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二次元外部ラプラス問題の FEM-CSM 近似解の誤差評価

An error estimate of an FEM-CSM combined method for planar exterior Laplace problems

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講演者の提案してきた外部ラプラス問題の FEM-CSM 結合解法の誤差評価に関しては、これまでいくつかの機会に結果を述べた (Reference 2、Reference 3 など)。吟味不足で、そこに述べた定理は正確では無かった。この報告でこの誤りを正したい。この報告の詳細は、Reference 4 に述べてある。

1. Introduction

Fix a simply connected bounded domain $\mathcal{O}$ in the plane. Assume that the boundary $\mathcal{C}$ of $\mathcal{O}$ is sufficiently smooth. The exterior domain of $\mathcal{C}$ is denoted by Ω. Let D_a be the interior of the disc with radius a having the origin as its center. Fix a function $f \in L^2(\Omega)$ whose support, $\mathrm{supp}(f)$, is bounded. Choose a so large that the open disc D_a may contain the union $\mathcal{O} \cup \mathrm{supp}(f)$ in its interior. The following Poisson equation (E) is employed as a model problem.

$$
\text{(E)} \qquad \begin{cases} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \mathcal{C}, \\ \sup\limits_{|\mathbf{r}|>a} |u| &< \infty. \end{cases}
$$

The intersection of the domain Ω and the disc D_a is said to be the interior domain, denoted by Ω_i: $\Omega_i = \Omega \cap D_a$. Consider the Dirichlet inner product $a(u,v)$ for $u, v \in H^1(\Omega_i)$:

$$
a(u,v) = \int_{\Omega_i} \mathrm{grad}u\, \mathrm{grad}v \, d\Omega.
$$

Let Γ_a be the boundary of the disc D_a. Since the trace $\gamma_a v$ on Γ_a is an element of $H^{1/2}(\Gamma_a)$ for any $v \in H^1(\Omega_i)$, the boundary bilinear form of Steklov type $b(u,v)$ is well defined for $u, v \in H^1(\Omega_i)$. The precise definition of $b(u,v)$ will be given in Section 3. Define a continuous symmetric bilinear form $t(u,v)$ for $u, v \in H^1(\Omega_i)$ through

$$t(u,v) = a(u,v) + b(u,v).$$

Let $F(v)$ be a continuous linear functional on $H^1(\Omega_i)$ defined through the following formula for $v \in H^1(\Omega_i)$:

$$F(v) = \int_{\Omega_i} fv \, d\Omega.$$

A function space V is defined as follows:

$$V = \left\{ v \in H^1(\Omega_i) : v = 0 \quad \text{on} \quad \mathcal{C} \right\}.$$

Using these notations, the following weak formulation problem (Π) is defined.

$$(\Pi) \qquad \begin{cases} t(u,v) = F(v), & v \in V, \\ u \in V. \end{cases}$$

We admit the equivalence between the equation (E) and the problem (Π).

2. A CSM approximate problem for Laplace equation in the exterior region of a disc

Let D_a be the interior of the disc with radius a having the origin as its center, and let Γ_a be the boundary of D_a. Let $\Omega_e = (D_a \cup \Gamma_a)^C$, which is said to be the exterior domain. We use the notation $\mathbf{r} = \mathbf{r}(\theta)$ for the point in the plane corresponding to the complex number $re^{i\theta}$ with $r = |\mathbf{r}|$ where $|\mathbf{r}|$ is the Euclidean norm of $\mathbf{r} \in \boldsymbol{R}^2$. Similarly we use $\mathbf{a} = \mathbf{a}(\theta)$, and $\vec{\rho} = \vec{\rho}(\theta)$, corresponding to $ae^{i\theta}$ with $a = |\mathbf{a}|$, and $\rho e^{i\theta}$ with $\rho = |\vec{\rho}|$, respectively.

Let $f(\mathbf{a}(\theta))$ be a continuous function on the circle Γ_a. The function $f(\mathbf{a}(\theta))$ is a 2π periodic function of θ. Denote the problem to find a harmonic function $u = u(\mathbf{r})$ coinciding with f on Γ_a, which is bounded in Ω_e, by ($\mathrm{E_f}$).

$$(\mathrm{E_f})\qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma_a, \\ \sup_{\Omega_e} |u| < \infty. \end{cases}$$

Fix a positive integer N. Set

$$\theta_1 = \frac{2\pi}{N}.$$

For any $j \in \boldsymbol{Z}$, denote $j\theta_1$ by θ_j. Fix a positive number ρ so as to satisfy $0 < \rho < a$. For the fixed positive integer N, set the points $\vec{\rho}_j, \mathbf{a}_j, 0 \le j \le N-1$, as follows:

$$\mathbf{a}_j = \mathbf{a}(\theta_j), \qquad \vec{\rho}_j = \vec{\rho}(\theta_j) \quad \text{with} \quad 0 < \rho < a.$$

The points $\vec{\rho}_j$, and $\mathbf{a}_j$, are said to be the charge, and the collocation, points, respectively. The arrangement of the set of points of charge points and collocation points introduced as above is called **the equi-distant equally phased arrangement of charge points and collocation points**.

Now we define a CSM approximate problem $(\mathrm{E_f^{(N)}})$ for the continuous problem $(\mathrm{E_f})$ as follows:

$$(\mathrm{E_f^{(N)}})\qquad \begin{cases} u^{(N)}(\mathbf{r}) = \sum_{j=0}^{N-1} q_j G_j(\mathbf{r}) + q_N, \\ u^{(N)}(\mathbf{a}_j) = f(\mathbf{a}_j), \quad 0 \le j \le N-1, \\ \sum_{j=0}^{N-1} q_j = 0, \end{cases}$$

where

$$G_j(\mathbf{r}) = E(\mathbf{r} - \vec{\rho}_j) - E(\mathbf{r}), \quad E(\mathbf{r}) = -\frac{1}{2\pi}\log r.$$

Let

$$N(\gamma) = \frac{\log 2}{-\log\gamma} \quad \text{with} \quad \gamma = \frac{\rho}{a}.$$

<u>**Theorem B**</u> (Cf. Katsurada-Okamoto[1].) *Fix a positive number $b, 0 < b < a$. Let $u(\mathbf{r})$ be harmonic in a domain containing the exterior domain of the disc with radius b having the origin as its center. Suppose that $N \ge N(\gamma)$. Let $u^{(N)}(\mathbf{r})$ be the solution of the problem $(\mathrm{E_f^{(N)}})$ with the data $f(\mathbf{a}(\theta)) = u(\mathbf{a}(\theta))$. Then there exist constants $B > 0$ and $\beta \in (0,1)$,*

dependent on parameters a, b and ρ, independent of u (with the property above) and N, such that the following two estimates are valid:

$$\max_{\mathbf{r}\in\overline{\Omega}_e} \left|u(\mathbf{r}) - u^{(N)}(\mathbf{r})\right| \leq B \cdot \beta^N \cdot \max_{|\mathbf{r}|=b} |u(\mathbf{r})|,$$

$$\max_{\mathbf{r}\in\overline{\Omega}_e} \left|\operatorname{grad} u(\mathbf{r}) - \operatorname{grad} u^{(N)}(\mathbf{r})\right|_{\mathbf{R}^2} \leq B \cdot \beta^N \cdot \max_{|\mathbf{r}|=b} |u(\mathbf{r})|.$$

3. Boundary bilinear form of Steklov type for exterior Laplace problems and its CSM-approximation form

Let $f(\theta)$ be a complex valued continuous 2π-periodic function of θ. For $n \in \boldsymbol{Z}$, a continuous Fourier coefficient f_n of the function $f(\theta)$ is defined through

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta.$$

For functions $u(\mathbf{a}(\theta))$ and $v(\mathbf{a}(\theta))$ of $H^{1/2}(\Gamma_a)$, let us introduce the boundary bilinear form of Steklov type for exterior Laplace problem through the following formula (1):

$$b(u,v) = 2\pi \sum_{n=-\infty}^{\infty} |n| f_n \overline{g_n}, \tag{1}$$

where f_n, and g_n, are continuous Fourier coefficients of $u(\mathbf{a}(\theta))$, and $v(\mathbf{a}(\theta))$, respectively.

The CSM approximate form for $b(u,v)$, which is denoted by $b^{(N)}(u,v)$, is represented through the following formula (2):

$$b^{(N)}(u,v) = -\frac{2\pi a}{N}\sum_{j=0}^{N-1} \frac{\partial u^{(N)}(\mathbf{a}_j)}{\partial r} v^{(N)}(\mathbf{a}_j), \tag{2}$$

where $u^{(N)}(\mathbf{r})$, and $v^{(N)}(\mathbf{r})$, are CSM-approximate solutions of the problem $(\mathrm{E}_{\mathrm{f}}^{(\mathrm{N})})$ with $f = u(\mathbf{a}(\theta))$, and $f = v(\mathbf{a}(\theta))$, respectively.

4. An FEM-CSM combined method for exterior Laplace problems

We say that the function $v(\mathbf{a}(\theta))$ is an equi-distant piecewise linear continuous 2π-periodic function with N nodal points if it is expressed in the following form:

$$v(\mathbf{a}(\theta)) = \frac{\theta_{j+1} - \theta}{\theta_1} v(\mathbf{a}(\theta_j)) + \frac{\theta - \theta_j}{\theta_1} v(\mathbf{a}(\theta_{j+1})),$$

$$\theta_j \le \theta \le \theta_{j+1}, \quad 0 \le j \le N-1.$$

And we use the notation, $a(v) = a(v,v)^{1/2}$, for $v \in V$.

A family of finite dimensional subspaces of V, $\{V_N : N = N_0, N_0+1, \cdots\}$ is supposed to have the following properties:

$(\mathrm{V_N}-1)$ $$V_N \subset C(\overline{\Omega_i}).$$

$(\mathrm{V_N}-2)$ $$\begin{cases} \textit{For any } v \in V_N,\ v(\mathbf{a}(\theta)) \textit{ is an equi-distant} \\ \textit{piecewise linear continuous } 2\pi\textit{-periodic} \\ \textit{function with } N \textit{ nodal points.} \end{cases}$$

$(\mathrm{V_N}-3)$ $$\begin{cases} \textit{There is a constant } C \textit{ independent of } N \\ \textit{such that for any } v \in V \cap H^2(\Omega_i) \\ \displaystyle\min_{v_N \in V_N} a(v - v_N) \le \frac{C}{N} ||v||_{H^2(\Omega_i)}. \end{cases}$$

For $u, v \in H^1(\Omega_i) \cap C(\overline{\Omega_i})$, we define bilinear forms $t^{(N)}(u,v)$ as follows.

$$t^{(N)}(u,v) = a(u,v) + b^{(N)}(u,v).$$

Now our approximate problem $(\Pi^{(N)})$ is stated as follows.

$(\Pi^{(N)})$. $$\begin{cases} t^{(N)}(u_N, v) = F(v), \quad v \in V_N, \\ u_N \in V_N. \end{cases}$$

Theorem 1 *Let u be the solution of the problem (Π), and let u_N be the solution of the problem $(\Pi^{(N)})$. Suppose that* $\mathrm{supp}(f)$ *is contained in a disc D_b with the radius $b(< a)$ having the origin as its center. Let the function $D(\xi)$ of $\xi \in (0,1)$ be defined through*

$$D(\xi) = \frac{4\xi}{(1-\xi)^3}.$$

Let $N \geq N(\gamma)$. Then there is a constant C such that

$$||u - u_N||_{H^1(\Omega_i)} \leq C\left\{B\beta^N + \frac{1 + D(\frac{b}{a})}{N}\right\}||f||_{L^2(\Omega_i)},$$

where the constants B and $\beta \in (0,1)$ are described in Theorem B for the set of parameters $\{a, b, \rho\}$. In the above, the constant C is independent of the inhomogeneous data f and N.

The proof of Theorem 1 is written in [4].

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