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On a class of fully nonlinear PDEs derived from variational problems of L^p norms

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1 Introduction

Let $\Omega\subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, $p>n$ and $f\in C(\overline{\Omega})$ such that $f>0$ in Ω.

At first, we shall consider the variational problem

$$\inf\{\|Dv\|^p-\int_\Omega fvdx|v\in W_0^{1,p}(\Omega)\}, \tag{1}$$

where $\|\cdot\|$ is the standard norm in $L^p(\Omega,R^n)$ defined as follows;

$$\|w\|=(\int_\Omega|w(x)|^p dx)^{\frac{1}{p}}$$

for $w\in L^p(\Omega,\mathbf{R}^n)$ and $|\cdot|$ is the Euclidean norm in $\mathbf{R}^n$.

T. Bhattacharya-E. DiBenedetto-J. Manfredi [5] and B. Kawohl [13] showed that the limit function of minimizers of the variational problem (1), as $p\to\infty$, is the distance function from the boundary of Ω.

We are interested in what is the limit function of minimizers of the variational problem with the norm equivalent to the standard one. For simplicity, we shall consider the following norm defined by

$$\|w\|_1=(\sum_{i=1}^n\|w_i\|_{L^p(\Omega)}^p)^{\frac{1}{p}}$$

for $w=(w_1,\ldots,w_n)\in L^p(\Omega,\mathbf{R}^n)$.

With this norm, we are concerned with the variational problem

$$\inf\{\|Dv\|_1^p-\int_\Omega fvdx|v\in W_0^{1,p}(\Omega)\}. \tag{2}$$

However, it seems hard for us to verify that by using a direct method as in [5] or [13], the limit function is a distance function corresponding to our norm.

On the other hand, to determine the limit function, we recall the following result by R. Jensen [10] for the limit PDE derived from (1); the limit function of minimizers of (1), as $p\to\infty$, satisfies

$$\min\{|Du(x)|-1,-\Delta_\infty u(x)\}=0\quad\text{in}\quad\Omega \tag{3}$$

in the viscosity sense, where the ∞-Laplacian is given by

$$\Delta_\infty u = \langle D^2 u Du, Du\rangle.$$

Since the above PDE (3) is not of divergence form, we need the notion of viscosity solutions as weak solutions.

We note that the ∞-Laplacian was introduced by G. Arronson to characterize the "absolutely minimizing Lipschitz extension" (AMLE for short). Recently, R. Jensen in [10] proved that the AMLE can be characterized as a unique viscosity solution of

$$-\Delta_\infty u(x) = 0 \quad \text{in} \quad \Omega$$

under the given inhomogenious Dirichlet boundary condition. To show the uniqueness of viscosity solutions of the above, R. Jensen treated (3)-type auxiliary equations.

Our strategy is as follows:
(1) Derive the limit PDE associated with (2).
(2) Obtain a uniqueness result for the PDE.
(3) Characterize the limit of minimizers of (2) as a unique solution of the PDE.
(4) Look for a distance function from $\partial\Omega$ which is also a solution of the PDE.

In the section 3, we prove the comparison principle for this limit PDE. However, as will be seen, this PDE has serious discountinuity, which violates the standard argument to show the comparison principle for viscosity solutions. We avoid this difficulty imposing an extra assumption for solutions.

In the section 4, we show that a distance function, which corresponds to our problem, satisfies the limit PDE.

In the section 5, we consider other equivalent norms in the variational problem and derive equations which the corresponding limit function satisfies. However, we cannot prove the comparison principle for this PDE in general.

2 Limit of minimizers as p$\to\infty$

In this section we derive the PDE for the limit function of minimizers of (2). First, we derive the Euler equation associated with the variational problem (2). It is not hard to show that the minimizer of (2) satisfies the Euler equation in the viscosity sense (c.f., Theorem 1.29 in [12]);

Proposition 1. *Let u_p be the minimizer of (2). Then, u_p satisfies the PDE*

$$-p(p-1)\sum_{i=1}^{n} |u_{x_i}(x)|^{p-2} u_{x_i x_i}(x) - f(x) = 0 \quad \text{in} \quad \Omega \tag{4}$$

in the viscosity sense.

First, we get the gradient estimate of the minimizer u_p uniformly in $p > n$; there exists a constant $C > 0$ such that

$$||Du_p||_{L^p(\Omega)} \leq C \quad \text{in} \quad \Omega$$

for all $p > n$. Hence, we can see that $\{u_p\}_{p>n}$ has a subsequence converging to some Lipschitz function uniformly in Ω. Dividing the PDE (4) by

$$p(p-1) \max_{i=1,\ldots,n} |u_{px_i}(x)|,$$

and then, sending $p \to \infty$, we can derive the limit PDE which the limit function of u_p satisfies in the viscosity sense.

Proposition 2. *Let u_p be the minimizer of (2). Then, there exist $u \in W^{1,\infty}(\bar{\Omega})$ and a subsequence $p_j \to \infty$ as $j \to \infty$ such that*

$$u_{p_j} \to u \quad \text{as} \quad j \to \infty \quad \text{uniformly in} \quad \Omega,$$

and that u satisfies the limit PDE

$$\min\{G(Du(x)) - 1, F(Du(x), D^2u(x)\} = 0 \quad \text{in} \quad \Omega \tag{5}$$

in viscosity sense. Here,

$$G(q) = \max_{i=1,\ldots,n} |q_i| \quad , \quad F(q, X) = -\sum_{i \in I[q]} X_{ii}$$

$$\text{and} \quad I[q] = \{i \in \{1, \ldots, n\} | G(q) = |q_i|\}$$

for $q = (q_1, \ldots, q_n) \in \mathbf{R}^n$ and $X = (X_{ij}) \in S^n$, where S^n denotes the set of all $n \times n$ symmetric realvalued matrices.

For the reader's convenience, we recall the definition of viscosity solutions. Consider functions $E : \mathbf{R}^n \times S^n \to \mathbf{R}$ and $w : \Omega \to \mathbf{R}$.

Definition. We call w a viscosity supersolution (respectively, subsolution) of $E(Dw, D^2w) = 0$ in Ω if and only if for any $x \in \Omega$ and $\psi \in C^2$,

$$E^*(D\psi(x), D^2\psi(x)) \geq 0$$

$$(\text{respectively,} \quad E_*(D\psi(x), D^2\psi(x)) \leq 0)$$

provided $u - \psi$ has a local minimum at x (respectively, a local maximum in x).

We also call w a viscosity solution of $E(Dw, D^2w) = 0$ in Ω if and only if it is a viscosity sub- and supersolution of it.

Here, E^* and E_* are, respectively, upper and lower countinuous envelopes, i.e.,

$$E^*(q, X) = \lim_{\epsilon \to 0} \sup\{E(\hat{q}, \hat{X}); |\hat{q} - q| < \epsilon \quad \text{and} \quad |\hat{X} - X| < \epsilon\},$$

$$E_*(q,X)=\lim_{\epsilon\to 0}\inf\{E(\hat{q},\hat{X});|\hat{q}-q|<\epsilon \quad \text{and} \quad |\hat{X}-X|<\epsilon\}$$

for all $p\in\mathbf{R}^n$ and $X\in S^n$.

We give an equivalent definition with the semi-jets. First, we define sub- and super-semijets of functions of second order.

Definition. For $w\in C(\Omega)$ and $x\in\Omega$,

$$J^{2,-}w(x)=\left\{ (q,X)\in\mathbf{R}^n\times S^n \;\middle|\; \begin{array}{ll} w(y)\geq & w(x)+\langle q,y-x\rangle \\ & +\frac{1}{2}\langle X(y-x),y-x\rangle \\ & +o(|x-y|^2) \quad \text{as} \quad y\to x \end{array} \right\},$$

$$J^{2,+}w(x)=\left\{ (q,X)\in\mathbf{R}^n\times S^n \;\middle|\; \begin{array}{ll} w(y)\leq & w(x)+\langle q,y-x\rangle \\ & +\frac{1}{2}\langle X(y-x),y-x\rangle \\ & +o(|x-y|^2) \quad \text{as} \quad y\to x \end{array} \right\},$$

Proposition 3. ([6]) *Let $E:\mathbf{R}^n\times S^n\to\mathbf{R}$. $w\in C(\Omega)$ is a viscosity subsolution of $E(Dw,D^2w)=0$ in Ω if and only if*

$$E_*(q,X)\leq 0$$

for all $x\in\Omega$ and $(q,X)\in\bar{J}^{2,+}w(x)$.

Similarly, $w\in C(\Omega)$ is a viscosity supersolution of $E(Dw,D^2w)=0$ in Ω, if and only if,

$$E^*(q,X)\leq 0$$

for all $x\in\Omega$ and $(q,X)\in\bar{J}^{2,-}w(x)$.

Here, $\bar{J}^{2,-}w(x)$ and $\bar{J}^{2,+}w(x)$ are the graph-closure of $J^{2,\pm}w(x)$;

$$\bar{J}^{2,\pm}w(x)=\left\{ (p,X)\in\mathbf{R}^n\times S^n \;\middle|\; \begin{array}{l} \exists(x_k,p_k,X_k)\in\Omega\times R^n\times S^n \\ \text{such that} \quad (p_k,X_k)\in J^{2,\pm}w(x_k) \\ \text{and} \quad \lim_{k\to\infty}(x_k,p_k,x_k)=(x,p,X) \end{array} \right\}.$$

We present representation formulas for F^* and F_*.

Lemma. *For $q=(q_1,\ldots,q_n)\in R^n$ and $X=(X_{ij})\in S^n$,*

$$F^*(q,X)=\max\{-\sum_{i\in I}X_{ii}|\emptyset\neq I\subset I[q]\},$$

$$F_*(q,X)=\min\{-\sum_{i\in I}X_{ii}|\emptyset\neq I\subset I[q]\}.$$

3 Comparison principle

In what follows, we often omit writing the terminology "viscosity".

Because of the discountinuity of F with respect to p-variables, the PDE for supersolutions is different from that for subsolutions in general. Thus, we cannot apply the standard argument to prove the comparison principle for (5). Avoiding this difficulty, we assume a concavity property for solutions in our comparison principle.

We remark that when Ω is convex and $f \equiv 1$, it is known that the power concavity of the minimizer of the variational problem (1), which is proved by S. Sakaguchi in [21]. Hence, throughout this and next sections, we assume $f \equiv 1$. Modifying the proof in [21], we obtain the following.

Theorem 4. *Let Ω be convex and u_p the minimizer of (2). Then, $u_p^{\frac{p-1}{p}}$ is concave in Ω.*

Idea of proof of Theorem 4. We first consider appropriately approximate equations, which is the Euler equation derived from the following variational problems,

$$\inf\{\int_\Omega \sum_{i=1}^{n}(\epsilon|v|^{\frac{2}{p}} + |v_{x_i}|^2)^{\frac{p}{2}}dx - \int_\emptyset v dx | v \in W_0^{1,p}(\Omega)\}.$$

Then, we apply Kennigton's maximum principle to $u_p^{(p-1)/p}$, where u_p is the weak solution of the associated Euler equation.

Since $u_p^{\frac{p-1}{p}}$ converges to $\lim_{p\to\infty} u_p$ uniformly in $\bar{\Omega}$, We can easily get the following.

Corollary 5. *Let Ω be convex, u_p the minimizer of (2), and $\{u_{p_j}\}_{j\in N}$ a subsequence constructed in Proposition 2. Then, the limit function*

$$u = \lim_{j\to\infty} u_{p_j}$$

is concave in Ω.

We shall restrict our comparison principle to the concave functions to characterize the limit function. More precisely, we can show the comparison principle under the local concavity assumption which is a weaker assumption than concavity.

Definition. Let $u \in C(\Omega)$ and $x \in \Omega$. Then, u is called locally concave at $x \in \Omega$ if and only if

$$\exists r > 0 \quad \text{s.t.} \quad u \quad \text{is concave in} \quad B_r(x)$$

Also, u is called locally concave in Ω if and only if u is locally concave at x for all $x \in \Omega$.

Remark. We note that the local concavity is defined even if Ω is not convex. Moreover, if Ω is convex and u is locally concave in Ω, then u is concave. To see this, we assume that there exists $(x, y) \in \Omega$ s.t.

$$S = \{t \in (0,1) | u(tx + (1-t)y) < tu(x) + (1-t)u(y)\} \neq \emptyset$$

Because of the countinuity of u, S is an open set. For $t_0 \in (0,1) \setminus S$, if it exists, using the local concavity of u, we see that there exists $r > 0$ such that $(t_0 - r, t_0 + r) \subset (0,1) \setminus S$, i.e., S is a closed set with a relative topology. Thus, we get $S = (0,1)$. But, this is a contradiction to the local concavity.

We prove the comparison principle under the local concavity restriction.

Theorem 6. *Let $u \in C(\bar{\Omega})$ be a subsolution of (5) and $v \in C(\bar{\Omega})$ be a supersolution of (5). Moreover, we impose an extra assumption; v is locally concave in Ω. Then, we have*

$$\sup_{\partial\Omega}(u - v) = \sup_{\Omega}(u - v).$$

Idea of proof.

Let us suppose $\sup_{\partial\Omega}(u - v) < \sup_{\Omega}(u - v)$.

We construct a strict subsolution $\bar{u}$ and a locally concave strict supersolution $\bar{v}$, which are sufficiently close to u and v, respectively. Thus, we may suppose $\sup_{\partial\Omega}(\bar{u}-\bar{v}) < \sup_{\Omega}(\bar{u}-\bar{v})$. At a maximum point x_0 of $\bar{u} - \bar{v}$, the gradient of $\bar{u}$ and $\bar{v}$ are equal at least formally; $D\bar{u}(x_0) = D\bar{v}(x_0)$. Moreover, we get

$$D^2\bar{u}(x_0) \leq D^2\bar{v}(x_0). \tag{6}$$

Since $\bar{v}$ is a strict supersolution, we have

$$G(D\bar{v}(x_0)) - 1 > 0.$$

Thus, we have $G(D\bar{u}(x_0)) - 1 > 0$. Hence, we get

$$- \sum_{i \in I[D\bar{u}(x_0)]} \bar{u}_{x_i x_i}(x_0) < 0.$$

On the other hand, (6) yields

$$D^2\bar{u}(x_0) \leq 0$$

in view of the local concavity of $\bar{v}$. This is a contradiction.

Remark. We note that the local concavity assumption in this comparison principle may be changed to a weaker one;

$$\forall x \in \Omega \quad \forall (q, X) \in J^{2,-}u(x) \quad \forall i \in I[q] \quad X_{ii} \leq 0.$$

In view of Theorem 6, we verify that the full sequence convergence to a unique Lipschitz continuous function.

Colloraly 7. *Let Ω be convex and u_p the minimizer of (2). Then, there exists a unique function $u \in W_0^{1,\infty}(\Omega)$ such that*

$$u_p \to u \quad \text{as} \quad p \to \infty \quad \text{uniformly in} \quad \bar{\Omega}$$

4 The limit function

In the variational problem (1), the limit function of minimizers is the distance function from $\partial\Omega$. In this section, we show that the limit function of our variational problem (2) also becomes a distance function from $\partial\Omega$.

Definition. We set

$$d_1(x) = \inf_{y\in\partial\Omega}\{\sum_{i=1}^{n}|x_i - y_i|\}$$

We expect d_1 to be the limit function. To check this, we use the previous comparison principle. We first list some properties on d_1.

Proposition 8. *Let Ω be convex. Then, d_1 is concave.*

It is easy to see that if Ω is convex, then d_1 satisfies the (local) concavity assumption in our comparison principle and the following inequality holds

$$-\sum_{i\in I[Du(x)]} u_{x_i x_i}(x) \geq 0$$

in the viscosity sense.

Proposition 9. *d_1 is a solution of*

$$G(Du(x)) - 1 = 0 \quad \text{in} \quad \Omega$$

From Propositions 8 and 9, we can easily prove that d_1 solves (5) in the viscosity sense;

Proposition 10. *Let Ω be convex. Then, d_1 is a viscosity solution of (5)*

Thus, in view of Theorem 6, we get the following.

Theorem 11. *Let Ω be convex and u_p the minimizer of (2). Then, we have*

$$\lim_{p\to\infty} u_p(x) = d_1(x) \quad \text{uniformly in} \quad \bar{\Omega}$$

5 Other norms

We consider other norms equivalent to the standard one in this section.

5.1 In the case of the norm $\|\cdot\|_\infty$.

We consider the norm $\|\cdot\|_\infty$ defined by

$$\|w\|_\infty = \max_{i=1,\ldots,n} \|w_i\|_{L^p(\Omega)} \quad \text{for} \quad w \in L^p(\mathbf{R}^n, \Omega).$$

We can see that the minimizer of the variational problem (1) with this norm satisfies the following inequalities

$$p(p-1) \max_{i=1,\ldots,n} \{-|u_{x_i}(x)|^{p-2} u_{x_i x_i}(x)\} - f(x) \geq 0 \quad \text{in} \quad \Omega$$

$$p(p-1) \min_{i=1,\ldots,n} \{-|u_{x_i}(x)|^{p-2} u_{x_i x_i}(x)\} - f(x) \leq 0 \quad \text{in} \quad \Omega$$

in the viscosity sense. We thus formally get the inequalities, as $p \to \infty$, which are satisfied by the limit function in the viscosity sense.

$$\min\{G(Du(x)) - 1, F_\infty^+(Du(x), D^2u(x))\} \geq 0 \quad \text{in} \quad \Omega,$$

$$\min\{G(Du(x)) - 1, F_\infty^-(Du(x), D^2u(x))\} \leq 0 \quad \text{in} \quad \Omega. \tag{7}$$

$$\text{Here,} \quad F_\infty^+(q, X) = \begin{cases} \max_{i\in I[q]}(-X_{ii}) & \text{provided} \quad I[q] = \{1, \ldots, n\}, \\ \max_{i\in I[q]}(-X_{ii} \vee 0) & \text{otherwise,} \end{cases}$$

$$\text{and} \quad F_\infty^-(q, X) = \begin{cases} \min_{i\in I[q]}(-X_{ii}) & \text{provided} \quad I[q] = \{1, \ldots, n\}, \\ \min_{i\in I[q]}(-X_{ii} \wedge 0) & \text{otherwise,} \end{cases} \tag{8}$$

for $q \in R^n$ and $X = (X_{ij}) \in S^n$.

However, if $u \in C^2(\Omega)$ such that $I[Du(x)] \neq \{1, \ldots, n\}$ for all $x \in \Omega$, then it is a subsolution of (6). Thus, the comparison principle for these inequalities cannot be expected (see [9]).

5.2 In the case of a mixed norm.

We next consider the norm which is the sum of above norms, $\|\cdot\|_1$, $\|\cdot\|$ and $\|\cdot\|_\infty$, i.e., we define the norm of functions $w \in L^p(\Omega, \mathbf{R}^n)$ as follows; for fixed sets $\emptyset \neq I_j \subset \{1, \ldots, n\}$ $(j = 1, 2, 3)$ such that $\cup_{j=1}^3 I_j = \{1, \ldots, n\}$,

$$\|w\|_* = (\|P_1(w)\|_{L^p(\Omega)}^p + \sum_{i\in I_2} \|w_i\|_{L^p(\Omega)}^p + \max_{i\in I_3} \|w_i\|_{L^p(\Omega)}^p)^{\frac{1}{p}},$$

where for $q = (q_1, \ldots, q_n) \in R^n$, $P_1(q) = (\sum_{i\in I_1} q_i^2)^{\frac{1}{2}}$.

Then, we formally get the inequalities, which the limit of minimizers of the corresponding variational problem solves.

$$\min\{G(Du(x)) - 1, F^+(Du(x), D^2u(x))\} \geq 0 \quad \text{in} \quad \Omega,$$

$$\min\{G(Du(x)) - 1, F^-(Du(x), D^2u(x))\} \le 0 \quad \text{in} \quad \Omega.$$

Here, for $q = (q_1, \ldots, q_n) \in \mathbf{R}^n$, $X = (X_{ij}) \in S^n$ and $J \subset \{1, \ldots, n\}$,

$$F^\pm(q, X) = -\sum_{k,l \in I_1} q_k q_l X_{kl} - \sum_{k \in I_2[q]} q_k^2 X_{kk} + f^\pm_{I_3[q]}(q, X),$$

$$I_k[q] = \{i \in I_k | G(q) = |q_i|\},$$

$$f_J^+(q, X) = \begin{cases} 0 & \text{provided} \quad J = \emptyset, \\ \max_{i \in J}(-q_i^2 X_{ii}) & \text{provided} \quad J = I_3, \\ \max_{i \in J}(-q_i^2 X_{ii} \vee 0) & \text{otherwise,} \end{cases}$$

$$f_J^-(q, X) = \begin{cases} 0 & \text{provided} \quad J = \emptyset, \\ \min_{i \in J}(-q_i^2 X_{ii}) & \text{provided} \quad J = I_3, \\ \min_{i \in J}(-q_i^2 X_{ii} \wedge 0) & \text{otherwise.} \end{cases}$$

Indeed, we may verify that the limit of minimizers is a solution of the above inequalities in the viscosity sense.

We can show that the comparison principle holds between a viscosity super- and subsolutions under certain assumptions for I_j. However, we cannot prove that the limit function of minimizers is concave even if Ω is convex (see [9]).

For the reader's convenience we give a list of papers on L^∞-Laplacian, which we do not mention here.

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