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Normal forms and cubic nonlinear Schrödinger equations in one space dimension

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1 Introduction and results

In this note we consider the Cauchy problem for the nonlinear Schrödinger equation in one space dimension

$$iu_t + \frac{1}{2}u_{xx} = F(u, \bar{u}, u_x, \bar{u}_x), \tag{1.1}$$

$$u(0, x) = u_0(x). \tag{1.2}$$

Here u is a complex-valued function of $(t, x) \in \mathbf{R} \times \mathbf{R}$ and F is a smooth function on a neighborhood of the origin such that for some integer $p \geq 2$

$$F(u, \bar{u}, q, \bar{q}) = O\left(|u|^p + |q|^p\right) \quad \text{near the origin.} \tag{1.3}$$

We are interested in finding some nonlinearities F such that the Cauchy problem (1.1)-(1.2) has a unique global solution which is asymptotically free.

It is known that if F satisfies

$$\mathrm{Re}\frac{\partial F}{\partial q}(u, \bar{u}, q, \bar{q}) \equiv 0, \tag{1.4}$$

then the usual energy method yields the local existence. When the nonlinearity F does not neccesarily satisfy (1.4) the local existence has also been established this decade (see [7] and [9]). Concerning the global existence of solutions, Klainerman-Ponce [10] and Shatah [13] showed that if F satisfies (1.3) with $p \geq 4$ and (1.4), then (1.1)-(1.2) possesses a unique global solution provided that the intial data u_0 is small enough in a certain Sobolev space. If the nonlinearity is of lower degree (*i.e.* quadratic or cubic), it seems difficult to prove the global existence in general. In spite of this, there are not a few papers on the global existence when the nonlinearity is cubic or quadratic. In particular, in the case where the nonlinearity F is cubic and *gauge invariant*, that is, F satisfies

$$F(\omega u, \overline{\omega u}, \omega q, \overline{\omega q}) = \omega F(u, \bar{u}, q, \bar{q}) \tag{1.5}$$

for any $\omega \in \mathbf{C}(|\omega| = 1), u, q \in \mathbf{C}$, much has been studied. For $F = \lambda|u|^2u$ or $F = i\lambda\partial_x(|u|^2u)$ with some $\lambda \in \mathbf{R}\setminus\{0\}$, the global existence is well known. Furthermore, for these nonlinearities, the asymptotic behavior of solutions is studied and the existence of modified scattering states is proved by Hayashi and Naumkin [4],[5]. They also established the asymptotic formula of a time-global solution for large time. Katayama and Tsutsumi [8] showed that if F satisfies (1.5) and "null gauge condition of order 3" (a typical example which satisfies these conditions is $F = \partial_x(|u|^2)(\lambda u + \mu u_x)$ with $\lambda, \mu \in \mathbf{C}$) then (1.1)-(1.2) has a unique global solution for small initial data u_0 and the usual scattering state exists. Recently, Hayashi and Naumkin [6] considered nonlinear Schrödinger equations with a derivative cubic nonlinearity which does not satisfy (1.5) and proved the global existence of solutions for small initial data and the existence of usual or modified scattering states. However, it still remains open what kind of cubic nonlinearities assures the global existence of solutions with a free profile in large time for small initial data. In the present note, we consider the global existence of a solution to the Cauchy problem (1.1)-(1.2) in the usual Sobolev spaces for small initial data and the existence of scattering states in a usual sense for $F = cuu_x^2$ or $F = c\bar{u}\bar{u}_x^2$ with $c \in \mathbf{C}$. To treat these critical cubic nonlinearities we use the techniques which transform them into harmless ones. These were developed by Shatah[14], Cohn[1],[2] and Ozawa[12] for quadratic nonlinearity. While they discussed quadratic nonlinear Schrödinger equations in [1],[2] and [12] (quadratic nonlinear Klein-Gordon equations in [14]), a class of cubic nonlinear Schrödinger equations will be treated in the present note. So, it should be emphasized that the transformation in the present paper will be more complicated than those for quadratic nonlinearities.

Before stating our results we give several notations.

Notation.

Let $[a]$ denote the largest integer less than or equal to a. Let $\hat{f}$ and $\mathcal{F}f$ denote the Fourier transform of f with respect to the space variable:

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbf{R}^n} f(x)e^{-ix\cdot\xi}dx.$$

For $1 \le p \le \infty$ and nonnegative integers m, we denote by $L^p = L^p(\mathbf{R})$ and $W^{m,p} = W^{m,p}(\mathbf{R})$ the standard Lebesgue space and Sobolev space, respectively. We also use the notation $H^m := W^{m,2}$ for the L^2-type Sobolev space. Let $C^k(I;B)$ denote the space of functions continuous with their derivatives up to k from a time interval $I \subset \mathbf{R}$ to a Banach space B, and let $C(I;B) := C^0(I;B)$. Let $U(t) = e^{\frac{it}{2}\partial_x^2}$ be the evolution operator associated with the free Schrödinger equation.

Our main results are the following. The first theorem gives a cubic nonlinear Schrödinger equation which is convertible into the free Schrödinger equation.

Theorem 1. *Let m be an integer with $m \geq 1$ and let $F = f(u)u_x^2$ where $f(u)$ is an entire function and satisfies $f(u) = O(u^k)$ at the origin. We put*

$$\varphi(u) = \int_0^u e^{-\int_0^z f(w)dw} dz.$$

Then there exist $\varepsilon_0 > 0$ such that for any $u_0 \in H^m$ with $\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon_0$, the Cauchy problem (1.1)-(1.2) has a unique global solution $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$. Moreover, the solution u is given explicitly by $u(t) = \varphi^{-1}\big(U(t)\varphi(u_0)\big)$. If in addition $u_0 \in L^1$, then

$$\|u(t)\|_{L^\infty} = O(|t|^{-\frac{1}{2}}) \quad \text{as } t \to \pm\infty \tag{1.6}$$

and there exists a unique $\phi \in H^m \cap L^1$ such that

$$\|u(t) - U(t)\phi\|_{H^m} = O(|t|^{-\frac{k+1}{2}}) \quad \text{as } t \to \pm\infty. \tag{1.7}$$

Furthermore, ϕ is given explicitly by $\phi = \varphi(u_0)$.

Remarks. (i) The assumption $\|\mathcal{F}\varphi(u_0)\|_{L^1} < \varepsilon_0$ is fulfilled if $\|u_0\|_{H^1}$ is sufficiently small.
(ii) For ε_0 in Theorem 1, we can take the radius of convergence of the Taylor expansion at the origin of the inverse function of φ.
(iii) The results in [12] covers the result of Theorem 1 if $f(u)$ is a constant. If $f(u) = O(u)$ at the origin, then Theorem 1 gives a cubic nonlinearity F which assures the global existence of solutions with a free profile to (1.1)-(1.2) .

We next state the theorem concerning a cubic nonlinear Schrödinger equation to which the normal form argument by Shatah[14] is applicable.

Theorem 2. *Let m be an integer with $m \geq 4$ and let $F = c\bar{u}\bar{u}_x^2$ where c is a complex constant. Then there exists $\varepsilon_0 > 0$ such that for any $u_0 \in H^m \cap W^{[(m+5)/2],1}$ with $\max\{\|u_0\|_{H^m}, \|u_0\|_{W^{[(m+5)/2],1}}\} < \varepsilon_0$ the Cauchy problem (1.1)-(1.2) has a unique global solution u satisfying*

$$u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2}),$$

$$\text{(1.8)} \qquad \|u(t)\|_{H^m} = O(1), \quad \|u(t)\|_{W^{[(m+1)/2],\infty}} = O(|t|^{-\frac{1}{2}}) \quad as\ t \to \pm\infty.$$

Moreover, there exist a unique $\phi_+ \in H^m$ and a unique $\phi_- \in H^m$ such that

$$\text{(1.9)} \qquad \|u(t) - U(t)\phi_+\|_{H^m} = O(|t|^{-1}) \quad as \quad t \to +\infty,$$

$$\|u(t) - U(t)\phi_-\|_{H^m} = O(|t|^{-1}) \quad as \quad t \to -\infty.$$

Remark. Recently, Naumkin[11] proved the global exitence of a solution to (1.1)-(1.2) with a fairly wide class of cubic nonlinearities F including two nonlinearities considered in Theorems 1 and 2. The results in [11], however, do not cover the results of Theorems 1 and 2 in the present note since the former require that the initial data should be small in a *weighted* Sobolev space while the latter do not.

2 Outline of the proof of Theorem 1

When the nonlinearity F is of low degree, it does not seem that we can prove a global existence result directly from the original equation. So we make use of a transformation which converts a solution of the original nonlinear Schrödinger equation into one of the linear Schrödinger equation. The first three Lemmas are devoted to prove that the function φ given in Theorem 1 is the helpful transformation.

The Lemma 2.1(a) and Lemma 2.2 show that φ is regular as a transformation on H^m.

Lemma 2.1 (a) *φ is an entire function on the whole complex plane.*
(b) *There exist a constant $\varepsilon > 0$ and a holomorphic function $\psi : B_\varepsilon \to \varphi^{-1}(B_\varepsilon)$ such that $\varphi \circ \psi = id_{B_\varepsilon}, \psi \circ \varphi = id_{\varphi^{-1}(B_\varepsilon)}$ and $\varphi^{-1}(B_\varepsilon)$ is bounded.*

Lemma 2.2 *Let m be an integer with $m \geq 1$. For any $u_0 \in H^m$, $\varphi(u_0) \in H^m$.*

The following Lemma shows that if u solves the original nonlinear Schrödinger equation (1.1)-(1.2) , then $v = \varphi(u)$ solves the homogeneous linear Schrödinger equation with $v(0) = \varphi(u_0)$.

Lemma 2.3 *Let m be an integer with $m \geq 1$. Let $u_0 \in H^m$ and let $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$ satisfy (1.1)-(1.2) with $F = f(u)u_x^2$. Then $\varphi(u(\cdot)) \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-}$ and*

$$\varphi(u(t)) = U(t)\varphi(u_0). \tag{2.1}$$

Next, we have to prove that the transformed function $v(t) = \varphi(u(t)) = U(t)\varphi(u_0)$ gives the solution $u(t)$ to the original nonlinear Schrödinger equation. Lemma 2.1(b) shows that this is true if $||U(t)\varphi(u_0)||_{L^\infty} < \varepsilon$.

From Lemma 2.1(b), we have the expansion

$$\varphi^{-1}(z) = \sum_{j=0}^{\infty} a_j z^j$$

with the radius of convergence larger than or equal to ε. We easily see that $a_0 = 0, a_1 = 1, a_i = 0$ $(2 \leq i \leq k+1)$ and $a_{k+2} \neq 0$. We put $\varepsilon_0 = \sqrt{2\pi}\varepsilon$. Then we have

$$\sup_{t\in\mathbf{R}} ||U(t)\varphi(u_0)||_{L^\infty} = \sup_{t\in\mathbf{R}} ||\mathcal{F}^{-1}e^{-it|\cdot|^2}\mathcal{F}\varphi(u_0)||_{L^\infty} \leq \frac{1}{\sqrt{2\pi}}||\mathcal{F}\varphi(u_0)||_{L^1} < \varepsilon.$$

Therefore the series

$$U(t)\varphi(u_0) + \sum_{j=k+2}^{\infty} a_j (U(t)\varphi(u_0))^j.$$

converges absolutely in L^2. This proves that $u(t) = \varphi^{-1}\big(U(t)\varphi(u_0)\big)$ makes sense and is in $C(\mathbf{R}; L^2)$. Some more calculations show that $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$, and u is a unique solution to (1.1)-(1.2) .

The decay estimate of the solution and the existence of a free profile in L^2 are shown by using the standard L^∞-decay estimates of the fundamental solution and following two inequalities

$$\begin{aligned}
||u(t)||_{L^\infty} &\leq ||U(t)\phi||_{L^\infty} + \sum_{j=k+2}^{\infty} |a_j|||(U(t)\phi)^j||_{L^\infty} \\
&\leq \frac{||\phi||_{L^1}}{(2\pi|t|)^{1/2}} \left(1 + \sum_{j=k+2}^{\infty} |a_j| \Big(\frac{||\mathcal{F}\phi||_{L^1}}{\sqrt{2\pi}}\Big)^{j-1}\right),
\end{aligned}$$

$$\begin{aligned}
||u(t) - U(t)\phi||_{L^2} &\leq \sum_{j=k+2}^{\infty} |a_j|||(U(t)\phi)^j||_{L^2} \\
&\leq \frac{||\phi||_{L^1}^{k+1}||\phi||_{L^2}}{(2\pi|t|)^{\frac{k+1}{2}}} \sum_{j=k+2}^{\infty} |a_j| \Big(\frac{||\mathcal{F}\phi||_{L^1}}{\sqrt{2\pi}}\Big)^{j-k-2}.
\end{aligned}$$

The existence of a free profile in H^m is proved after a somewhat complicated calculation.

3 Outline of the proof of Theorem 2

The crucial part of the proof of Theorem 2 is to establish a priori estimates of the solution to (1.1)-(1.2) . The global existence result is obtained by combining a local existence theory and a priori estimates. Since the nonlinearity $F = c\bar{u}\bar{u}_x^2$ satisfies (1.4), the local existence is an immediate consequence of the usual energy method. But we cannot derive sufficient time decay estimates to prove the global existence directly from the original equation since F is cubic. In order to obtain good a priori estimates, we use the argument of normal forms introduced by Shatah (see [1],[2],[14]).

Following Shatah [14], we introduce a new unknown function v:

$$v = u + K(\bar{u}, \bar{u}, \bar{u}), \tag{3.1}$$

where K is thought of as a distribution and the representaion of the cubic term is given by

$$K(f, g, h)(x) = \int_{\mathbf{R}^3} K(x-y, x-z, x-w) f(y) g(z) h(w)\, dy dz dw. \tag{3.2}$$

After some calculations, we obtain

$$K(f, g, h)(x) = (2\pi)^{3/2} \int_{\mathbf{R}^3} \widehat{K}(p, q, r) \hat{f}(p) \hat{g}(q) \hat{h}(r) e^{ix(p+q+r)}\, dp dq dr, \tag{3.3}$$

$$\begin{aligned} i\partial_t v + \frac{1}{2}\partial_x^2 v = c\bar{u}\bar{u}_x^2 &+ \left[(\partial_y^2 + \partial_z^2 + \partial_w^2 + \partial_y\partial_z + \partial_z\partial_w + \partial_w\partial_y)K\right](\bar{u}, \bar{u}, \bar{u}) \\ &- K(\bar{c}uu_x^2, \bar{u}, \bar{u}) - K(\bar{u}, \bar{c}uu_x^2, \bar{u}) - K(\bar{u}, \bar{u}, \bar{c}uu_x^2). \end{aligned} \tag{3.4}$$

All cubic terms in (3.4) cancel out, when we take K as follows:

$$\widehat{K}(p, q, r) = -\frac{c}{3}\frac{pq + qr + rp}{p^2 + q^2 + r^2 + pq + qr + rp}.$$

Then the function v defined by the transformation (3.1) satisfies

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \frac{|c|^2}{3}\Big(\Omega(uu_x^2, \bar{u}, \bar{u}) + \Omega(\bar{u}, uu_x^2, \bar{u}) + \Omega(\bar{u}, \bar{u}, uu_x^2)\Big), \tag{3.5}$$

where we put $\Omega = -\frac{3}{c}K$. We remark that the nonlinear term in the right hand side of (3.5) is of degree five. This new equation (3.5) with a nonlinearity of higher degree is called a *normal form*.

We have to prove that the transformation (3.1) is regular in the space where we consider the Cauchy problem in order to establish sufficient a priori estimates to prove the global existence result. The following Lemma on Fourier multipliers due to Coifman and Meyer ([3]) plays an important role for this purpose.

Lemma 3.1 *Let*

$$\Lambda(f,g,h)(x)=\int_{\mathbf{R}^3}\lambda(p,q,r)\widehat{f}(p)\widehat{g}(q)\widehat{h}(r)e^{ix(p+q+r)}dpdqdr,$$

and let

$$|\partial_p^j\partial_q^k\partial_r^l\lambda(p,q,r)|\le C_{j,k,l}(|p|+|q|+|r|)^{-(j+k+l)} \tag{3.6}$$

for all nonnegative integers j,k,l such that $0\le j+k+l\le 1$. Then

$$\|\Lambda(f,g,h)\|_{L^p}\le C_{p_1,p_2,p_3}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}}$$

where $\frac{1}{p}=\sum_{j=1}^{3}\frac{1}{p_j}$, $1<p_j\le\infty$ $(j=1,2)$ *and* $1<p_3<\infty$.

The following estimate for $K(\cdot,\cdot,\cdot)$ defined by (3.2) follows immediately from (3.3) and Lemma 3.1.

Lemma 3.2 *Let p,p_j $(j=1,2,3)$ satisfy* $\frac{1}{p}=\sum_{j=1}^{3}\frac{1}{p_j}$, $1<p_j\le\infty$ $(j=1,2)$ *and* $1<p_3<\infty$. *If $\widehat{K}$ is a Coifman-Meyer kernel (that is, $\lambda=\widehat{K}$ satisfies (3.6)), then*

$$\|K(f,g,h)\|_{L^p}\le C_{p_1,p_2,p_3}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}}.$$

The following lemma gives several formulas which are useful to simplify the representation of nonlinear terms of (3.5).

Lemma 3.3 (a) $\Omega(f,g,h)=\Omega(f,h,g)=\Omega(g,f,h)$.
(b) $\partial_x\Omega(f,g,g)=M(f,g,g_x)$, *where $\widehat{M}$ is a Coifman-Meyer kernel.*

From Lemma 3.3(a), (3.5) is rewritten as follows:

$$i\partial_t v+\frac{1}{2}\partial_x^2 v=|c|^2\Omega(uu_x^2,\bar{u},\bar{u}). \tag{3.7}$$

We will derive a priori estimates of u via this equation.

The inequalities in the following lemma are needed to estimate nonlinear terms of (3.7) when we derive a priori estimates.

Lemma 3.4 *The estimates* (a) *and* (b) *hold for $m\ge 1$, and* (c) *holds for $m\ge 0$.*
(a) $\|\Omega(f,g,g)\|_{H^m}\le C\Big(\|f\|_{H^{m-1}}\|g\|^2_{W^{[(m+1)/2],\infty}}+\|f\|_{W^{[(m-1)/2],\infty}}\|g\|_{H^m}\|g\|_{W^{[(m+1)/2],\infty}}\Big)$,
(b) $\|\Omega(f,g,g)\|_{W^{m,1}}\le C\Big(\|f\|_{H^{m-1}}\|g\|_{H^m}\|g\|_{W^{[(m-1)/3]+1,\infty}}+\|f\|_{W^{[(m-1)/3],\infty}}\|g\|^2_{H^m}\Big)$,
(c) $\|\Omega(f,f,f)\|_{H^m}\le C\|f\|_{H^m}\|f\|^2_{W^{[m/2],\infty}}$,

The proof of Lemma 3.4(a) and (b) is based on the result in Lemma 3.3(b). A priori energy and decay estimates of u will be derived via (3.1) and (3.7) by using Lemma 3.4(a) and (b) with $f = uu_x^2$ and $g = \bar{u}$.

For $m \geq 4$ and $T > 0$, we define

$$\|u\|_{m,T} = \sup_{t\in[0,T]} \Big(\|u(t)\|_{H^m} + (1+t)^{\frac{1}{2}}\|u(t)\|_{W^{[\frac{m+1}{2}],\infty}}\Big).$$

Lemma 3.5 (a priori energy estimate) *Let $m \geq 4$ and let $u_0 \in H^m$. Assume that the initial value problem (1.1)-(1.2) with $F = c\bar{u}\bar{u}_x^2$ has a solution $u \in C([0,T];H^m) \cap C^1([0,T];H^{m-2})$. Then the following inequality holds for any $t \in [0,T]$:*

$$\|u(t)\|_{H^m} \leq C\Big(\|u_0\|_{H^m} + \|u_0\|_{H^m}^3 + \|u\|_{m,T}^3 + \|u\|_{m,T}^5\Big),$$

where C is independent of T and u_0.

Lemma 3.6 (a priori decay estimate) *Let $m \geq 4$ and let $u_0 \in H^m \cap W^{[(m+5)/2],1}$. Assume that the Cauchy problem (1.1)-(1.2) with $F = c\bar{u}\bar{u}_x^2$ has a solution $u \in C([0,T];H^m)$. Then the following inequality holds for any $t \in [0,T]$:*

$$(1+t)^{1/2}\|u(t)\|_{W^{[(m+1)/2],\infty}} \leq C\Big(\|u_0\|_{W^{[(m+1)/2]+2,1}} + \|u_0\|_{H^m}^3 + \|u\|_{m,T}^3 + \|u\|_{m,T}^5\Big),$$

where C is independent of T and u_0.

Combining the local existence of solutions and the a priori estimates, we obtain the global existence of solutions to (1.1)-(1.2) with $F = c\bar{u}\bar{u}_x^2$ and the existence of a free profile by the standard argument.

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