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Pattern formation described by the Swift-Hohenberg equation

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1. The Swift-Hohenberg equation

The Swift-Hohenberg equation is a fundamental equation in the study of spatio-temporal pattern formation in extended systems. It was first put forward in 1976 by Swift and Hohenberg [SH] as a simple model for the Rayleigh-Bénard instability of roll waves. However, since then it has proved an effective model equation for a variety of systems in physics and mechanics.

The Swift-Hohenberg equation is the evolution equation associated with the gradient system

$$u_t = -\frac{\delta\mathcal{L}}{\delta u} \tag{1.1}$$

in which $u_t = \partial u/\partial t$ and the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L}(u) = \int \left\{\frac{1}{2}(u_{xx})^2 - (u_x)^2 + \frac{1}{2}F_\kappa(u)\right\} dx. \tag{1.2}$$

Here the potential function F_κ is given by

$$F_\kappa(s) = \frac{1-\kappa}{2}s^2 + \frac{1}{4}s^4, \tag{1.3}$$

and has been normalised so that $F_\kappa(0) = 0$. The constant κ is viewed as an eigenvalue parameter. Note that if $\kappa < 1$, then F_κ is a single well potential, and if $\kappa > 1$, it is a double well potential, with the bottoms of the wells located at $s = \pm a$, where

$$a = \sqrt{\kappa - 1}, \qquad \kappa > 1. \tag{1.4}$$

The corresponding evolution equation is the parabolic equation

$$\frac{\partial u}{\partial t} = \kappa u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3. \tag{1.5}$$

involving a fourth order spatial derivative.

The Swift-Hohenberg equation has been studied a great deal, both analytically and numerically. Many references can be found in the book by Collet and Eckmann [CE], in the extensive survey paper by Cross and Hohenberg [CH] and in the recent review by Bodenschatz, Pesch and Ahlers [BPA]. It also arises in a variety of other contexts, such as the study of incommenurate materials [DB1,2, DBCR], and in problems in the mechanics of second order materials [LM, MPT], and the study of travelling waves in nonlinearly supported beams [LMcK, PT2].

In this paper we focus on stationary periodic solutions, that is on periodic solutions $u(x)$ of the ODE

$$u^{iv} + 2u'' + f_\kappa(u) = 0, \tag{1.6}$$

in which the source function $f_\kappa = F_\kappa'$ is given by

$$f_\kappa(s) = (1-\kappa)s + s^3. \tag{1.7}$$

It has been shown [CE, EP] that a branch of periodic solutions, which are odd with respect to their zeros and even with respect to their critical points, bifurcates from the trivial solution at $\kappa = 0$. Its local behaviour near the bifurcation point is given by

$$\|u(\cdot,\kappa)\|_\infty \sim 2\sqrt{\frac{\kappa}{3}} \qquad \text{as} \quad \kappa \to 0^+. \tag{1.8}$$

In this paper we follow this branch and prove that it continues at least as far as the line $\kappa = 1$ in the (κ, u)-plane. Inside the half strip

$$\Sigma_0 = \{(\kappa, M) : 0 \le \kappa \le 1,\ M \ge 0\}, \qquad M = \|u(\cdot,\kappa)\|_\infty,$$

the periodic solutions preserve their qualitative properties: they remain odd with respect to their zeros and even with respect to their critical points. No critical points emerge out of the blue and no critical points disappear.

In addition to following this first branch of single bump periodic solutions, we follow branches of periodic solutions with more complex graphs: *n-lap* periodic solutions for any $n \ge 1$. When n is *odd* then these solutions are odd with respect to their zeros. These branches bifurcate from the trivial solution at critical values $\kappa_n \in (0,1)$, and they all traverse the region Σ_0 preserving their qualitative properties.

In Section 2 we present the main results and illustrate them with a series of numerically obtained bifurcation pictures. In Section 3 we give the flavour of some of the proofs, and in Section 4 we present a few generalisations. In particular we discuss there the continuation of the branches and their solutions in the region beyond $\kappa = 1$:

$$\Sigma_1 = \{(\kappa, M) : \kappa > 1,\ M \ge 0\}.$$

We shall find that, unlike in Σ_0, in Σ_1 the qualitative properties of periodic solutions does change along branches. Details of the proofs of Theorems 2.2 - 2.4 in Section 2, and further numerical results can be found in Chapter 9 of [PT3], and for related results in [BPT].

The numerical results presented in this paper have been obtained with the ODE software *Phase Plane* of Ermentrout [E], and the bifurcation software *AUTO97* of Doedel *et.al.* [D].

2. The main results

We begin with the important observation that the equation

$$u^{iv} + 2u'' + (1-\kappa)u + u^3 = 0, \tag{2.1}$$

has a first integral, the *energy identity*:

$$\mathcal{E}(u) \stackrel{\text{def}}{=} u'u''' - \frac{1}{2}(u'')^2 + (u')^2 + F_\kappa(u) = E, \tag{2.2}$$

where E, the *energy*, is a constant. Plainly, the trivial solution $u = 0$ has energy $E = 0$. Being interested in branches of solutions which bifurcate from $u = 0$, we shall focus here on periodic solutions which have zero energy.

In our first result we show that equation (2.1) has no periodic solutions if $\kappa \le 0$. This allows us to restrict our attention to positive values of κ.

Theorem 2.1. *If $\kappa \le 0$, then there exist no nontrivial periodic solutions of equation* (2.1).

The proof relies on a functional $\mathcal{H}(u)$, which is of some interest in its own right. We therefore give the proof right away.

Proof of Theorem 2.1. We introduce the functional

$$\mathcal{H}(u) = \frac{1}{2}(u'')^2 + (u')^2 + F_\kappa(u),$$

and we set $H(x) = \mathcal{H}(u(x))$. For any solution u of equation (2.1) we have

$$\begin{aligned} H'(x) &= u''u''' + 2u'u'' + f_\kappa(u)u', \\ H''(x) &= (u''')^2 + 2u'u''' + f_\kappa'(u)(u')^2. \end{aligned}$$

Thus we see that H'' is a quadratic polynomial in u'''; it is positive if the discriminant D is negative. For D we find

$$D = (u')^2\{4 - 4f_\kappa'(u)\} = 4(u')^2(\kappa - 3u^2).$$

Hence, if $\kappa < 0$, then $D < 0$ whenever $u' \neq 0$, and H is strictly concave. In fact, even when $\kappa = 0$, then H is concave when $u \neq 0$ and $u' \neq 0$. Since u is nontrivial, this means that

$$H'(x) > H'(0) \qquad \text{for} \quad x > 0. \tag{2.3}$$

However, if u is periodic with period ℓ, we should have $H'(\ell) = H'(0)$, which contradicts (2.3).

Next, we turn to the branch of single bump periodic solutions which bifurcates from the origin of the (κ, u) plane.

Theorem 2.2. *For each $\kappa \in (0,1)$ there exists a zero-energy periodic solution of equation* (2.1)*, which is symmetric with respect to its critical points and odd with respect to its zeros, such that $u'(0) > 0$.*

In view of the symmetry of equation (2.1), we see that if u is a solution, then so is $-u$. Note that if u is odd, then it can be transformed into $-u$ by translation over half the period.

We denote the branch of odd single bump periodic solutions in the (κ, M)-plane by $\mathcal{S}_1$. Here

$$M = \max\{u(x) : x \in \mathbf{R}\}.$$

In Figure 2.1 we present the numerically computed branch $\mathcal{S}_1$ in the (κ, M)-plane.

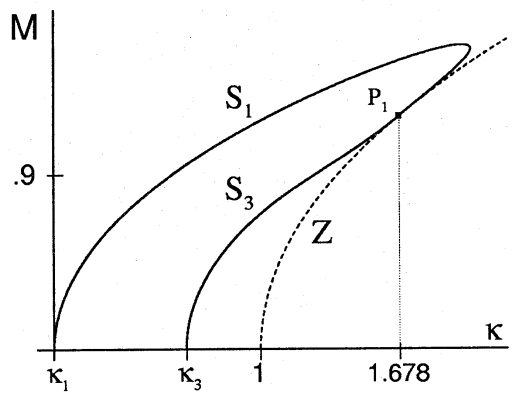
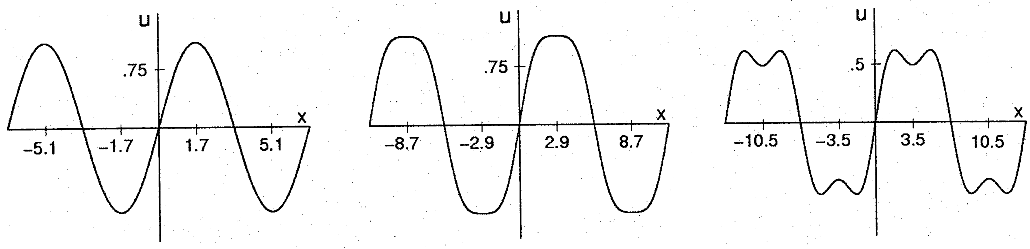


Fig. 2.1. The branch $\mathcal{S}_1$ of odd *1-Lap* periodic solutions
and the branch $\mathcal{S}_3$ of odd *3-Lap* solutions

The curve Z in Figure 2.1 will be explained in Section 4.

We see that $\mathcal{S}_1$ bifurcates from the point $(\kappa, M) = (0,0)$, traverses the strip $0 < \kappa < 1$ and continues into the regime $\kappa \geq 1$. In this regime it loops back and enters the strip $0 < \kappa < 1$ again to end on the $M = 0$ axis. We see from the solution graphs shown in Figure 2.2 that by the time the branch reenters the strip $0 < \kappa < 1$ the solution graphs have acquired a *dimple* near the maxima and the minima. This happens as κ passes through the critical value $\kappa = 1.678$. At this value we have $u' = 0$ and $u'' = 0$ at the maximum of the solution, as indicated in Figure 2.2.



on $\mathcal{S}_1$ at $\kappa = 0.96$ at P_1 ($\kappa = 1.678$) on $\mathcal{S}_3$ at $\kappa = 0.96$

Fig. 2.2. Solution graphs on the loop shown in Figure 2.1

In the strip $0 < \kappa < 1$, solutions on $\mathcal{S}_1$ are symmetric with respect to their critical points. Therefore, they have *one* monotone segment – or one *Lap* – between two consecutive points of symmetry, i.e. critical points ζ with the property

$$u(\zeta + y) = u(\zeta - y) \quad \text{for all} \quad y \in \mathbf{R}.$$

Solutions on the lower branch of the loop, where it passes through the strip $0 < \kappa < 1$, have *three* monotone segments between two consecutive points of symmetry. We therefore denote this branch by $\mathcal{S}_3$. More generally:

We call a periodic solution with m monotone segments between points of symmetry an m-Lap solution. A branch of m-Lap periodic solutions such that $u' > 0$ in a right neighbourhood of the origin will be denoted by $\mathcal{S}_m$.

We see that the branch $\mathcal{S}_3$ also bifurcates from the trivial solution at

$$\kappa_3 = \frac{14}{25}.$$

To understand why it bifurcates just there we need to understand the local behaviour near the trivial solution. Linearising equation (1.6) about $u = 0$, we obtain

$$v^{iv} + 2v'' + (1 - \kappa)v = 0. \tag{2.4}$$

Thus, the characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 - \kappa = 0,$$

and, when $\kappa > 0$, its eigenvalues are

$$\lambda_\pm^2 = -1 \pm \sqrt{\kappa}. \tag{2.5}$$

Since $\kappa \in (0,1)$, they are purely imaginary:

$$\lambda = \pm ia \quad \text{and} \quad \lambda = \pm ib, \tag{2.6a}$$

where

$$a = \sqrt{1+\sqrt{\kappa}} \quad \text{and} \quad b = \sqrt{1-\sqrt{\kappa}}. \tag{2.6b}$$

Resonance occurs if a is an integral multiple of b ([GH], p.397), i.e. when

$$\frac{a}{b} = m \geq 1 \qquad \Longleftrightarrow \qquad \kappa = \kappa_m \stackrel{\text{def}}{=} \left(\frac{m^2-1}{m^2+1}\right)^2. \tag{2.7}$$

It is at these points of resonance that branches $\mathcal{S}_m$ bifurcate from the trivial solution. Specifically, we prove the following result.

Theorem 2.3. *Let $n \geq 1$. For each $\kappa \in (\kappa_{2n-1}, 1)$ there exists an odd zero-energy periodic solution $u_n(x)$ of equation (1.6) which has $2n-1$ laps in each half-period, such that $u_n'(0) > 0$. Its first positive point of symmetry is ζ_n, and*

$$u_n(x) > 0 \qquad \textit{for} \quad 0 < x < 2\zeta_n.$$

Computations show that these solutions also lie on loop shaped branches in the (κ, M) plane. The branch emanating from κ_5 and returning to κ_7 on the $M = 0$ axis is shown in Figure 2.3.

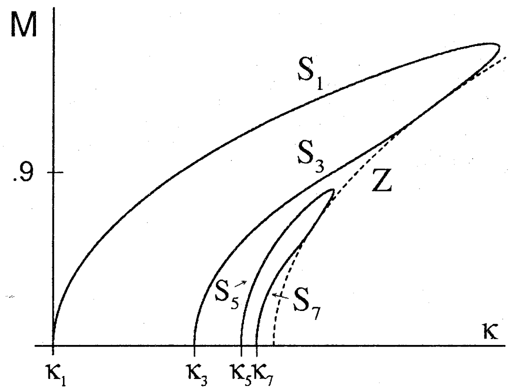


Fig. 2.3. The branches $\mathcal{S}_1$, $\mathcal{S}_3$, $\mathcal{S}_5$ and $\mathcal{S}_7$

Since Theorem 2.3 establishes the existence of odd solutions such that $u'(0) \neq 0$, it is concerned with solutions with an *odd* number of laps, that is solutions on branches $\mathcal{S}_m$ for m odd.

For solutions with an even number of laps we focus on even solutions. Their existence is the subject of Theorem 2.4.

Theorem 2.4. *Let $n \geq 1$. For each $\kappa \in (\kappa_{2n}, 1)$ there exists two even zero-energy periodic solution, u_{na} and u_{nb} of equation (1.6) which have $2n$ laps in each half-period such that $u_{na} < 0$, $u''_{na} > 0$ and $u_{nb} < 0$, $u''_{nb} < 0$ at the origin. Their first positive point of symmetry is ζ_{2n}, and they have precisely one zero on their half period $(0, \zeta_{2n})$.*

In Figure 2.4 we see how two branches bifurcate from the point κ_2, both with solutions having two laps. On one of the branches – denoted by $\mathcal{S}_{2a}$ – the solutions are convex at the origin, and on the other – denoted by $\mathcal{S}_{2b}$ – the solutions are concave at the origin.

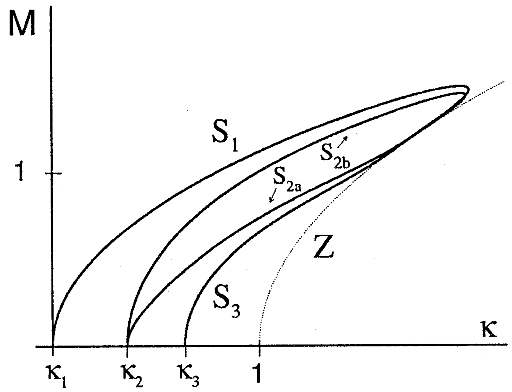
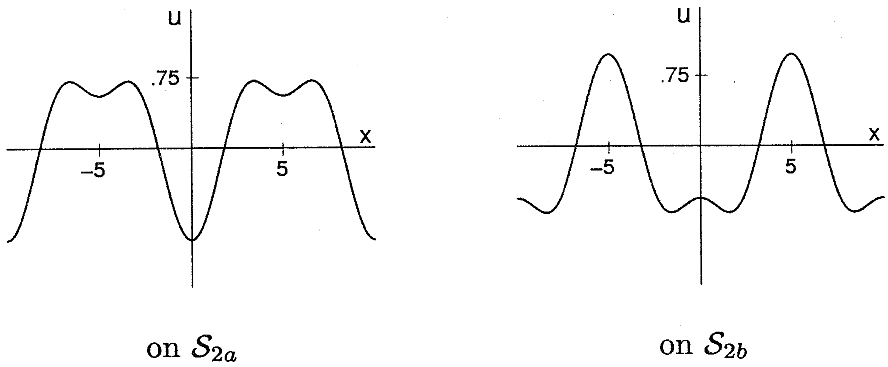


Fig. 2.4. The branches $\mathcal{S}_1$, $\mathcal{S}_{2a}$, $\mathcal{S}_{2b}$ and $\mathcal{S}_3$

In Figure 2.5 we give two solution graphs, one on $\mathcal{S}_{2a}$ and one on $\mathcal{S}_{2b}$.



on $\mathcal{S}_{2a}$ on $\mathcal{S}_{2b}$

Fig. 2.5. Solutions of Theorem 2.4

Since $f_\kappa(-s) = f_\kappa(s)$, it is clear that if u is a solution, then so is $-u$. In addition, since $u''_{2a} > 0$ at the origin, and the number of laps is even, we must have $u''_{2a} > 0$ at the point of symmetry, i.e. the point of symmetry is a local minimum. Therefore, if we reflect u_{2a} and then shift it over half a period L, the resulting function

$$u_{2b}(x) = -u_{2a}(x - L)$$

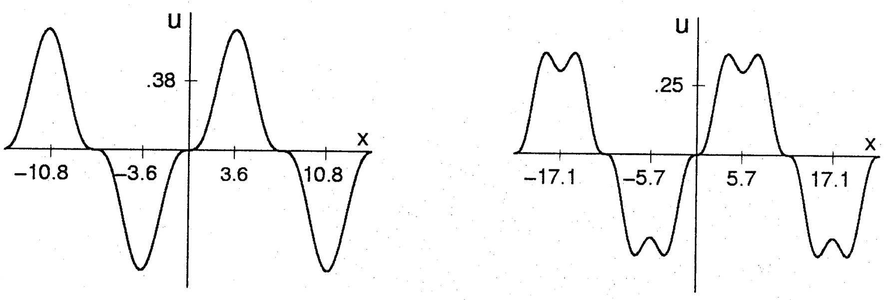
is a solution of equation (1.6) such that $u_{2b} < 0$ and $u''_{2b} < 0$ at the origin. Since, unlike with the odd solutions,

$$\max\{u_{2a}(x) : x \in \mathbf{R}\} \neq \min\{u_{2a}(x) : x \in \mathbf{R}\} = -\max\{u_{2b}(x) : x \in \mathbf{R}\},$$

these two solutions show up as two different branches.

Note that in contrast to the branches of odd solutions, the branches of even solutions are loops which begin and end at the same value of κ on the M-axis.

We conclude with a second family of periodic solutions. They all have the property that they have zero slope at the origin. Solutions on the two first branches are shown in Figure 2.6.



on $\mathcal{W}_3$ on $\mathcal{W}_5$

Fig. 2.6. Solutions on branches $\mathcal{W}_3$ and $\mathcal{W}_5$ at $\kappa = .96$

These solutions are odd, and if we regard the points of inflection on the u-axis as a lap which has collapsed to a point, then the first such solution has 3 laps, the second 5 and so on. We denote these branches by $\mathcal{W}_m$, where m is now an odd integer. The branches $\mathcal{W}_3$ and $\mathcal{W}_5$ are shown in Figure 2.7.

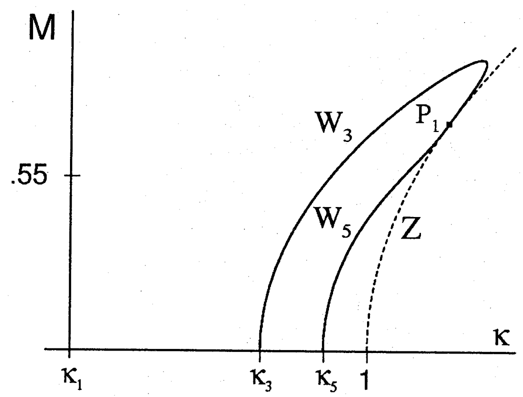


Fig. 2.7. The branches $\mathcal{W}_3$ and $\mathcal{W}_5$

Note that $\mathcal{W}_3$ and $\mathcal{W}_5$ bifurcate from the trivial solution at respectively κ_3 and κ_5. One can prove the general result that the family of branches $\{\mathcal{W}_{2n+1} : n \geq 1\}$, indeed exists, with $\mathcal{W}_{2n+1}$ bifurcating at κ_{2n+1}.

3. Ideas of the proofs

The proofs of Theoerms 2.2, 2.3 and 2.4 are based on a shooting technique. We look for odd or even solutions of equation (1.6), and hence it suffices to construct solutions on $\mathbf{R}^+$ and continue them to $\mathbf{R}^-$ as odd or even functions. For simplicity we focus here on *odd* solutions; *even* solutions are constructed in a similar manner. We then study the initial value problem

$$\begin{cases} u^{iv} + 2u'' + u^3 + (1-\kappa)u = 0, \quad x > 0, & (3.1a) \\ (u, u', u'', u''')(0) = (0, \alpha, 0, \beta). & (3.1b) \end{cases}$$

When we substitute (3.1b) into the energy identity (2.2), and remember that we assume that $E = 0$, we find that the initial data are related through the equation

$$\alpha\beta = -\alpha^2. \tag{3.2}$$

One can distinguish two cases:

$$\text{(i) } \alpha \neq 0, \quad \beta = -\alpha \qquad \text{and} \qquad \text{(ii) } \alpha = 0, \quad \beta \neq 0. \tag{3.1c}$$

By symmetry, solutions come in pairs: if u is a solution, then so is $-u$. In Case (i) we may therefore assume without loss of generality that $u'(0) = \alpha > 0$, and in Case (ii) that $\beta > 0$. In Case (i) we denote the solution by $u = u(x, \alpha)$ and in Case (ii) by $u(x, \beta)$.

The key idea of the proofs is to follow the local maxima and minima on $\mathbf{R}^+$ as α varies over $\mathbf{R}^+$. We denote the successive critical points by $\zeta_k(\alpha)$, the maxima by $\xi_k(\alpha)$ and the minima by $\eta_k(\alpha)$. One can show that for $0 < \kappa < 1$ these critical points exist for all $\alpha > 0$ and all $k \geq 1$, and that they depend continuously on α.

We are particularly interested in the position of the critical points on the solution graph with respect to the u-axis: above or below, and in the function:

$$\phi_k(\alpha) \stackrel{\text{def}}{=} u'''(\zeta_k(\alpha), \alpha).$$

The continuity properties of u and ζ_k with respect to α imply that if $0 < \kappa < 1$, then ϕ_k is a continuous function of α.

If α^* is a zero of ϕ_k, then the graph of $u(x, \alpha^*)$ on $[0, \zeta_k]$ can be continued as a symmetric solution on $[0, 2\zeta_k]$ and then continued as an odd solution on $[-2\zeta_k, 2\zeta_k]$, and finally as a periodic solution of (1.6) on $\mathbf{R}$. Thus the construction of a periodic solution, which is symmetric with respect to ζ_k, is reduced to finding a zero of ϕ_k.

Let us now construct a periodic solution which is symmetric with respect to $\zeta_1 = \xi_1$, i.e. a *single bump* periodic solution which is even with respect to its critical points and odd with respect to its zeros. We thereto inspect $\phi_1(\alpha)$ for large and small values of $\alpha > 0$.

For large values of α one can prove the following properties of $u(\xi_1)$:

Lemma 3.1. *For $\alpha \in \mathbf{R}^+$ large enough we have*

$$u(\xi_1(\alpha), \alpha) > 0 \qquad \textit{and} \qquad u'''(\xi_1(\alpha), \alpha) < 0.$$

The first inequality is evident. The proof of the second inequality is based on a scaling argument [PT3].

To determine the sign of $\phi_1(\alpha)$ for small values of α we inspect the local behaviour near $u = 0$ and study the solution v of the associated linear problem

$$\begin{cases} v^{iv} + 2v'' + (1 - \kappa)v = 0, \quad x > 0, & \text{(3.3a)} \\ (v, v', v'', v''')(0) = (0, 1, 0, -1). & \text{(3.3b)} \end{cases}$$

An elementary computation shows that v has the following poperties at its critical points ζ_j [PT3]:

Lemma 3.2. *Let $n \geq 1$ and $\kappa \in (\kappa_{2n-1}, 1)$. Then*

$$v(\zeta_j) > 0 \quad \textit{and} \quad v'''(\zeta_j) > 0 \qquad \textit{for} \quad j = 1, 2, \ldots, n. \tag{3.4}$$

Thus, if we set $n = 1$ in Lemma 3.2 we find that

$$v(\xi_1) > 0 \quad \textit{and} \quad v'''(\xi_1) > 0 \qquad \text{for any} \quad \kappa \in (0, 1).$$

This implies, by the continuity of u and ξ_1 with respect to α, that there exists a $\delta > 0$ such that

$$u'''(\xi_1(\alpha), \alpha) > 0 \quad \text{for} \quad 0 < \alpha < \delta. \tag{3.5}$$

Thus, we have shown that $\phi_1(\alpha) < 0$ for α large and $\phi_1(\alpha) > 0$ for α small. Therefore $\phi_1(\alpha)$ must have a zero on $\mathbf{R}^+$. Denoting this zero by α_1 we have

$$\phi_1(\alpha_1) = u'''(\xi_1(\alpha_1), \alpha_1) = 0,$$

so that we can continue $u(x, \alpha_1)$ to obtain a periodic solution of (1.6) which is even with respect to its critical points and odd with respect to its zeros, and has period $4\xi_1$. This completes the proof of Theorem 2.2.

Solutions on $\mathcal{S}_3$ are symmetric with respect to $\zeta_2 = \eta_1$. Therefore, to construct them we need to follow $u(\eta_1)$ and find a zero of

$$\phi_2(\alpha) = u'''(\eta_1(\alpha), \alpha).$$

Plainly, given any $\kappa \in (0,1)$, we have

$$u(\eta_1) = -u(\xi_1) \qquad \text{at} \quad \alpha_1.$$

On the other hand, we have seen in Lemma 3.2 that if $\kappa \in (\kappa_3, 1)$, then

$$u(\eta_1) > 0 \qquad \text{for} \quad \alpha \text{ small.}$$

Thus, for such κ we know that

$$\tilde{\alpha} = \sup\{\alpha > 0 : u(\eta_1) > 0 \quad \text{on} \quad (0, \alpha)\}$$

is well defined, and that $0 < \tilde{\alpha} < \alpha_1$. In additon, by continuity,

$$u(\eta_1) = 0 \quad \text{and} \quad u'''(\eta_1) < 0 \quad \text{at} \quad \tilde{\alpha}, \tag{3.6}$$

which means that $\phi_2(\tilde{\alpha}) < 0$. We have also seen in Lemma 3.2 that if $\kappa \in (\kappa_3, 1)$, then $\phi_2(\alpha) > 0$ for α small. Therefore, we conclude that ϕ_2 has a zero, say α_2 on the interval $(0, \tilde{\alpha})$, so that $u(x, \alpha_2)$ is an odd periodic which is symmetric with respect to η_1.

Proceeding in an analogous manner we can construct succesively the branches $\mathcal{S}_{2n-1}$ of $(2n-1)$-lap periodic solutions for every $n \geq 1$.

4. The region where $\kappa > 1$

As we have seen in Section 2, at every critical value κ_n a branch $\mathcal{S}_n$ bifurcates from the trivial solution which curls to the right and enters the region $\kappa > 1$ of the (κ, M) plane. Numerical results, as presented in the bifurcation graphs in Section 2, suggest that for every $n \geq 1$ the projection of the loop emanating from the $u = 0$ axis at κ_n is of the form $[\kappa_n, \kappa_n^*]$, where $\kappa_n^* > 1$. The solutions on $\mathcal{S}_n$ have n laps between points of symmetry, as long as $\kappa \in (\kappa_n, 1)$.

We see from the computed graphs of the solutions, that upon entering the region $\Sigma_1 = \{\kappa > 1\}$, the number of laps may change. The reason is that if $\kappa > 1$, the potential F_κ is a double well potential, and the bottom of the wells have dropped below the energy level $E = 0$. In fact, we have

$$F(u) \begin{cases} < E & \text{for} \quad 0 < |s| < z, \\ > E & \text{for} \quad |s| > z, \end{cases} \tag{4.1}$$

where

$$z = Z(\kappa) \stackrel{\text{def}}{=} \sqrt{2(\kappa - 1)}. \tag{4.2}$$

The graph of Z has been included in the different bifurcation diagrams.

It follows from the energy identity (2.2) that if $u(\zeta) = z$ at a critical point ζ of u, then $u''(\zeta) = 0$, so that at such points zeros of u' can appear or disappear. We see this happen when follow for instance the branch $\mathcal{S}_5$, starting from the trivial solution, and return along $\mathcal{S}_7$, and inspect the solution graphs at various points along the loop. The loop and its solutions are shown in Figures 4.1 and 4.2.

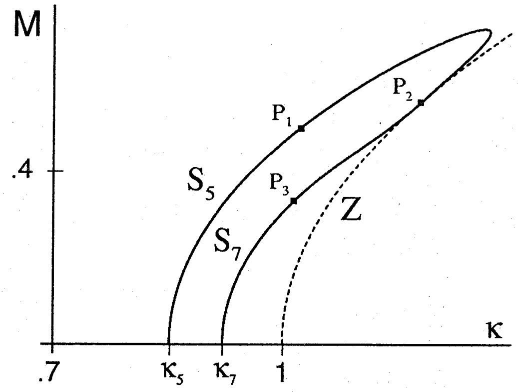
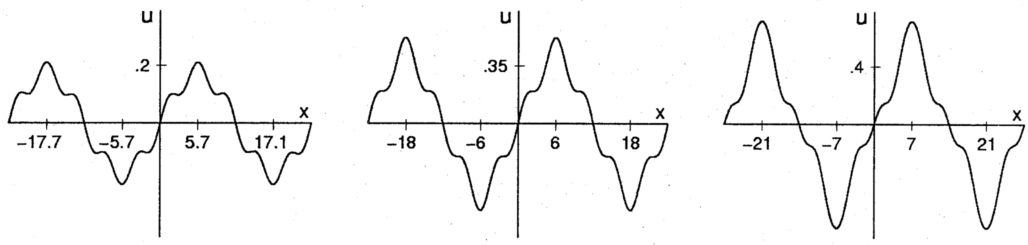


Fig. 4.1. The branches $\mathcal{S}_5$ and $\mathcal{S}_7$

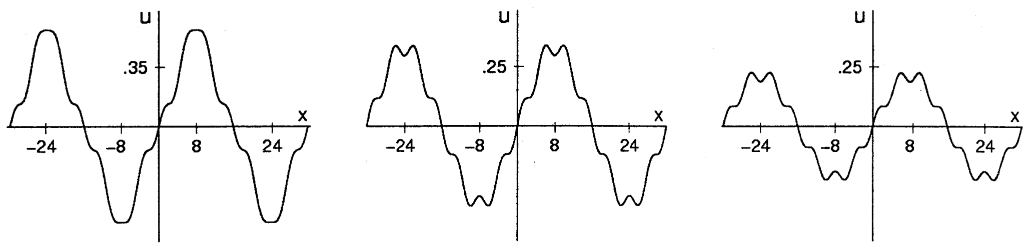
Starting on the $M = 0$ axis at κ_5, the solutions have 5 laps. At the point P_1, which lies in Σ_1, two laps disappear and the remaining monotone segments join up to form a *1-Lap* periodic solution. This sequence is shown in Figure 4.2a.



on $\mathcal{S}_5$ at $\kappa = 0.9$ at P_1 ($\kappa = 1.02$) at the tip ($\kappa = 1.27$)

Fig. 4.2a. The first three solutions

At the point P_2, which lies on the parabola $Z(\kappa)$ defined in (4.2), the tip acquires a dimple and as we progress towards κ_7 a pair of laps is formed at P_3, still in Σ_1, and we end up in the region $0 < \kappa < 1$ with 7 laps. The corresponding solutions, starting with the one at P_2, are shown in Figure 4.2b.



at P_2 at $\kappa = 1.169$ at P_3 ($\kappa = 1.01$) on $\mathcal{S}_7$ at $\kappa = 0.96$

Fig. 4.2b. The last three solutions

We conclude with a few remarks about problems involving equation (1.6) but with different nonlinearities f. We mention the work of Kramers and Paape [PK] in which

$$f_\kappa(s) = (1-\kappa)s \qquad \text{and} \qquad |s| \leq 1. \tag{4.3}$$

In this problem the corresponding potential F_k is for $\kappa \in (0,1)$ a single well potential with vertical walls at $s = \pm 1$. Other nonlinearities that have been studied [BSC, GL, NTV] include

$$f_\kappa(s) = (1-\kappa)s - \mu s^3 + s^5, \qquad \kappa \in \mathbf{R}, \quad \mu \in \mathbf{R}^+. \tag{4.4}$$

The branch $\mathcal{S}_1$ now first buckles back before it turns to positive values of κ, i.e. we have a subcritical bifurcation at the origin. The branches $\mathcal{S}_1$ and $\mathcal{S}_3$ are shown in Figure 4.3 for $\mu = 0.7$.

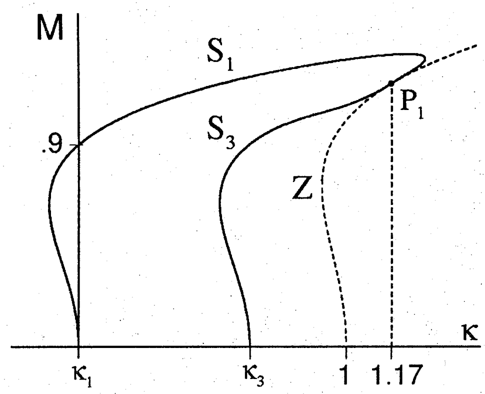


Fig. 4.3. The branches $\mathcal{S}_1$ and $\mathcal{S}_3$, and the graph of Z

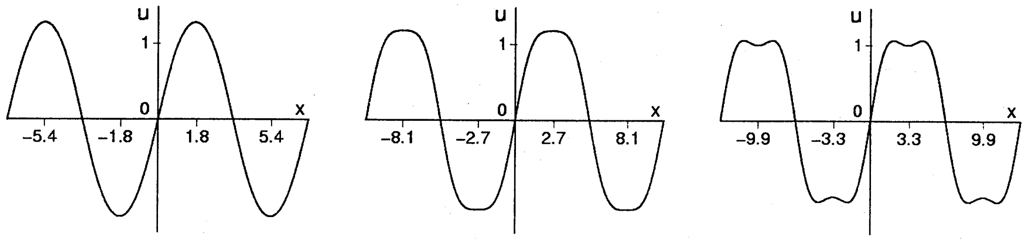
In Figure 4.3 we also show the graph Z of the zeros of the primitive $F_\kappa(s,\mu)$ of the source function $f_k(s,\mu)$ introduced in (4.4). It is given by

$$F_\kappa(s,\mu) = \frac{1-\kappa}{2}s^2 - \frac{\mu}{4}s^4 + \frac{1}{6}s^6. \tag{4.5}$$

The curve Z in the (κ, M)-plane is now given implicity by the relation

$$\kappa = 1 - \frac{\mu}{2}M^2 + \frac{1}{3}M^4.$$

Solutions on the loop exhibited in Figure 4.3 are shown in Figure 4.4.



on $\mathcal{S}_1$ at P_1 on $\mathcal{S}_3$

Fig. 4.4. Three solutions on the branch in Figure 4.3

We see that, as in Figures 2.1 and 2.2 the solution graphs acquire a dimple at their maxima and minima when the branch touches the curve Z at the point P_1, where $\kappa = 1.17$.

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