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On the discrete Morse flow as a numerical tool

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Dedicated to Professor Norio Kikuchi for his 60th birthday

1 Introduction

The **discrete Morse flow(DMF)** have been developed by Professor Norio Kikuchi(at Keio Univ.) for constructing solution of heat type equation related to a minimizing problem ([3], [4], [7], [8], [9], [10] and [11]). Kikuchi introduced the time-semidiscretized energy form below: For fixed positive constant h, consider the functional:

$$J_m(u) := \int_\Omega \frac{|u-u_{m-1}|^2}{h} dx + I(u), \tag{1.1}$$

for an elliptic functional $I(u)$. Here $\Omega(\subset \boldsymbol{R}^n)$ be a domain with Lipschitz boundary ($n \geq 1$). $I(u)$ is a functional which usually has the form

$$I(u) := \int_\Omega F(x, u, \nabla u) dx, \quad \text{on} \quad \mathcal{K} := \{u : \Omega \to \boldsymbol{R}^N; u \in \text{suitable space}\} \tag{1.2}$$

where $N \geq 1$ and $F(x,z,p) : \Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN} \to \boldsymbol{R}$ with suitable regularity. Usually F is required to have good properties such as the elliptic structure for the first variation (See [2] for example.) We determine a sequence $\{u_m^h\}$ as minimizers of J_m in $\mathcal{K}$, called the **DMF**, inductively: Firstly, for an initial data $u_0 \in \mathcal{K}$ with $I(u_0) < \infty$, we define u_1, as a minimizer of J_1 in $\mathcal{K}$. The next function $u_2 \in \mathcal{K}$ is determined a minimizer of J_2 in $\mathcal{K}$, and so on. $\{u_m^h\}$ is an approximate solution of a heat equation(See section 2) which possibly describes the Morse (semi)flow for $I(u)$.

Historically, minimizing method for heat equation was firstly introduced by Rektorys ([14]). Kikuchi rediscovered this type of method and introduced the form (1.1). By Kikuchi's rediscovery, many authors treated the problem of this type. For example, Bethuel et.al. treated harmonic mapping([1]), Nagasawa and Omata treated free boundary problem to seek a stationary point of $I(u)$ without passing the limit of $h \to 0$ ([12]). Koji Kikuchi constructed a varifold solution for non-convex functional $I(u)$. Hyperbolic type problems are also studied. (See [6] and [7].) Recently, Omata developed a numerical method using minimizing method (See [13] and its references.)

2 Convergence and asymptotic behavior

The important estimate on this flow is based on the following property:

$$J_m^h(u_m^h) \equiv \int_\Omega \frac{|u_m^h - u_{m-1}^h|^2}{h} dx + I(u_m^h) \leq J_m^h(u_{m-1}^h) \equiv I(u_{m-1}^h),$$

and therefore we have $\int_\Omega |u_m^h - u_{m-1}^h|^2/h dx \le I(u_{m-1}^h) - I(u_m^h)$. Summing up from $m = 1$ to M, we have a estimate:

$$I(u_M^h) + \sum_{m=1}^{M} \int_\Omega \frac{|u_m^h - u_{m-1}^h|^2}{h} dx \le I(u_0). \tag{2.1}$$

This estimate is a basic estimate of this flow, from which main properties are obtained.

For showing a convergence theory, we define an approximate solution of a heat equation.

DEFINITION 2.1 *We define functions $\bar{u}^h$ and u^h on $\Omega \times (0, \infty)$ by*

$$\bar{u}^h(x,t) = u_m^h(x); \qquad u^h(x,t) = \frac{t-(m-1)h}{h} u_m^h(x) + \frac{mh-t}{h} u_{m-1}^h(x),$$

for $(x,t) \in \Omega \times ((m-1)h, mh]$.

Firstly, we mention convergence theory when h tends to zero. By use of (2.1), if F is **coercive** in $H^1(\Omega)$, we can easily obtain facts that the following norms are uniformly bounded with respect to h:

$$\|\frac{\partial u^h}{\partial t}\|_{L^2((0,\infty)\times\Omega)}, \|\nabla \bar{u}^h\|_{L^\infty((0,\infty);L^2(\Omega))}, \|\nabla u^h\|_{L^\infty((0,\infty);L^2(\Omega))},$$
$$\|u^h\|_{L^\infty((0,\infty);L^2(\Omega))}, \|\bar{u}^h\|_{L^\infty((0,\infty);L^2(\Omega))}, \|u^h\|_{W^{1,2}((0,T)\times\Omega)}, (\text{for all } T > 0).$$

By use of these estimates, we see that by a suitable choice of subsequence $\{h_j\}$ ($h_j \to 0$, $j \to \infty$), approximate solutions may converge to a weak solution u (a limit function) in a some topology. Thus we can say:

THEOREM 2.2 *If F satisfies good conditions, then a limit function is a weak solution, i.e.*

$$\begin{aligned}\int_\Omega u_0 \eta(x,0) dx = \int_0^T \int_\Omega D_t u \eta dx dt \\ + \int_0^T \int_\Omega (F_{p_\alpha^i}(x,u,\nabla u) D_\alpha \eta^i + F_{z^i}(x,u,\nabla u)\eta^i) dx dt\end{aligned} \tag{2.2}$$

for all $\eta \in \overset{\circ}{W}{}_2^{1,1}((0,T)\times\Omega))$ with $\eta(x,T) = 0$, where $\overset{\circ}{V}_2((0,T)\times\Omega) = \{u \in L^2(Q_T), u_x \in L^2(Q_T); |u|_{Q_T} =$ ess $\sup_{0\le t\le T} \|u(x,t)\|_{L^2(\Omega)} + \|u_x\|_{L^2(Q_T)} < \infty\}$.

For above theorem, very strong conditions for F is requested, so some mathematicians are trying to relax them. (See [9] for example.)

Secondly, we can get a result on asymptotic behavior of the D.M.F. $\{u_m\}$ as $m \to \infty$. From (2.1), we easily have $\|u_m - u_{m-1}\|_{L^2(\Omega)} \to 0$ as $m \to \infty$. Again, by (2.1), for any subsequence $\{u_{m_j}\} \subset \{u_m\}$, there exists a subsequence $\{u_{m_{j_\nu}}\} \subset \{u_{m_j}\}$ and a function u_∞ on Ω such that $u_{m_{j_\nu}} \to u_\infty$ weakly in $W^{1,2}(\Omega)$, and strongly in $L^2(\Omega)$, as $\nu \to \infty$. Moreover, we have:

THEOREM 2.3 *If F satisfies good conditions, the limit function u_∞ is a minimizer of the functional*

$$J_\infty(u) = \int_\Omega \left(\frac{|u-u_\infty|^2}{h} + F(x,u,\nabla u) \right) dx$$

in $\mathcal{K}$, hence, u_∞ satisfies $-\int_\Omega (F_{p_\alpha^i}(x,u,\nabla u) D_\alpha \phi^i + F_{z^i}(x,u,\nabla u)\phi^i) dx = 0$ for any $\phi \in C_0^\infty(\Omega)$.

Sometimes regularity for u is obtained from this minimality ([12]). In other word, u_∞ is not a simple stationary point to $I(u)$, but expected to have some regular properties.

3 Numerical Examples

By use of **DMF**, we can construct a simple algorithm (i.e. Nonlinear Optimization) for solving heat type equations numerically ([13]). Compared with the Gralerkin method, in some problem, it works well and usually the structure would be simpler. We show two types of examples.

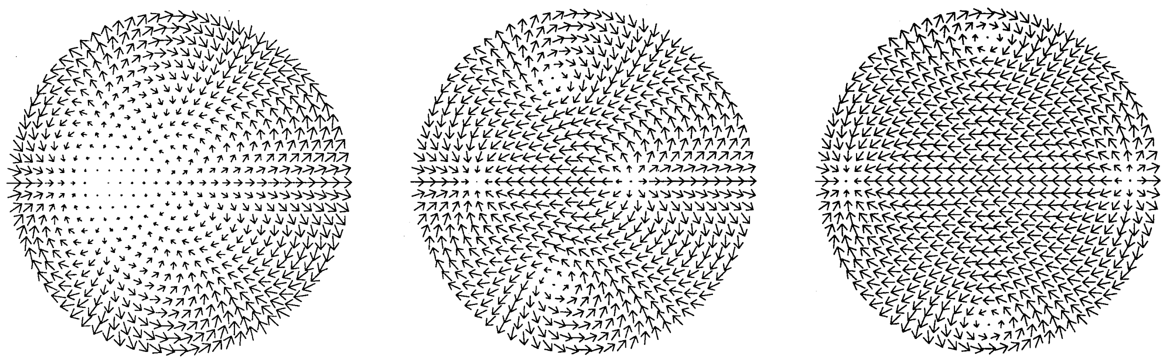
3.1 Ginzburg-Landau energy

In this experiment, we treat ($u : B_1(0) \to \mathbf{R}^2$)

$$I(u) = \int_{B_1(0)} \left(|\nabla u|^2 + \frac{1}{\delta}(|u|^2 - 1) \right) dx.$$

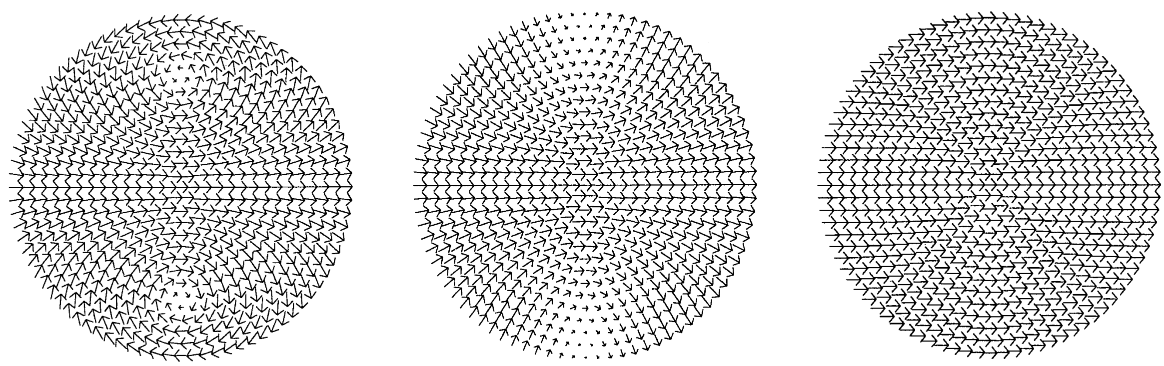
The boundary data is Dirichlet with 4 degrees and as a initial data we put 1 vortex with 4 degrees. The vortex splits into 4 vortices with 1 degree.

t=0 t=0.08 t=∞



In this case, the boundary data is Neumann, and put 2 vortices with 1 degree as a initial data. They push each other and eventually they are going out.

t=0 t=0.12 t=∞



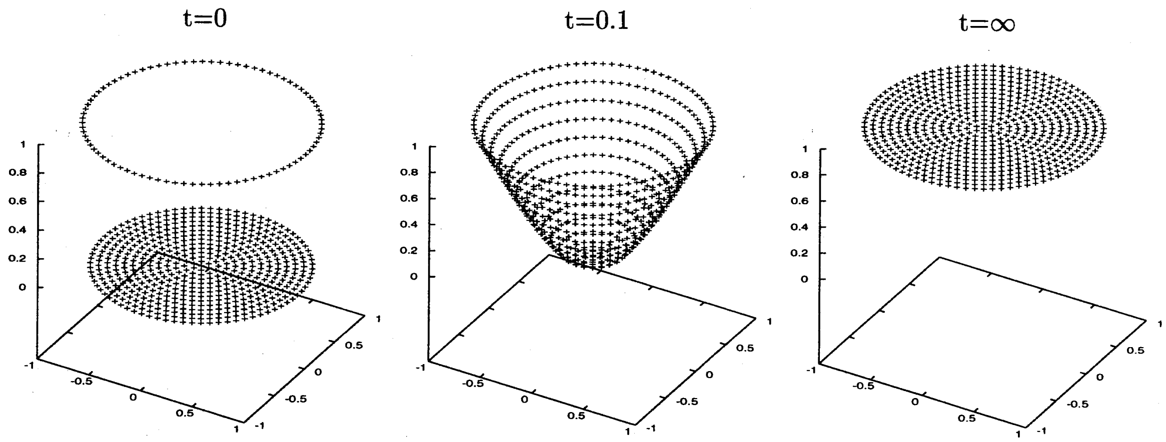
3.2 A Free Boundary Problem

In these experiment, we treat $(u : B_1(0) \to \mathbf{R})$ the following free boundary $\big(\partial\{u > 0\}\big)$ problem

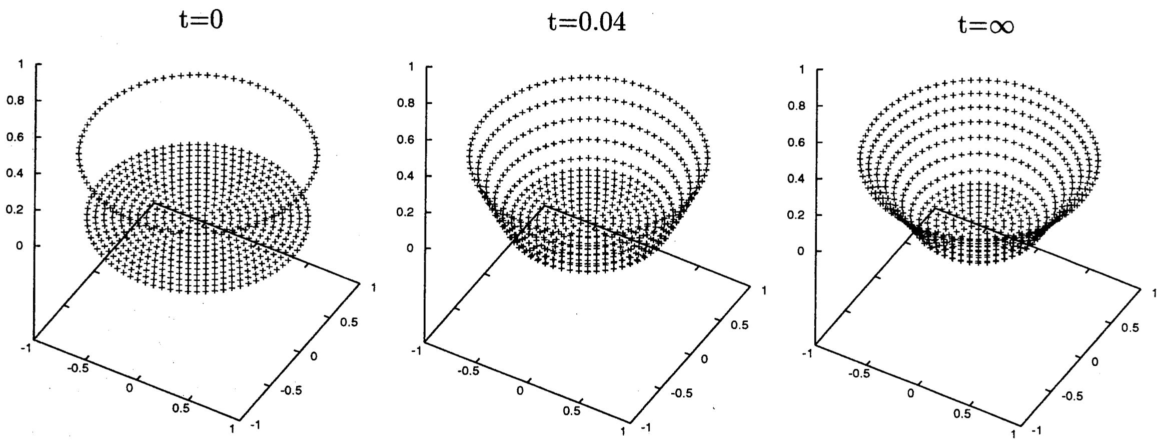
$$I(u) = \int_{B_1(0)} \Big(|\nabla u|^2 + \chi_{u>0}\Big)dx.$$

The difference of boundary data causes a different conclusions.

In this experiment boundary data is high enough and the limit function becomes constant.



In this experiment the boundary data is low so that the limit function keeps the free boundary.



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