterministic Büchi automaton is regarded as a Markov chain in their proof. Unfortunately, their method cannot be applied to non-deterministic Büchi automata. We prove the theorem only on the closures of the languages of non-deterministic Büchi automata. A characterisation for the measure of the languages of non-deterministic Büchi automata is still open.

Lemma 3.3 *For any Büchi automaton B, we can construct a deterministic Büchi automaton $\bar{B}$ such that $\overline{L(B)} = L(\bar{B})$.*

Theorem 3.4 (Main Result) *For each Büchi automaton B, the measure of the closure of the language $\mu(\overline{L(B)})$ is rational.*

Proof. By Theorem 3.1 and Lemma 3.3 above. ∎

Corollary 3.5 *For each Büchi automaton B with $\mathbf{2}^2$ as its character set, the area of the closure $\phi(\overline{L(B)})$ is rational.*

References

[JS'99] Jurgensen, H. & Staiger, L.: Finite Automata Encoding Geometric Figures, the pre-proceedings of the Workshop on Imprementing Automata 1999.

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