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Correlations of measurement information and noise in quantum measurements with finite resolution

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Abstract

The original purpose of measurements is to provide us with information about a previously unknown physical property of the system observed. In the Hilbert space formalism of quantum mechanics, this physical meaning of measurement information is not immediately apparent. In order to study the relationship between the Hilbert space coherence of the quantum state and the measurement information obtained in the laboratory, we introduce a generalized measurement postulate for finite resolution measurements. With this measurement model, correlations between non-commuting observables can be investigated. These experimentally accessible correlations reveal nonclassical features in their dependence on the operator ordering, reflecting the particular measurement context by which they are determined.

1 Introduction

The interpretation of quantum mechanics has been controversial from the very beginning, and even one hundred years after Planck's quantum hypothesis, the physical reality behind the formalism is still being debated [1]. At the heart of this confusion is the issue of quantum measurements [2]. The original definitions of physical quantities were based on the expectation that the quantities observed in measurements are effectively identical to the fundamental mathematical elements of the theory. The implicit assumption behind this expectation is that knowledge and fact are independent and seperable. An ideal measurement simply provides information about facts without changing the facts. Reality would then be defined by an infinitely precise set of observable quantities. However, this assumption had to be abandoned in order to describe atomic and optical phenomena. Starting with Planck's quantum hypothesis, quantum theory completely abandoned the constraints of classical physics. In order to justify this radical departure from these highly successful concepts, Bohr and Heisenberg argued that no reality needs to be attributed to properties which are not being observed. In the famous Bohr-Einstein dialogue, the decisive argument was provided by the uncertainty principle which places a fundamental restriction on our ability to know observable properties.

At the same time, a highly precise mathematical description of quantum mechanics emerged. This mathematical description introduces the concept of probability amplitudes, which combine the epistemological nature of probabilities with the deterministic concept of interference. It is this Hilbert space formalism which has firmly established quantum mechanics in 20th century physics. Nevertheless interpretational problems remain because the physical meaning of the Hilbert space formalism is purely statistical. The quantum state of a single system cannot be measured. This situation gives rise to a dualism between the mathematical description and the physical meaning in quantum mechanics [3] which is illustrated in figure 1. For a long time, this dualism was not recognized as experiments mostly revealed sta-

Physical principles Observable properties in space and time Uncertainty relations $\Downarrow$ Quantum Noise	Mathematical principles Abstract formalism in Hilbert space States and Operators $\Downarrow$ Quantum Information

Figure 1: Illustration of the dualism between the mathematical principles and the physical principles of quantum mechanics.

tistical averages of only a few observable properties. As modern technology allows a more detailed investigation of quantum phenomena, however, the interpretational problems associated with a theory that is not based upon directly observable facts have reemerged. It may therefore be necessary to investigate the role of measurement in quantum mechanics with respect to the possible manipulations of single quantum objects in order to clarify the physical meaning of quantum states. In the light of the present discussion on the usefulness of quantum effects for computation and data transmission, a clarification of the process by which information is extracted from a quantum system may also provide a better understanding of the technical requirements for quantum information processing.

In the following, the relationship between the measurement information obtained at finite resolution and the noise introduced by the measurement interaction is investigated. It is shown that the noise is strongly correlated with the measurement result, even if the eigenvalues of the observed property would not permit such correlations. These correlations represent operator ordering dependent properties which emerge only in the regime where the measurement resolution is high enough to reveal quantization, yet low enough to permit coherence. Thus, finite resolution measurements reveal more about the physical properties of quantum systems than precise measurements. Hopefully, the investigation of these nonclassical correlations will help to bridge the gap between the mathematical principles and the physical principles of quantum mechanics by providing a physical interpretation of quantum coherence and operator ordering.

2 Uncertainty in quantum measurements

The connection between the mathematical formalism and the physical observables is established by the measurement. The squared probability amplitudes of Hilbert space formalism can then be interpreted as real probabilities. The probabilities can be summarized in the density matrix $\hat{\rho}$ which may be expressed in terms of the eigenstates of $\hat{n}$ as

$$\hat{\rho} = \sum_{n,m} \rho_{nm} \mid n\rangle\langle m \mid . \tag{1}$$

While the diagonal elements ρ_{nn} clearly define the probability of obtaining a measurement result of n in a precise measurement of $\hat{n}$, the relationship between $\hat{n}$ and the off-diagonal elements ρ_{nm} with $n \neq m$ is less clear. The physical meaning of the off-diagonal elements only emerges in a transformation to a different set of eigenstates, where ρ_{nm} modifies the new probabilities as an interference term. However, a precise measurement of $\hat{n}$ destroys this interference information, increasing the uncertainty in the probability distributions associated with eigenstates of observables other than $\hat{n}$ in accordance with the uncertainty principle.

Uncertainty requires the dissapearance of ρ_{nm} before n can be distinguished from m. The information obtained about $\hat{n}$ therefore requires that the measurement interaction introduces decoherence in the off-diagonal elemnts ρ_{nm}. If the noise does not reduce ρ_{nm} to zero, then the resolution δn is not sufficient to completely distinguish n from m. This situation is typical for optical quantum nondemolition measurements of photon number [4, 5, 6]. In such experiments, the information obtained about the photon number $\hat{n}$ is given by a measurement result n_m and a resolution δn which corresponds to the Gaussian uncertainty of the measurement. However, this reduced measurement resolution is not just a limitation but helps to preserve the coherence of the quantum state [7, 8].

Instead of selecting a well-defined photon number component $\mid n\rangle$, the measurement adjusts the statistical weight of each component $\mid n\rangle$ by a factor dependent on the difference between n_m and n. This effect of the measurement can be represented by a generalized measurement operator $\hat{P}_{\delta n}(n_m)$ [9] given by

$$\hat{P}_{\delta n}(n_m) = (2\pi\delta n^2)^{-1/4} \exp\left(-\frac{(n_m - \hat{n})^2}{4\delta n^2}\right). \tag{2}$$

For a given initial state $\hat{\rho}_i$, the probability distribution $P(n_m)$ over measurement results n_m and the density matrix $\hat{\rho}_f(n_m)$ after the measurement read

$$\begin{aligned} P(n_m) &= Tr\left\{\hat{P}_{\delta n}(n_m)\ \hat{\rho}(\mathrm{in})\ \hat{P}_{\delta n}(n_m)\right\} && (3)\\ \hat{\rho}_f(n_m) &= \frac{1}{P(n_m)}\hat{P}_{\delta n}(n_m)\ \hat{\rho}(\mathrm{in})\ \hat{P}_{\delta n}(n_m). && (4)\end{aligned}$$

By describing the effects of a finite measurement resolution δn on all physical properties of the system the generalized measurement postulate defined by the operator $\hat{P}_{\delta n}(n_m)$ provides an expression of the role of uncertainty in quantum measurements.

It is then possible to quantify the changes in coherence associated with a photon number measurement result n_m in detail.

3 Photon number measurement statistics

The interference properties of the light field are described by the coherent amplitude operator $\hat{a}$. This operator is also referred to as annihilation operator because its effect on photon number states is given by

$$\hat{a} \mid n\rangle = \sqrt{n} \mid n-1\rangle. \tag{5}$$

However, the physical meaning of this photon number property is simply that the interference properties of the light field given by the expectation value $\langle \hat{a} \rangle$ depend on the density matrix elements ρ_{nm} with $m = n-1$. Photon number information therefore destroys the interference properties by reducing $\langle \hat{a} \rangle$.

For a coherent light field such as that emitted by a typical laser, the initial density matrix is given by

$$\hat{\rho}_i = \sum_{n,m} \exp(-|\alpha|^2) \frac{1}{\sqrt{n!m!}} (\alpha^*)^m (\alpha)^n \mid n\rangle\langle m \mid . \tag{6}$$

This coherent state has an average amplitude of $\langle \hat{a} \rangle = \alpha$ and an average photon number of $\langle \hat{n} \rangle = |\alpha|^2$. The optical coherence of this state is maximal and any reduction in the photon number fluctuations leads to a reduction of the coherence given by $\langle \hat{a} \rangle = \alpha$. By applying the general measurement postulate according to equation (3), one obtains the probability distribution for photon number measurement results n_m at a finite resolution of δn as

$$P(n_m) = \frac{\exp(-|\alpha|^2)}{\sqrt{2\pi\delta n^2}} \sum_n \frac{|\alpha|^{2n}}{n!} \exp\left(-\frac{(n-n_m)^2}{2\delta n^2}\right). \tag{7}$$

This probability distribution is simply a sum of Gaussians centered around the respective integer photon numbers. At low resolution (high δn), the Gaussians merge to form a continuous probability distribution. At high resolution (low δn), the Gaussians are completely separate. However, coherence is only possible in the regions where neighbouring Gaussians overlap. This becomes clear if the coherence $\langle\hat{a}\rangle_f(n_m)$ after a measurement of n_m is determined. It reads

$$\langle\hat{a}\rangle_f(n_m) = \alpha\ \exp\left(-\frac{1}{8\delta n^2}\right)\frac{\sum_n \frac{|\alpha|^{2n}}{n!}\exp\left(-\frac{(n+\frac{1}{2}-n_m)^2}{2\delta n^2}\right)}{\sum_n \frac{|\alpha|^{2n}}{n!}\exp\left(-\frac{(n-n_m)^2}{2\delta n^2}\right)}. \tag{8}$$

This fraction of two sums of Gaussians has its maxima at half integer photon numbers, where the denominator representing the measurement probability is minimal. Figure 2 shows the probability distribution and the coherence after the measurement for $\alpha = 3$ and $\delta n = 0.3$. A direct comparison of the probability and the coherence near $n_m = 9$ is shown in figure 3 It is obvious that integer measurement results n_m are correlated with strong decoherence and half integer results with weak decoherence.

4 Nonclassical correlation

If the quantization Q of a measurement result n_m is defined as

$$Q = \cos\left(2\pi n_m\right), \tag{9}$$

then the correlation between quantization and coherence observed in the finite resolution measurement of photon number may be expressed as

$$C(Q,\langle\hat{a}\rangle_f) = \int\left(Q(n_m)\,\langle\hat{a}\rangle_f(n_m)\right)dn_m - \left(\int Q(n_m)\,dn_m\right)\left(\int\langle\hat{a}\rangle_f(n_m)\,dn_m\right). \tag{10}$$

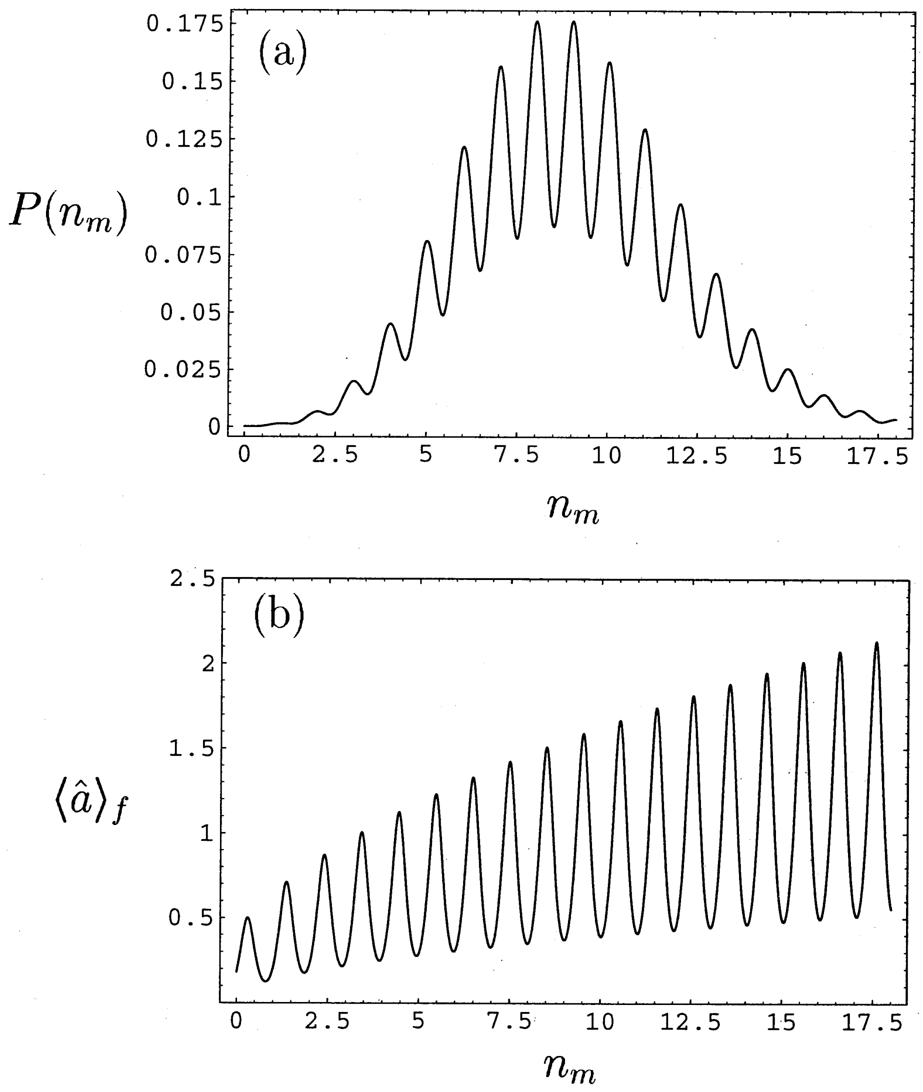

Figure 2: Photon number measurement statistics of a coherent state with an average amplitude of $\alpha = 3$ at a photon number resolution of $\delta n = 0.3$. (a) shows the probability distribution over measurement results and (b) shows the expectation value $\langle \hat{a} \rangle_f(n_m)$ after the measurement.

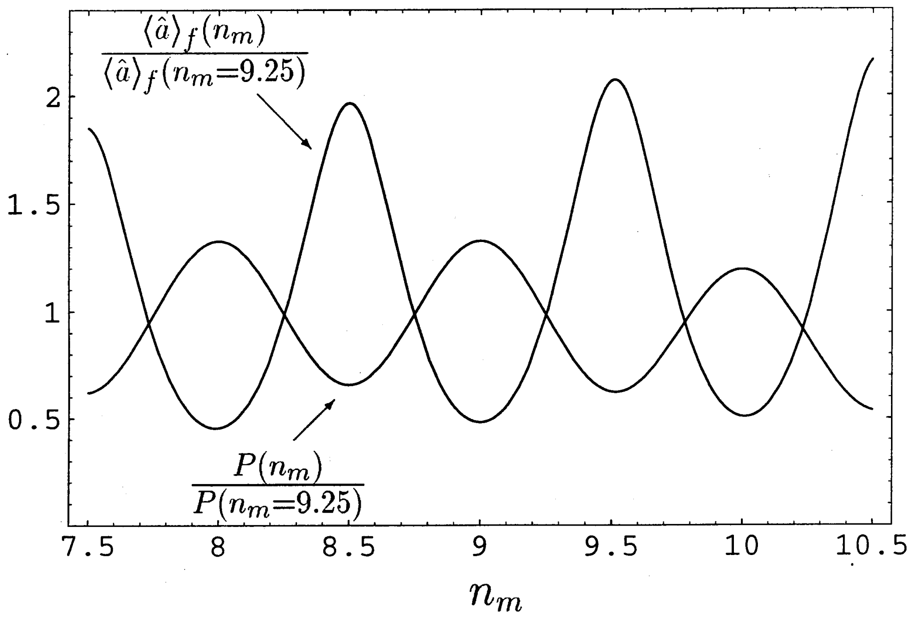

Figure 3: Details of the quantum mechanical modulations of measurement probability and coherence after the measurement near $n_m = 9$, normalized by the results at $n_m = 9.25$.

In the case of a coherent state, the averages of quantization Q, of coherence $\langle\hat{a}\rangle_f$, and of their product are given by

$$\begin{aligned} \int Q(n_m)\, dn_m &= \exp\left(-2\pi^2 \delta n^2\right) && (11) \\ \int \langle\hat{a}\rangle_f(n_m)\, dn_m &= \exp\left(-\frac{1}{8\delta n^2}\right)\alpha && (12) \\ \int \left(Q(n_m)\, \langle\hat{a}\rangle_f(n_m)\right) dn_m &= -\exp\left(-2\pi^2\delta n^2\right)\exp\left(-\frac{1}{8\delta n^2}\right)\alpha. && (13) \end{aligned}$$

Quantization and coherence are therefore exactly anti-correlated, with $C(Q, \langle\hat{a}\rangle_f)$ being equal to two times the negative product of average quantization and average coherence,

$$\begin{aligned} C(Q,\langle\hat{a}\rangle_f) &= -2\left(\int Q(n_m)\, dn_m\right)\left(\int \langle\hat{a}\rangle_f(n_m)\, dn_m\right) \\ &= -2\exp\left(-2\pi^2\delta n^2\right)\exp\left(-\frac{1}{8\delta n^2}\right)\alpha. && (14) \end{aligned}$$

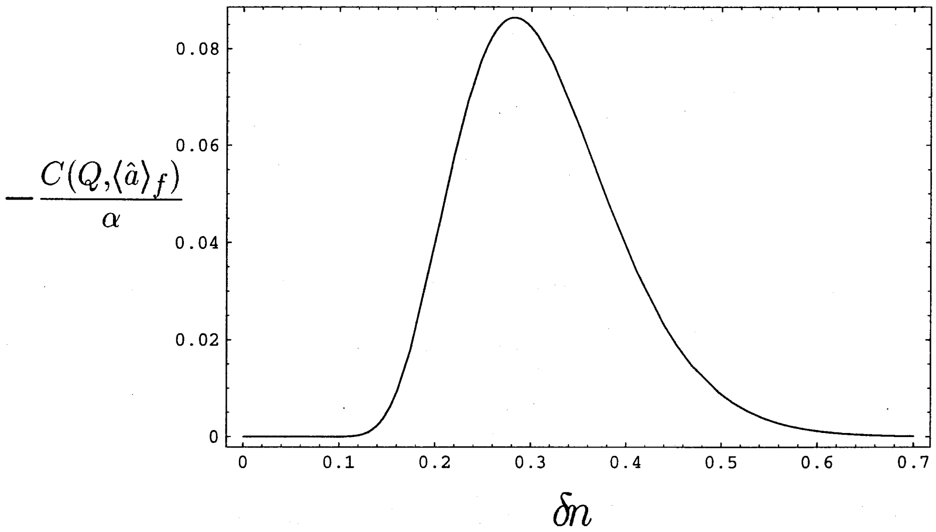

Figure 4: Normalized anticorrelation of the quantization Q of the measurement result n_m and the coherence $\langle\hat{a}\rangle_f(n_m)$ after the measurement as a function of measurement resolution δn.

Figure 4 shows this anti-correlation as a function of measurement resolution δn. Note that the anti-correlation is maximal near $\delta n = 0.3$. For lower δn, decoherence reduces the average value of $\langle\hat{a}\rangle_f(n_m)$ to zero. For higher δn, quantization is not resolved in the measurement and the average value of Q is zero.

The anti-correlation of quantization and coherence is a direct consequence of quantum coherence. It originates from the fact that the operator $\hat{a}$ connects photon number states $\mid n\rangle$ with photon number states $\mid n-1\rangle$. This property means that $\hat{a}$ does not commute with functions of $\hat{n}$. If the quantization operator $\hat{Q}$ is defined as

$$\begin{aligned}\hat{Q} &= (\cos(\pi\hat{n}))^2 - (\sin(\pi\hat{n}))^2 \\ &= (-1)^{2\hat{n}}, \end{aligned} \tag{15}$$

then the anti-correlation between quantization and coherence can be obtained by "splitting" $\hat{Q}$ into two operators and inserting $\hat{a}$ in the middle:

$$\begin{aligned} C(\hat{Q},\hat{a}) &= \langle(-1)^{\hat{n}}\hat{a}(-1)^{\hat{n}}\rangle - \langle(-1)^{2\hat{n}}\rangle\langle\hat{a}\rangle \\ &= -2\langle\hat{a}\rangle. \end{aligned} \tag{16}$$

Of course, this operator ordering is only justified because the actual measurement first obtained n_m and then $\langle\hat{a}\rangle$. The position of $\hat{a}$ between the parity operators $(-1)^{\hat{n}}$ is a consequence of this measurement context.

5 Context dependence of information

In the example above, the measurement information obtained is not information about the quantum state before the measurement, since this state is already known to be a coherent state. Nevertheless, the quantum state does not provide sufficient information to describe the physical properties of the light field in every possible context. Therefore, relevant new information about a physical property may be extracted even from the pure state. There seems to be no classical analogy to this process of extracting new information from a well defined state. Possibly, the generalization of classical information concepts to quantum mechanics is not as straightforward as the concepts of entropy and qbits seem to suggest [10]. In particular, the measurement interaction is responsible for selecting the relevant information. If information has been encoded in a different variable, then this information becomes physically irrelevant and is lost. However, this loss of information correponds to a smooth and continuous transition from one type of context to the other, leaving some room for correlations such as the one between quantization and coherence.

In fact, it seems to be more natural to consider situations in which the context is not given by a well defined orthogonal set of states, but by a combination of non-commuting properties which could be described by a sequence of finite resolution measurements. The problem of avoiding decoherence in quantum computation probably arises because it is difficult to limit the measurement context to precise measurements at well defined times. If the wrong type of information leaks out, the artificial concepts of quantum information break down. On the other hand, the full wealth of possibilities naturally inherent in the quantum properties of physical properties may only be explored by embracing the correlations between non-commuting variables. Specifically, the nonclassical correlations expressed by context dependent operator ordering may provide a clearer understanding of the physical properties behind such applications of quantum effects as quantum computation, quantum teleportation, and quantum cryptography.

6 Conclusions

The uncertainty relations severely restrict the physical information available about a quantum system. Nevertheless, new information may even be extracted from a pure state by measuring a previously unknown property. This extraction of new information requires the loss of information about the known system properties and therefore constitutes an exchange of one type of physical information for another. Yet this exchange process is more precise than the notion of uncertainty would seem to suggest.

In the case discussed above, the coherent field information is gradually exchanged for photon number information. However, an accidental observation of an "incorrect" half integer photon number does not result in a loss of information - instead the original phase information is retained. On the other hand, even the accidental observation of the "correct" integer photon number does not provide more information on the system - instead, there is an increased loss of phase coherence to compensate the precision of the photon number measurement result. Thus the non-classical correlation of quantization and coherence preserve the total information available about the system while changing the physical context.

It seems to be the major difference between classical physics and quantum physics that quantum physics establishes a fundamental relationship between the information available about a system and the physical properties of the system. In particular, bit like quantized information is only obtained if the property of coherence has been lost. The anti-correlation of quantization and coherence demonstrates that quantization itself is a property which depends on the measurement context and thus on the information presently available about the light field. This ambiguity in the physical properties appears to be the fundamental reason for the dualism between the physical and the mathematical principles of quantum mechanics.

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