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The Navier-Stokes exterior problem in the Lorentz spaces

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Introduction.

Most of the ingredients of this article is based on a joint work with Yoshihiro Shibata of Waseda University.

Let Ω be an exterior domain in $\mathbb{R}^n$ for $n \geq 3$ with smooth boundary $\partial\Omega$. We are concerned with the stationary Navier-Stokes equation with the Dirichlet boundary condition in Ω as follows:

$$
\begin{aligned}
-\Delta_x w(x) + (w\cdot\nabla)\,w(x) + \nabla\pi(x) &= f(x) \text{ in } \Omega, && (0.1)\\
\nabla\cdot w(x) &= 0 \quad\ \text{ in } \Omega, && (0.2)\\
w(x) &= 0 \quad\ \text{ on } \partial\Omega, && (0.3)\\
w(x) &\to u_\infty \ \text{ as } |x|\to\infty, && (0.4)
\end{aligned}
$$

where u_∞ is a small constant vector. We are particularly interested in the behavior of the solution $w(x)$ as $u_\infty \to 0$ with fixed $f(x)$.

We are also concerned, either in the case $n = 3$ or in the case $n \geq 4$ and $u_\infty = 0$, with the non-stationary Navier-Stokes equation with the Dirichlet boundary condition in Ω on the whole time interval $\mathbb{R}^n$ as follows:

$$
\begin{aligned}
\frac{\partial v}{\partial t}(t,x) - \Delta_x v(x) + (v\cdot\nabla)\,v(x) + \nabla p(x) &= f(t,x) \text{ in } \mathbb{R}\times\Omega, && (0.5)\\
\nabla\cdot v(t,x) &= 0 \qquad \text{ in } \mathbb{R}\times\Omega, && (0.6)\\
v(t,x) &= 0 \qquad \text{ on } \mathbb{R}\times\partial\Omega, && (0.7)\\
v(t,x) &\to u_\infty \quad \text{ as } |x|\to\infty, && (0.8)
\end{aligned}
$$

and the Cauchy problem of the above non-stationary problem for the above system as follows:

$$\frac{\partial v}{\partial t}(t,x) - \Delta_x v(x) + (v\cdot\nabla)\,v(x) + \nabla p(x) = f(t,x) \text{ in } (0,\infty)\times\Omega, \qquad (0.9)$$

$$\begin{aligned} \nabla\cdot v(t,x) &= 0 && \text{in } (0,\infty)\times\Omega, && (0.10)\\ v(t,x) &= 0 && \text{on } (0,\infty)\times\partial\Omega, && (0.11)\\ v(t,x) &\to u_\infty && \text{as } |x|\to\infty, && (0.12)\\ v(0,x) &= v_0(x) && \text{on } \Omega. && (0.13) \end{aligned}$$

Here we assume that the external force $f(t,x)$ depends on the time variable t and does not decay as $t\to\infty$ in general. For example, we consider time-periodic functions or almost periodic functions as $f(t,x)$. If the external force $f(t,x)$ is independent of t, the problem (0.5)–(0.8) reduces to the stationary problem (0.1)–(0.4) above, and the problem (0.9)–(0.13) with $a(x)$ near $u(x)$ above concerns the stability of the stationary solution $u(x)$.

We first review previous results on (0.1)–(0.4). Shibata [37] considered the stationary problem (0.1)–(0.4) with small $u_\infty\neq 0$ in the case $n=3$, and showed that, if $f(x)$ is small enough in an appropriate function space, then there uniquely exists a small solution $w(x)\in L^3(\Omega)$ of the problem above. However, if the vector u_∞ tends to 0, the assumption on the smallness of the external force $f(x)$ becomes stronger, and hence one cannot tell the asymptotic behavior of the solution $w(x)$ of (0.1)–(0.4) in $L^3(\Omega)$ as $u_\infty\to 0$ for $f(x)\not\equiv 0$.

Recently, Galdi and Rabier [12] considered, among others, the same problem for $u_\infty\neq 0$ by using anisotropic spaces of Sobolev type. However, the vector u_∞ is fixed in their argument, and hence one cannot derive the asymptotic behavior as $u_\infty\to 0$.

The difficulty above naturally arises from the fact that, in the case $n=3$, the solution $w(x)$ of the problem (0.1)–(0.4) with $u_\infty=0$, even if it is small enough, does not belong to the space $L^3(\Omega)$ in general, contrary to the case $u_\infty\neq 0$. In fact, Borchers and Miyakawa [6, Theorem 2.4], Kozono and Sohr [22, Theorem C] and Kozono, Sohr and Yamazaki [23, Theorem 2, (1)] showed that the solution $w(x)$ of (0.1)–(0.4) belongs to $L^3(\Omega)$ only in very restricted situations. More detailed references are found in [25, 26]. It follows that one cannot find the limit of the solution $w(x)$ in the space $L^3(\Omega)$ in general as $u_\infty\to 0$.

On the other hand, in the case $u_\infty=0$, the problem (0.1)–(0.4) is considered by many authors. Novotny and Padula [35], Galdi and Simader [13] and Borchers and Miyakawa [6] proved the following: If the external force enjoys the condition $|f(x)|\leq c|x|^{-m}$ with sufficiently small c for $m\in[3,n]$, then there exists a unique solution $w(x)$ of (0.1)–(0.4) such that $|w(x)| \leq C|x|^{2-m}$ and that $|\nabla w(x)| \leq C|x|^{1-m}$. Estimates of this type involving higher order terms are recently obtained by Šverák and Tsai [42]. In other words, for three-dimensional exterior domains, they proved the unique existence of physically reasonable solutions in the sense of Finn [7], and obtained sharp estimates of the solutions and their derivatives. Furthermore, Nazarov and Pileckas [33, 34] obtained the asymptotic expansion of the solution, the principal term in which is homogeneous of order -1. Hence the solution $w(x)$ does not belong to the standard L^3 space $L^3(\Omega)$ in general, but belongs to the weak-L^3 space $L^{3,\infty}(\Omega)$, which is slightly larger than $L^3(\Omega)$. Similarly, the derivative $\nabla w(x)$ belongs to $L^{3/2,\infty}(\Omega)$ but not to $L^{3/2}(\Omega)$ in general.

Later on, by introducing the weak-L^p spaces and modifying the L^p-theory of Kozono and Sohr [21] for $n \geq 4$ accordingly, Kozono and Yamazaki [25] showed that, for $f(x)$ of the form $\nabla \cdot F(x)$ such that $F(x)$ is sufficiently small in $L^{n/2,\infty}(\Omega)$, the unique existence of the solution $w(x)$ of the problem (0.1)–(0.4) such that $w(x) \in L^{n,\infty}(\Omega)$ and that $\nabla w(x) \in L^{n/2,\infty}(\Omega)$ with norms bounded by a definite constant. The assumption on the external force in this result generalizes the assumption of [35, 13, 6].

Hence this result implies that, in the case $n = 3$, the class $L^{3,\infty}$ is a natural generalization of the class of physically reasonable solutions satisfying $w(x) \to 0$ as $|x| \to \infty$. However, the argument employed in [35, 13, 6, 25] is essentially different from that of [37] and hence the relationship between these works still remains unclear. Hence it is very difficult to obtain the pointwise estimate of the difference of the solution for small u_∞ and the solution for $u_\infty = 0$. This difficulty is partly due to the fact that, in the case $u_\infty = \lambda a$ with some vector $a \neq 0$, the decay rate of the fundamental solution of the stationary Oseen equation remain unchanged when λ tends to $+0$.

In order to consider this problem, we give a unified approach for the case $u_\infty \neq 0$ and for the case $u_\infty = 0$ based on functional analysis and the Lorentz spaces in this paper. Then we show that, if $|u_\infty|$ is sufficiently small and $F(x)$ is sufficiently small in $L^{n/2,\infty}(\Omega)$, then there uniquely exists a solution $w(x)$ of (0.1)–(0.4) such that $w(x)$ is small in $L^{n,\infty}(\Omega)$ and that $\nabla w(x)$ and $\pi(x) - c$ are small in $L^{n/2,\infty}(\Omega)$ with some constant c. The smallness imposed on $F(x)$ is uniform as $u_\infty \to 0$. Namely, we generalize the results of [35, 13, 6, 25] to the case $u_\infty \neq 0$, and at the same time we generalize the result of [37] to general dimension $n \geq 3$ and relax the conditions on the smallness and the regularity of $f(x)$. As a consequence, we show that the solution $w(x)$ converges to the solution given by [25] in the weak-$*$ topology of the space $L^{n,\infty}(\Omega)$, and $\nabla w(x)$ and $\pi(x)$ converges in the same way in the weak-$*$ topology of the space $L^{n/2,\infty}(\Omega)$ as $u_\infty \to 0$.

In this paper we assume that the external force $f(x)$ is of potential type;

namely, $f(x)$ is represented as

$$f(x) = \left(f_j(x)\right)_{j=1,\ldots,n} = \nabla \cdot F(x) = \left(\sum_{k=1}^{n} \frac{\partial F_{jk}}{\partial x_k}(x)\right)_{j=1,\ldots,n}$$

with a tensor function $\left(F_{jk}(x)\right)_{j,k=1,\ldots,n}$, and we put $v(x) = w(x) - u_\infty$. As we shall see in Remark 1.3, many external forces enjoy this assumption. Then the system (0.1)–(0.4) is transformed into the following system for $v(x)$;

$$-\Delta_x v(x) + (u_\infty \cdot \nabla)\, v(x) + (v(x) \cdot \nabla)\, v(x) + \nabla \pi(x) = \nabla \cdot F(x) \text{ in } \Omega, \tag{0.14}$$

$$\nabla \cdot v(x) = 0 \qquad \text{in } \Omega, \tag{0.15}$$

$$v(x) = -u_\infty \qquad \text{on } \partial\Omega, \tag{0.16}$$

$$v(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{0.17}$$

In order to solve (0.14)–(0.17), we consider the linearization of this system with the homogeneous boundary condition, which is called the stationary Oseen equation, as follows:

$$-\Delta_x u(x) + (u_\infty \cdot \nabla)\, u(x) + \nabla \pi(x) = f(x) \text{ in } \Omega, \tag{0.18}$$

$$\nabla \cdot u(x) = 0 \quad \text{in } \Omega, \tag{0.19}$$

$$u(x) = 0 \quad \text{on } \partial\Omega, \tag{0.20}$$

$$u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{0.21}$$

and we make a functional analytic treatment of the system above in the framework of the Lorentz spaces. In the case $u_\infty = 0$, Kozono and Yamazaki [25] made such a treatment by modifying the duality argument employed in Kozono and Sohr [21]. This argument is based on the homogeneity of the Stokes operator, and hence is not applicable to our situation. Instead we construct the parametrix of the stationary Oseen equation from the fundamental solution on the whole space by way of the standard cut-off procedure. Our argument is also useful to the study of the situation where the well-posedness of the stationary Oseen equation fails. (See Section 2.)

We next review previous results on (0.5)–(0.8) and (0.9)–(0.13). Kozono and Nakao [19] considered the problem (0.5)–(0.8) on Ω, where Ω is the whole space $\mathbb{R}^n$ or the half space $\mathbb{R}^n_+$ for $n \geq 3$ or an exterior domain in $\mathbb{R}^n$

for $n \geq 4$, and constructed time-periodic solutions for time-periodic $f(t,x)$ satisfying the assumption

$$f(t,\cdot) \text{ is small in } L^\infty\left(\mathbb{R} : L^r(\Omega) \cap \dot{H}_p^{-1}(\Omega)\right) \tag{0.22}$$

with some $p < n/2$ and $r > n/3$. Although in the previously mentioned works on the stationary problem (0.1)–(0.4), the conditions on the smallness of $f(x)$ are given in terms of norms invariant under the scaling $(u,\pi,f) \to (u_\lambda,\pi_\lambda,f_\lambda)$ such that $u_\lambda(t,x) = \lambda u(\lambda^2 t,\lambda x)$, $\pi_\lambda(t,x) = \lambda^2\pi(\lambda^2 t,\lambda x)$, $f_\lambda(t,x) = \lambda^3 f(\lambda^2 t,\lambda x)$, the condition (0.22) is not in the form above and much stronger than those for the stationary solutions.

Then Taniuchi [43] proved the stability of the periodic solutions constructed in [19] in the space $L^n(\Omega)$. These works treated solutions belonging to suitable L^p spaces. Yamazaki [46] considered the problem on $\mathbb{R}^n$ for $n \geq 3$, and generalized the results of [19, 43] for Morrey spaces.

On the other hand, Salvi [36] considered the problem (0.5)–(0.8) on three-dimensional exterior domains Ω, and proved the existence of a time-periodic weak solution with period T for time-periodic $f(t,x)$ with period T satisfying the assumption

$$f(t,\cdot) \in L^2\left([0,T]; L^2(\Omega) \cap \dot{H}_2^{-1}(\Omega)\right). \tag{0.23}$$

He also showed the existence of a time-periodic strong solution with period T under the assumption that $f(t,x)$ is small in the class above. Actually he considered a more general situation; he solved the problem above on three-dimensional exterior domains with boundary moving periodically with period T. However, the uniqueness of the periodic solution is not known.

For the existence of weak solutions of the problem (0.1)–(0.4) in the sense of Leray [28], it suffices to assume $f(x) = \nabla_x F(x)$ with some $F(x) \in L^2(\Omega)$, and no smallness is necessary. The condition in Salvi [36] seems to be the composition of this one (condition for the existence of stationary weak solution) and the condition for the existence of non-stationary weak solution. (See Leray [28, 29].)For the stationary problem (0.1)–(0.4), Galdi [11, Chapter 9, Theorem 9.4] and Miyakawa [31] showed that, if $u(x)$ is a weak solution and if $\sup_{x\in\Omega}(|x|+1)|u(x)|$ is sufficiently small, then $u(x)$ enjoys the energy identity, and every weak solution enjoying the energy inequality coincides with $u(x)$ above. Kozono and Yamazaki [27] proved the same result under the more general assumption that $\|u\|_{n,\infty}$ is sufficiently small. In other words, the uniqueness of weak solutions is proved only for small physically reasonable solutions, or small solutions in the class generalizing physically reasonable solutions. Hence it seems to be very difficult to prove the uniqueness of the solutions given by Salvi [36] without assuming conditions as above.

More detailed references, including results for bounded domains, the whole spaces and the half spaces, are given in [45, 19, 36].

We next describe the idea to treat (0.5)–(0.8) and (0.9)–(0.13). Let $w(x)$ denote the solution of the stationary problem (0.1)–(0.4) with the external force $f(x)$ replaced by $f_0(x)$, and put $u(t,x) = v(t,x) - w(x)$. Then the systems (0.5)–(0.8) and (0.9)–(0.13) are rewritten into the following systems respectively:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) + (u_\infty \cdot \nabla)\, u(t,x) \\
&\quad + \big(w(x) \cdot \nabla\big) u(t,x) + (u(t,x) \cdot \nabla)\, w(x) \\
&\qquad + (u(t,x) \cdot \nabla)\, u(t,x) + \nabla q(t,x) = g(t,x) \quad \text{in } \mathbb{R} \times \Omega, && (0.24)\\
&\nabla \cdot u(t,x) = 0 \quad \text{in } \mathbb{R} \times \Omega, && (0.25)\\
&u(t,x) = 0 \quad \text{on } \mathbb{R} \times \partial\Omega, && (0.26)\\
&u(t,x) \to 0 \quad \text{as } |x| \to \infty, && (0.27)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) + (u_\infty \cdot \nabla)\, u(t,x) \\
&\quad + \big(w(x) \cdot \nabla\big) u(t,x) + (u(t,x) \cdot \nabla)\, w(x) \\
&\qquad + (u(t,x) \cdot \nabla)\, u(t,x) + \nabla q(t,x) = g(t,x) \quad \text{in } (0,\infty) \times \Omega, && (0.28)\\
&\nabla \cdot u(t,x) = 0 \quad \text{in } (0,\infty) \times \Omega, && (0.29)\\
&u(t,x) = 0 \quad \text{on } (0,\infty) \times \partial\Omega, && (0.30)\\
&u(t,x) \to 0 \quad \text{as } |x| \to \infty, && (0.31)\\
&u(0,x) = u_0(x) \quad \text{on } \Omega && (0.32)
\end{aligned}$$

respectively, where $g(t,x) = f(t,x) - f_0(x)$ and $u_0(x) = v_0(x) - w(x)$. Throughout this paper we assume that $g(t,x)$ is represented as

$$g(t,x) = \big(g_j(t,x)\big)_{j=1,\dots,n} = \nabla \cdot G(t,x) = \left(\sum_{k=1}^{n} \frac{\partial G_{jk}}{\partial x_k}(t,x) \right).$$

As a modification of the method of Fujita and Kato [8], Kozono and Nakao [19] rewrote the system of differential equations (0.24)–(0.27) above into the integral equation on the interval $(-\infty, t)$ for every $t \in \mathbb{R}$, and showed the unique solvability of this integral equation under appropriate assumptions by successive approximation method which will be discussed later. If $f(x)$ is independent of t, it suffices to consider the linearization of this system around the stationary solution $u(x)$. However, if $f(t,x)$ depends on t, the linearization of this system around the solution of (0.24)–(0.27) depends on t, and hence the linearization as above becomes difficult to handle. Instead, they solved the integral equation by regarding the Stokes operator as the principal part and everything else as the perturbation. However, for the integral on the infinite interval should converge so that the iteration scheme associated with the viewpoint above should work, the external force must enjoy decay property and regularity stronger than those in the case (0.1)–(0.4). Namely, under our weaker assumption, the convergence is difficult to prove.

Moreover, for three-dimensional exterior domains, the integral in question does not converge in $L^3(\Omega)$ in general even under the stronger condition in [19], as is understood from the results of [6, 22, 23]. Hence we must work on the space $L^{3,\infty}(\Omega)$ instead, as is stated in [6, 27]. But the weak-L^p spaces contain nontrivial homogeneous functions, and the integral in question fails to converge in the strong topology in any of the weak-L^p spaces when the integrand contains such homogeneous functions.

In fact, [19] employed the iteration scheme

$$\begin{aligned}\frac{\partial u_{j+1}}{\partial t}(t,x) &- \Delta_x u_{j+1}(t,x) + (u_\infty\cdot\nabla)\,u_{j+1}(t,x) + \big(w(x)\cdot\nabla\big)u_j(t,x)\\ &+ (u_j(t,x)\cdot\nabla)\,w(x) + (u_j(t,x)\cdot\nabla)\,u_j(t,x) + \nabla q_{j+1}(t,x) = g(t,x) \qquad (0.33)\end{aligned}$$

in order to solve (0.24)–(0.27). This scheme is also employed in Shibata [37] in order to solve (0.28)–(0.32). On the other hand, in order to solve the system above, Borchers and Miyakawa [6] and Kozono and Yamazaki [26] employed a somewhat different iteration scheme

$$\begin{aligned}\frac{\partial u_{j+1}}{\partial t}(t,x) &- \Delta_x u_{j+1}(t,x) + (u_\infty\cdot\nabla)\,u_{j+1}(t,x) + \big(w(x)\cdot\nabla\big)u_{j+1}(t,x)\\ &+ (u_{j+1}(t,x)\cdot\nabla)\,w(x) + (u_j(t,x)\cdot\nabla)\,u_j(t,x) + \nabla q_{j+1}(t,x) = 0. \qquad (0.34)\end{aligned}$$

Namely, we regard the convection terms as part of the principal terms, and apply the perturbation theory of linear operators. It is hard to apply the scheme (0.34) to the case of [19] because of the dependence of $g(t,x)$ on t, and to the case of [37] because the spectrum of the operator $-\Delta + (u_\infty\cdot\nabla)$ is tangent to the imaginary axis. On the other hand, it was thought that strong decay conditions are necessary to employ the scheme (0.33). Indeed, in the case of [37] the term $w(x)$ in the case $u_\infty \neq 0$ decays faster than in the case $u_\infty = 0$ outside the wake region, and in the case of [19] stronger decay conditions are imposed on $g(t,x)$.

Our method is similar to the one in [19] in spirit, but in order to get around the difficulty above, we show that the integral in question does converge in the weak-$*$ topology of certain weak-L^p spaces. By using this convergence we can show that the iteration scheme (0.33) works in all of the cases above. For this purpose we employ duality argument, which leads naturally to the notion of mild solutions. Roughly speaking, a mild solution is a function, bounded in an appropriate function space, which solves the integral equation associated with the Navier-Stokes equation in the sense of distributions. As in Kozono and Yamazaki [25, 26], the duality between the Lorentz spaces $L_{\sigma}^{n/(n-1),1}(\Omega)$ and $L_{\sigma}^{n/2,\infty}(\Omega)$ plays the most important role. In order to employ in the duality argument above, we prove a sharp version of the L^p-L^q estimates of the non-stationary Oseen semigroup formulated in the Lorentz spaces. This estimate itself seems to be of interest.

As a result, for three-dimensional exterior domains as well, we can construct bounded solutions in the whole time, including time-periodic and almost periodic solutions under an appropriate assumptions on $f(t,x)$, which is unique in a small ball in $L^{n,\infty}(\Omega)$ and depending continuously on $f(t,x)$. We can also show their stability under small initial perturbation in the same class $L^{n,\infty}(\Omega)$, which is exactly the same as the unique existence and the stability class of stationary solutions. Our class of time-dependent solutions is equipped with a norm invariant under the scaling above, and is a natural generalization of the class of stationary solutions introduced in [25, 26, 27], and hence of the class of reasonable stationary solutions satisfying $u(x) \to 0$ as $|x| \to \infty$.

As is seen above, our assumption is more general than those in [19] possibly except the smallness. On the other hand, neither of our assumption or the assumption of [36] implies the other. In particular, we need not assume the square summability of $f(t,x)$.

The outline of this article is as follows. In Section 1 we state our main results on the stationary problem (0.1)–(0.4). These results are derived from the results on the linear stationary Oseen equation (0.18)–(0.21), which will be stated in Section 2. Then we state our main results on the non-stationary problems (0.24)–(0.27) and (0.28)–(0.32) in Section 3. These results are derived from sharp estimates of L^p-L^q type for the Oseen semigroup in the Lorentz spaces, which will be described in Section 4.

1 Results on the stationary problem.

Before stating our results, we introduce some function spaces. For $1 < p < \infty$ and $1 \le q \le \infty$, let $L^{p,q}(\Omega)$ denote the Lorentz space on Ω defined by

$$L^{p,q}(\Omega) = \left\{ u(x) \in L^1_{\mathrm{loc}}(\Omega) \mid \|u\|_{p,q} < +\infty \right\},$$

where

$$\|u\|_{p,q} = \left(\int_0^{+\infty} \left(s\mu\left(\{x \in \Omega \mid |u(x)| > s\}\right)^{1/p} \right)^q \frac{ds}{s} \right)^{1/q}$$

for $1 \le q < \infty$ and

$$\|u\|_{p,q} = \sup_{s>0} s\mu\left(\{x \in \Omega \mid |u(x)| > s\}\right)^{1/p}.$$

Although the function $\|u\|_{p,q}$ above does not satisfy the triangle inequality, there exists a norm equivalent to $\|u\|_{p,q}$, and with this norm the space $L^{p,q}(\Omega)$ becomes a Banach space. Note that the space $L^{p,p}(\Omega)$ is equivalent to the standard space $L^p(\Omega)$. For these spaces, real interpolation yields the equivalence $(L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q} = L^{p,q}(\Omega)$, where $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ satisfy $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1 \le q \le \infty$. Note that this space is determined independently of the choice of p_0 and p_1 up to equivalent norms. (See Bergh and Löfström [2] or Triebel [44] for example.) We remark that, if $1 \le q < \infty$, the dual of the space $L^{p,q}(\Omega)$ coincides with the space $L^{p/(p-1),q/(q-1)}(\Omega)$. Note furthermore that, for $1 \le q < \infty$, the space $C_0^\infty(\Omega)$ is dense in $L^{p,q}(\Omega)$, while it is not so for $L^{p,\infty}(\Omega)$. Let $L^{p,\infty-}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $L^{p,\infty}(\Omega)$. Then the dual of $L^{p,\infty-}(\Omega)$ coincides with the space $L^{p/(p-1),1}(\Omega)$. For every $p \in (1,\infty)$, we equip the space $L^{p,\infty}(\Omega)$ with the weak-$*$ topology as the dual of the space $L^{p/(p-1),1}(\Omega)$.

Next, for $1 < p < \infty$, put

$$\dot{H}^1_p(\Omega) = \{u(x) \in L^p_{\mathrm{loc}}(\Omega) \mid \nabla u \in L^p(\Omega)\},$$

and let this spece equip with the norm $\|\nabla\cdot\|_p$, where $\|\cdot\|_p$ is the norm of the usual L_p space on Ω. Then the set

$$\{\varphi(x)|_\Omega \mid \varphi(x) \in C_0^\infty(\mathbb{R}^n)\}$$

is dense in $\dot{H}^1_p(\Omega)$. Since Ω is an exterior domain, the space $\dot{H}^1_p(\Omega)$ is strictly larger than the usual Sobolev space $H^1_p(\Omega)$. It follows that

$$\dot{H}^1_p(\Omega) \subset L^{np/(n-p)}(\Omega) \quad \text{for} \quad p \in (1,n). \tag{1.1}$$

Furthermore, for $1 < p < \infty$ and $1 \le q \le \infty$, we define the function spaces $H^1_{p,q}(\Omega)$ and $\dot{H}^1_{p,q}(\Omega)$ respectively by way of real interpolation as follows:

$$H^1_{p,q}(\Omega) = \left(H^1_{p_0}(\Omega), H^1_{p_1}(\Omega)\right)_{\theta,q} \quad \text{and} \quad \dot{H}^1_{p,q}(\Omega) = \left(\dot{H}^1_{p_0}(\Omega), \dot{H}^1_{p_1}(\Omega)\right)_{\theta,q},$$

where $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ satisfy $1/p = (1-\theta)/p_0 + \theta/p_1$. Note that these spaces are determined independently of the choice of p_0 and

p_1 up to equivalent norms. Note furthermore that, for $1 \leq q < \infty$, the spaces $H^1_{p,q}(\Omega)$ and $\dot{H}^1_{p,q}(\Omega)$ coincide with the completion of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla\cdot\|_{p,q}+\|\cdot\|_{p,q}$ and that with respect to the norm $\|\nabla\cdot\|_{p,q}$ respectively. From (1.1) and real interpolation we have the inclusion relation

$$\dot{H}^1_{p,q}(\Omega) \subset L^{np/(n-p),q}(\Omega) \quad \text{for} \quad p \in (1,n) \quad \text{and} \quad q \in [1,\infty]. \tag{1.2}$$

Even in the case $q = p$ this relation improves (1.1). We moreover have

$$\dot{H}^1_{n,1}(\Omega) \subset L^\infty(\Omega) \tag{1.3}$$

We next define the notion of solutions of (0.14)–(0.17) employed in this paper.

Definition 1. Suppose that $v(x) = \big(v_1(x), \ldots, v_n(x)\big)$ is a vector-valued function on Ω such that $v(x) \in \big(L^2_{\mathrm{loc}}(\overline{\Omega})\big)^n$, $\nabla v(x) \in \big(L^2_{\mathrm{loc}}(\overline{\Omega})\big)^{n^2}$ and $\pi(x) \in L^2_{\mathrm{loc}}(\overline{\Omega})$. We moreover assume that the functions $v_1(x), \ldots, v_n(x), \pi(x)$ can be extended to tempered distributions on $\mathbb{R}^n$. Then we say that the pair $\big(v(x), \pi(x)\big)$ is a solution of (0.14)–(0.17) if they enjoy (0.14) in the sense that the identity

$$\begin{aligned}\big(v(x), \Delta\varphi(x)\big)+\big(v(x), (u_\infty\cdot\nabla)\varphi(x)\big)+\big(v(x)\otimes v(x), \nabla\varphi(x)\big)+\big(\pi(x), \nabla\cdot\varphi(x)\big)\\ = \big(F(x), \nabla\varphi(x)\big) \quad (1.4)\end{aligned}$$

holds for every $\varphi(x) \in (C_0^\infty(\Omega))^n$, and if $v(x)$ enjoys (0.15) in the sense of distributions on Ω, (0.16) in the usual sense, and if (0.17) in the sense that

$$\lim_{R\to\infty} R^{-n} \int_{R\leq|x|\leq 2R} |v(x)|^r \, dx = 0 \tag{1.5}$$

holds for some $r \in (1,\infty)$. Here $v(x) \otimes v(x)$ and $\nabla\varphi(x)$ denote the tensors $\big(v_j(x)v_k(x)\big)^n_{j,k=1}$ and $\big(\partial\varphi_k(x)/\partial x_j\big)^n_{j,k=1}$ respectively.

Remark 1.1. If (1.5) holds for some $r = r_0$, then (1.5) holds for every $r \in (1, r_0]$. Indeed, Hölder's inequality implies that

$$\begin{aligned}R^{-n} \int_{R\leq|x|\leq 2R} |v(x)|^r \, dx &\leq R^{-n}\,(CR^n)^{1-r/r_0} \left(\int_{R\leq|x|\leq 2R} |v(x)|^{r_0} \, dx\right)^{r/r_0}\\ &\leq C\left(R^{-n} \int_{R\leq|x|\leq 2R} |v(x)|^{r_0} \, dx\right)^{r/r_0}.\end{aligned}$$

Then we have the following uniqueness theorem.

Theorem 1.1. *There exist positive constants* C_0 *and* ε_1 *such that, for every* $u_\infty \in \mathbb{R}^n$ *such that* $|u_\infty| < \varepsilon_1$ *and every* $F(x)$*, the solution* $\big(v(x), \pi(x)\big)$ *of* (0.14)–(0.17) *such that* $v(x) \in L^{n,\infty}(\Omega)$ *satisfying the inequality*

$$\|v\|_{n,\infty} \leq C_0 \tag{1.6}$$

is at most unique up to the constant difference of $\pi(x)$.

On the other hand, we have the following existence theorem.

Theorem 1.2. *There exist positive constants* δ_1 *and* $\varepsilon_2 \leq \varepsilon_1$ *such that, for every* $u_\infty \in \mathbb{R}^n$ *such that* $|u_\infty| < \varepsilon_2$ *and every* $F(x) \in \big(L^{n/2,\infty}(\Omega)\big)^{n^2}$ *such that* $\|F\|_{n/2,\infty} < \delta_1$*, there uniquely exists a solution* $\big(v(x), \pi(x)\big) \in \Big(\dot{H}^1_{n/2,\infty}(\Omega)\Big)^n \times L^{n/2,\infty}(\Omega)$ *of* (0.14)–(0.17) *satisfying the estimate* (1.6). *Furthermore, this solution enjoys the stronger estimate*

$$\|v\|_{n,\infty} + \|\nabla v\|_{n/2,\infty} + \|\pi\|_{n/2,\infty} \leq C_1. \tag{1.7}$$

Remark 1.2. Theorem 1.2 holds both for $u_\infty = 0$ and $u_\infty \neq 0$. Moreover, the constants δ_1 can be taken uniformly in u_∞ as $|u_\infty| \to 0$.

In order to verify the assumptions of the theorems above, it is worth finding conditions on $f(x)$ sufficient for the existence of $F(x) \in (L^{p,q}(\mathbb{R}^n))^{n^2}$ such that $f(x) = \nabla \cdot F(x)$. If $f(x) \in L^{r,q}(\mathbb{R}^n)$ holds with some $r \in (1, n)$ and $q \in [1, \infty]$, then we see by Young's inequality and real interpolation that the function $g(x) = \big(g_1(x), \ldots, g_n(x)\big)$ defined by $g(x) = c_n(x/|x|^n) * f$ satisfies $\nabla \cdot g(x) = f(x)$, and we have $g(x) \in \big(L^{nr/(n-r),q}(\mathbb{R}^n)\big)^n$. In the same way, if $f(x) \in L^1(\mathbb{R}^n)$ we have $g(x) \in L^{n/(n-1),\infty}(\mathbb{R}^n)$, and if $f(x) \in L^{n,1}(\mathbb{R}^n)$ we have $g(x) \in L^\infty(\mathbb{R}^n)$. Hence we see the next remark.

Remark 1.3. We have the following assertions:

(1) If $f(x) \in (L^1(\Omega))^n$, then there exists a function $F(x) \in \big(L^{n/(n-1),\infty}(\Omega)\big)^{n^2}$ such that $f(x) = \nabla \cdot F(x)$.

(2) If $f(x) \in (L^{p,q}(\Omega))^n$ with some $p \in (1, n)$ and $q \in [1, \infty]$, then there exists a function $F(x) \in \big(L^{pn/(n-p),q}(\Omega)\big)^{n^2}$ such that $f(x) = \nabla \cdot F(x)$.

(3) If $f(x) \in (L^{n,1}(\Omega))^n$, then there exists a function $F(x) \in (L^\infty(\Omega))^{n^2}$ such that $f(x) = \nabla \cdot F(x)$.

If the external force $F(x)$ has better regularity, or decay property in the case $n \geq 4$, the solution $\big(v(x), \pi(x)\big)$ has better regularity or decay property accordingly. Namely, we have the following theorem.

Theorem 1.3. *Let p and q satisfy either one of the following conditions:*

(1) $p = n/(n-1)$, $q = \infty$.

(2) $n/(n-1) < p < n$, $1 \le q \le \infty$.

(3) $p = n$, $q = 1$.

Then there exist positive constants $\delta_2 \le \delta_1$ and $\varepsilon_3 \le \varepsilon_2$ such that, if $|u_\infty| \le \varepsilon_3$ and if the external force $F(x)$ enjoys $F(x) \in \left(L^{n/2,\infty}(\Omega) \cap L^{p,q}(\Omega)\right)^{n^2}$ and $\|F\|_{n/2,\infty} < \delta_2$, the solution $\left(v(x), \pi(x)\right)$ given in Theorem 1.2 enjoys $v(x) \in \left(\dot{H}^1_{p,q}(\Omega)\right)^n$ and $\pi(x) \in L^{p,q}(\Omega)$ as well.

Remark 1.4. Either in the case $p < n/2$ or in the case $p = n/2$ and $q < \infty$, Theorem 1.3 asserts that $v(x)$ decays better than in the conclusion of Theorem 1.2, and either in the case $p > n/2$ or in the case $p = n/2$ and $q < \infty$, Theorem 1.3 asserts that $v(x)$ is more regular than in the conclusion of Theorem 1.2. In the case $n = 3$, we must have $p \ge n/(n-1) = n/2$, and the equality holds only in the case $q = \infty$. Hence we cannot expect better decay result which holds uniformly in u_∞ as $u_\infty \to 0$.

Remark 1.5. In particular, we can take $p = q = n/2$ in the case $n \ge 4$, and in this case our results reads as follows: If $F(x) \in \left(L^{n/2}(\Omega)\right)^{n^2}$ and is sufficiently small in $\left(L^{n/2,\infty}(\Omega)\right)^{n^2}$, then there uniquely exists a solution $\left(v(x), \pi(x)\right) \in \left(\dot{H}^1_{n/2}(\Omega)\right)^n \times L^{n/2}(\Omega)$ of (0.14)–(0.17) which is sufficiently small in $\left(\dot{H}^1_{n/2,\infty}(\Omega)\right)^n \times L^{n/2,\infty}(\Omega)$. Putting $u_\infty = 0$ as a particular case of this result, we obtain a slight improvement of the result of Kozono and Sohr [21] on the smallness of external forces and solutions.

Either in the case $n \ge 4$ or in the case $n = 3$ and $u_\infty \ne 0$, we have the following proposition, which is another slight improvement of Theorem 1.2,

Proposition 1.4. *Suppose that either $n \ge 4$, $|u_\infty| < \varepsilon_2$ or $n = 3$, $0 < |u_\infty| < \varepsilon_2$. Suppose moreover that $F(x) \in \left(L^{n/2,\infty-}(\Omega)\right)^{n^2}$ and $\|F\|_{n/2,\infty} < \delta_1$, and let $\left(v(x), \pi(x)\right)$ denote the solution of* (0.14)–(0.17) *given in Theorem 1.2. Then we have $v(x) \in \left(L^{n,\infty-}(\Omega)\right)^n$.*

Remark 1.6. We cannot generalize Proposition 1.4 to the case $n = 3$ and $u_\infty = 0$, as we shall see in Proposition 1.6.

As is stated in the Introduction, we can show the weak-$*$ continuous dependence of the stationary solution on u_∞ including the case $u_\infty = 0$. Namely, we have the following theorem.

Theorem 1.5. *Fix $F(x) \in \left(L^{n/2,\infty}(\Omega)\right)^{n^2}$ and $a \in \mathbb{R}^n$ such that $\|F\|_{n/2,\infty} < \delta_1$ and that $|a| < \varepsilon_2$. For $u_\infty \in \mathbb{R}^n$ such that $|u_\infty| < \varepsilon_2$, let $\left(v_{u_\infty}(x), \pi_{u_\infty}(x)\right)$ denote the solution given in Theorem 1.2. Then the function $v_{u_\infty}(x)$ converges to $v_a(x)$ in the weak-$*$ topology of $L^{n,\infty}(\Omega)$, and the functions $\nabla v_{u_\infty}(x)$ and $\pi_{u_\infty}(x)$ converge to $\nabla v_a(x)$ and $\pi_a(x)$ respectively in the weak-$*$ topology of $L^{n/2,\infty}(\Omega)$ as $u_\infty \to a$. Moreover, for every $p < n$, the function $v_{u_\infty}(x)$ converges to $v_a(x)$ strongly in $L^p_{\mathrm{loc}}(\Omega)$ as $u_\infty \to a$; namely, for every bounded open set U such that $\overline{U} \subset \Omega$, the function $v_{u_\infty}(x)$ converges to $v_a(x)$ strongly in $L^p(U)$ as $u_\infty \to a$.*

It is natural to ask whether the weak-$*$ convergence can be replaced by the strong convergence or the weak convergence in the conclusion of the theorem above, but it seems to be impossible in the case $n = 3$ and $a = 0$, as can be seen from Proposition 1.4 and the next proposition, together with the fact that a strongly closed subspace of a Banach space is also weakly closed, which is a direct consequence of the Hahn-Banach theorem. This proposition is a slight generalization of Theorem 2 of Kozono, Sohr and Yamazaki [23].

Proposition 1.6. *Suppose that $n = 3$, $u_\infty = 0$ and $F(x) \in \left(L^2(\Omega)\right)^{3^2}$, and let $\left(v(x), \pi(x)\right)$ be a weak solution of (0.14)–(0.17); namely, the identity (1.4) holds for every $\varphi(x) \in (C_0^\infty(\Omega))^n$. Define $T(x) = \{T_{jk}(x)\}_{j,k=1}^3$ by the formula*

$$T_{jk}(x) = \frac{\partial v_k}{\partial x_j}(x) + \frac{\partial v_j}{\partial x_k}(x) - \delta_{jk}\pi(x).$$

Then we have the following assertions:

(1) *The boundary integral*

$$S = \int_{\partial\Omega} \left(T(x) + F(x)\right) \cdot \nu(x) dS(x)$$

is well-defined in a generalized sense. Here $\nu(x)$ denotes the outer unit normal vector to $\partial\Omega$ at x.

(2) *If $F(x)$ belongs to the class $\left(L^{3/2,\infty-}(\Omega)\right)^{3^2}$ as well and if $v(x) \in \left(L^{3,\infty-}(\Omega)\right)^3$, then we have $S = 0$.*

The results in this section are derived from the results on the linear stationary Oseen equation (0.18)–(0.21) on the exterior domain given in the next section. Detailed methods of derivation, together with the proofs of the results in the next section, are given in Shibata and Yamazaki [38].

2 Solvability of the Oseen equation.

This section is devoted to the proof of the solvability and the uniqueness of the Oseen system (0.18)–(0.21) in the exterior domain Ω in $\mathbb{R}^n$. We first prove the following uniqueness theorem for this system.

Theorem 2.1. *Let $\big(u(x), \pi(x)\big)$ be a solution of the system* (0.18)–(0.20) *with $F(x) \equiv 0$ such that $u(x) \in (L^r_{\mathrm{loc}}(\Omega))^n$, $\nabla u(x) \in (L^r_{\mathrm{loc}}(\Omega))^{n^2}$ and $\pi(x) \in L^r_{\mathrm{loc}}(\Omega)$ hold with some $r > 1$, and that $u_1(x), \ldots, u_n(x), \pi(x)$ can be extended to tempered distributions on $\mathbb{R}^n$. Suppose moreover that* (0.21) *holds in the sense that the condition* (1.5) *with $v(x)$ replaced by $u(x)$; namely, the condition*

$$\lim_{R\to\infty} R^{-n} \int_{R\leq|x|\leq 2R} |u(x)|^r \, dx = 0 \tag{2.1}$$

holds for some $r \in (1, \infty)$. Then we have $u(x) \equiv 0$ and $\pi(x) \equiv c$ with some constant c.

The next theorem is a general existence theorem.

Theorem 2.2. *Suppose that $1 < p < \infty$ and $1 \leq q \leq \infty$. Then there exist positive numbers $C = C(n, p, q, \Omega)$ and ε_0 such that, for every $u_\infty \in \mathbb{R}^n$ such that $|u_\infty| \leq \varepsilon_0$ and for every $F(x) \in (L^{p,q}(\Omega))^{n^2}$, there exists a solution $\big(u(x), \pi(x)\big)$ of* (0.18)–(0.20) *of the form $u(x) = u_1(x) + u_2(x)$ and $\pi(x) = \pi_1(x) + \pi_2(x)$ satisfying the estimates*

$$\|\nabla u_1\|_{p,q} + \|\pi_1\|_{p,q} \leq C\|F\|_{p,q} \tag{2.2}$$

and

$$\begin{aligned} &\|\nabla u_2\|_{n/(n-1),\infty} + \|u_2\|_{n/(n-2),\infty} + \|\pi_2\|_{n/(n-1),\infty} \\ &\qquad + \left\|\nabla^2 u_2\right\|_{p,q} + \|\nabla \pi_2\|_{p,q} \leq C\|F\|_{p,q}. \end{aligned} \tag{2.3}$$

Remark 2.1. The solution above is not uniquely determined without the condition (0.21). However, for general p and q, none of the solutions of (0.18)–(0.20) enjoy (0.21) in general. In other words, the problem above is not well-posed, with or without the boundary condition at infinity, for all p and q.

Either in the case $1 < p < n$ or the case $p = n$ and $q = 1$, the problem above becomes well-posed if we add the condition (0.21) as the boundary condition at infinity. Namely, we have the following theorem.

Theorem 2.3. *Suppose that either $1 < p < n$ and $1 \le q \le \infty$, or $p = n$ and $q = 1$. Then there exist positive numbers $C = C(n, p, q, \Omega)$ and ε_0 such that, for every $u_\infty \in \mathbb{R}^n$ such that $|u_\infty| \le \varepsilon_0$ and for every $F(x) \in (L^{p,q}(\Omega))^{n^2}$, there uniquely exists a solution $(u(x), \pi(x))$ of (0.18)–(0.20) of the form $u(x) = u_1(x) + u_2(x)$ and $\pi(x) = \pi_1(x) + \pi_2(x)$ satisfying the estimates*

$$\begin{aligned} \|\nabla u_1\|_{p,q} + \|u_1\|_{np/(n-p),q} + \|\pi_1\|_{p,q} &\le C\|F\|_{p,q} \text{ if } 1 < p < n, \\ \|\nabla u_1\|_{n,1} + \|u_1\|_\infty + \|\pi_1\|_{n,1} &\le C\|F\|_{n,1} \text{ if } p = n, q = 1 \end{aligned} \tag{2.4}$$

and (2.3). Moreover, the solution $u(x)$ enjoys (2.1) for every r such that $1 < r < np/(n-p)$ provided $1 < p < n$, and for every r such that $1 < r < \infty$ if $p = n$ and $q = 1$.

If p is not very near to 1, then we see that the functions $u_2(x)$ and $\pi_2(x)$ enjoy the same estimates as $u_1(x)$ and $\pi_1(x)$. As a result we have the following corollary.

Corollary 2.4. *Suppose that one of the following conditions holds:*

(1) $p = n/(n-1)$, $q = \infty$.

(2) $n/(n-1) < p < n$, $1 \le q \le \infty$.

(3) $p = n$, $q = 1$.

Then there exists a positive number $C' = C'(n, p, q, \Omega)$ such that, for every u_∞ and every $F(x)$ as in Theorem 2.3, there uniquely exists a solution $(u(x), \pi(x))$ of (0.18)–(0.20) satisfying the estimates

$$\begin{aligned} \|\nabla u\|_{p,q} + \|u\|_{np/(n-p),q} + \|\pi\|_{p,q} &\le C'\|F\|_{p,q} \text{ in the case (1) or (2),} \\ \|\nabla u\|_{n,1} + \|u\|_\infty + \|\pi\|_{n,1} &\le C'\|F\|_{n,1} \text{ in the case (3).} \end{aligned} \tag{2.5}$$

and (2.3). Moreover, the solution $u(x)$ enjoys (2.1) for every r as in Theorem 2.3.

3 Results on the non-stationary problems.

In this section we assume either $n = 3$, or $n \ge 4$ and $u_\infty = 0$. Before stating our result, we introduce some notations. For every $1 < p < \infty$, we have the Helmholtz decomposition $(L^p(\Omega))^n = L^p_\sigma(\Omega) \oplus G^p(\Omega)$, where

$$L^p_\sigma(\Omega) = \{u(x) \in (L^p(\Omega))^n \mid \operatorname{div} u(x) \equiv 0 \text{ in } \Omega \text{ and } \nu \cdot u(x) \equiv 0 \text{ on } \partial\Omega\}$$

$$G^p(\Omega) = \{u(x) = \operatorname{grad} f(x) \in (L^p(\Omega))^n \text{ for some } f(x) \in L^p_{\mathrm{loc}}(\Omega)\}.$$

For the proof, see Fujiwara and Morimoto [9], Miyakawa [30] and Simader and Sohr [39]. Let P_p denote the projection operator from $(L^p(\Omega))^n$ onto $L^p_\sigma(\Omega)$ along $G^p(\Omega)$. Then the dual of the operator P_p coincides with $P_{p/(p-1)}$. In particular, the operator P_2 is an orthogonal projection in the Hilbert space $(L^2(\Omega))^n$.

We next generalize the Helmholtz decomposition to the Lorentz spaces following Miyakawa and Yamada [32]. We have $P_p = P_q$ on $(L^p(\Omega))^n \cap (L^q(\Omega))^n$ and hence we can extend P_p as a projection operator P in $\left(\sum_{1<p<\infty} L^p(\Omega)\right)^n$. It follows that P is also a projection in $(L^{p,q}(\Omega))^n$. Let $(L^{p,q}(\Omega))^n = L^{p,q}_\sigma(\Omega) \oplus G^{p,q}(\Omega)$ denote the associated direct sum decomposition. Then, for $u_\infty \in \mathbb{R}^n$, we define the Oseen operator A_{u_∞} by the formula $A_{u_\infty} = P\big(-\Delta + (u_\infty \cdot \nabla)\big)$. In particular, the operator A_0 is called the Stokes operator.

Note furthermore that, for $1 \le q < \infty$, the space $C^\infty_{0,\sigma}(\Omega)$ consisting of all the smooth solenoidal vector fields with compact support in Ω is dense in $L^{p,q}_\sigma(\Omega)$, and we can regard $L^{p/(p-1),q/(q-1)}_\sigma(\Omega)$ as the dual space of $L^{p,q}_\sigma(\Omega)$. The dual of the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^{p,\infty}(\Omega)$ coincides with the space $L^{p/(p-1),1}_\sigma(\Omega)$. In view of the duality above, the dual of the Oseen operator A_{u_∞} coincides with A_{-u_∞}.

In order to introduce the notion of mild solution, we define some function classes. Put $\mathcal{K} = BUC\left(\mathbb{R}, L^{n,\infty}_\sigma(\Omega)\right)$ and $\mathcal{L} = BUC\left(\mathbb{R}, (L^{n/2,\infty}(\Omega))^{n^2}\right)$, where $BUC(\mathbb{R}, X)$ denotes the set of bounded and uniformly continuous functions on $\mathbb{R}$, equipped with the norm $\|f\,|BUC(\mathbb{R},X)\| = \sup_{t\in\mathbb{R}} \|f(t,\cdot)\,|X\|$. Next, put $\mathcal{K}_+ = BC\left(\mathbb{R}_+, L^{n,\infty}_\sigma(\Omega)\right)$ and $\mathcal{L}_+ = BC\left(\mathbb{R}_+, (L^{n/2,\infty}(\Omega))^{n^2}\right)$, where $BC(\mathbb{R}_+, X)$ denotes the set of bounded continuous functions on $\mathbb{R}_+$ with values in the Banach space X, equipped with the norm $\|f\,|BC(\mathbb{R}_+,X)\| = \sup_{t\in\mathbb{R}_+} \|f(t,\cdot)\,|X\|$.

Definition 2. A function $u(t,x) \in \mathcal{K}$ is said to be a *mild solution* of the system (0.24)–(0.27) if the identity

$$(u(t,\cdot),\varphi) = \sum_{j,k=1}^{n} \int_0^{+\infty}$$
$$\Big(w_j(\cdot)u_k(t-\tau,\cdot) + u_k(t-\tau,\cdot)w_k(\cdot) + u_j(t-\tau,\cdot)u_k(t-\tau,\cdot) - G_{jk}(t-\tau,\cdot),$$
$$\frac{\partial}{\partial x_j}\big(\exp(-\tau A_{-u_\infty})\varphi\big)_k\Big)\,d\tau \quad (3.1)$$

holds for every $\varphi \in L^{n/(n-1),1}_{\sigma}(\Omega)$ and every $t \in \mathbb{R}$.

Definition 3. A function $u(t,x) \in \mathcal{K}_+$ is said to be a *mild solution* of the system (0.28)–(0.32) if the identity

$$(u(t,\cdot),\varphi) = (u_0, \exp(-tA_{-u_\infty})\varphi) + \sum_{j,k=1}^{n} \int_0^t$$
$$\Big(w_j(\cdot)u_k(t-\tau,\cdot) + u_k(t-\tau,\cdot)w_k(\cdot) + u_j(t-\tau,\cdot)u_k(t-\tau,\cdot) - G_{jk}(t-\tau,\cdot),$$
$$\frac{\partial}{\partial x_j}\big(\exp(-\tau A_{-u_\infty})\varphi\big)_k\Big)d\tau \quad (3.2)$$

holds for every $\varphi \in L^{n/(n-1),1}_{\sigma}(\Omega)$ and every $t > 0$.

Remark 3.1. As is explained in the introduction, the relations (3.1) and (3.2) are the weak form of the integral equations

$$u(t) = \int_0^{+\infty} \exp\left(-\tau A_{u_\infty}\right)\big[-P\big[(w\cdot\nabla)u(t-\tau,\cdot) + \big(u(t-\tau,\cdot)\cdot\nabla\big)w$$
$$+ \big(u(t-\tau,\cdot)\cdot\nabla\big)u(t-\tau,\cdot) - \nabla F(t-\tau,\cdot)\big]d\tau \quad (3.3)$$

and

$$u(t) = \exp\left(-tA_{u_\infty}\right)u_0 +$$
$$\int_0^{t} \exp\left(-\tau A_{u_\infty}\right)\big[-P\big[(w\cdot\nabla)u(t-\tau,\cdot) + \big(u(t-\tau,\cdot)\cdot\nabla\big)w$$
$$+ \big(u(t-\tau,\cdot)\cdot\nabla\big)u(t-\tau,\cdot) - \nabla F(t-\tau,\cdot)\big]d\tau \quad (3.4)$$

respectively. if we regard the terms $(w\cdot\nabla)u(t-\tau,\cdot)$ and so forth as an element of the space above by way the duality pairing $\big((w\cdot\nabla)u(t-\tau,\cdot),\varphi\big) = -\big(w\otimes u(t-\tau,\cdot),\nabla\varphi\big)$ and so forth for $\varphi \in C^{\infty}_{0,\sigma}(\Omega)$.

Then our main result is the following theorem.

Theorem 3.1. *There exist positive numbers A, ε and C_0 depending on n and Ω such that, if $w(x)$ is the stationary solution of* (0.1)–(0.4) *with $f(x)$ replaced by $f_0(x)$ such that $w(x) - u_\infty \in L^{n,\infty}(\Omega)$ with the estimate $\|w - u_\infty\|_{n,\infty} < \varepsilon$, then the following statements hold:*

(1) *For every* $G(t,x) \in \mathcal{L}$ *such that* $\|F\,|\mathcal{L}\| < \varepsilon$, *there exists one and only one mild solution* $u(t,x) \in \mathcal{K}$ *of the system* (0.24)–(0.27) *with* $g(t,x) = \nabla G(t,x)$ *such that* $\|u\,|\mathcal{K}\| < A$. *Moreover, for every* $\delta \in (0,\varepsilon)$, *the mapping* T *from the closed ball in* $\mathcal{L}$ *centered at the origin with radius* δ *to* $\mathcal{K}$ *defined by* $T(G) = u$ *is uniformly continuous. Furthermore, the function* $u(t,x)$ *is the only solution of* (0.24)–(0.27) *in the sense of distributions in* $\mathbb{R}\times\Omega$ *such that* $u(t,x) \in \mathcal{K}$ *with* $\|u\,|\mathcal{K}\| < A$. *Namely, the function* $u(t,x)$ *is the only one satisfying the estimate* $\|u\,|\mathcal{K}\| < A$ *and the identity*

$$\frac{d}{dt}(u(t,\cdot),\varphi) = \big(u(t,\cdot),\Delta\varphi\big) + \sum_{j,k=1}^{n}\left(w_j u_k(t,\cdot) + u_j(t,\cdot)w_k + u_j(t,\cdot)u_k(t,\cdot) - G_{jk}(t,\cdot), \frac{\partial\varphi_k}{\partial x_j}\right) \tag{3.5}$$

for every $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$ *and every* $t \in \mathbb{R}$.

(2) *For every* $G(t,x) \in \mathcal{L}_+$ *and every* $u_0(x) \in L^{n,\infty}_\sigma$ *such that* $C_0\|u_0\|_{n,\infty} + \|G\,|\mathcal{L}_+\| < \varepsilon$, *there exists one and only one mild solution* $u(t,x) \in \mathcal{K}_+$ *of the system* (0.28)–(0.32) *with* $g(t,x) = \nabla G(t,x)$ *such that* $\|u\,|\mathcal{K}_+\| < A$. *Moreover, for every* $\delta \in (0,\varepsilon)$, *the mapping* T_+ *from the set* $\Big\{\big(G(t,x),u_0\big) \mid \|G\,|\mathcal{L}_+\| + C_0\|u_0\|_{n,\infty} \le \delta\Big\}$ *to* $\mathcal{K}_+$ *defined by* $T_+(G,u_0) = u$ *is uniformly continuous. Furthermore, the function* $u(t,x)$ *is the only solution of the* (0.28)–(0.32) *in the sense of distributions in* $\mathbb{R}_+\times\Omega$ *such that* $u(t,x) \in \mathcal{K}_+$ *with* $\|u\,|\mathcal{K}_+\| < A$. *Namely, the function* $u(t,x)$ *is the only one satisfying the estimate* $\|u\,|\mathcal{K}_+\| < A$, *the identity* (3.5) *for every* $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$ *and every* $t > 0$, *and*

$$\big(u(t,\cdot),\varphi\big) \to (u_0,\varphi) \text{ as } t \to +0 \tag{3.6}$$

for every $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$.

As an application to the unique existence of time periodic and almost periodic solutions, we have the following result.

Corollary 3.2. *Suppose that* $u(t,x) \in \mathcal{K}$ *is the unique mild solution such that* $\|u\,|\mathcal{K}\| < A$. *Then we have the following assertions:*

(1) *If* $G(t,\cdot)$ *is time-periodic with period* T, *then the unique mild solution* $u(t,x)$ *such that* $\|u\,|\mathcal{K}\| < A$ *is also time-periodic with period* T.

(2) *If* $G(t,\cdot)$ *is almost periodic with respect to* $t \in \mathbb{R}$, *then the unique mild solution* $u(t,x)$ *such that* $\|u\,|\mathcal{K}\| < A$ *is also almost periodic with respect to* $t \in \mathbb{R}$.

Remark 3.2. For three-dimensional exterior domains, the best spatial decay condition expected in general is $u(t,\cdot) \in L^{3,\infty}(\Omega)$. On the other hand, if we put $u(t,x) = U(x)$ with some homogeneous function $U(x)$ of degree -1 on $\mathbb{R}^3$ such that $U(x) \in L^{3,\infty}(\mathbb{R}^3)$, the function $V(\tau,x)$ defined by the formula $V(\tau,\cdot) = \exp(-\tau A)\nabla(U \otimes U)$ is forward self-similar; namely, it enjoys the equality $V(\lambda^2\tau, \lambda x) = \lambda^{-3}V(\tau,x)$ for every λ, $\tau > 0$ and $x \in \mathbb{R}^3$. It follows that

$$\|V(\tau,\cdot)\|_{q,\infty} = \left(\frac{1}{\sqrt{\tau}}\right)^3 \left\|V\left(1, \frac{\cdot}{\sqrt{\tau}}\right)\right\|_{q,\infty} = C\tau^{3/2q-3/2}$$

for every $q \in (1,\infty)$. We thus conclude that the right-hand side of (3.3) in Remark 3.1 is not Bochner integrable in $L^{q,\infty}$ for any $q \in (1,\infty)$.

Remark 3.3. Assertion (2) of Theorem 3.1 implies the Lyapunov stability of the solution given in Assertion (1) of Theorem 3.1. In particular, if $G(t,x)$ is independent of t, then by the same reasoning as in Corollary 3.2, the solution given in Assertion (1) becomes the stationary solution given in Kozono and Yamazaki [25], and Assertion (2) implies the stability of this stationary solution under small initial perturbation. This result removes the technical assumption $\nabla u(x) \in L^{q,\infty}(\Omega)$ with some $q > n$ on the stationary solution $u(x)$ posed in Kozono and Yamazaki [26].

Remark 3.4. Even in the trivial case $F(x) \equiv w(x) \equiv 0$ and $G(t,x) \equiv u(t,x) \equiv 0$, we cannot expect the asymptotic stability in the space $L^{n,\infty}_\sigma$ itself. This is observed in the following fact. Suppose that $\Omega = \mathbb{R}^3$, and put $b(x) = (0,0,\log|x|)$ and

$$a(x) = c \operatorname{rot} b(x) = c\left(\frac{x_2}{|x|^2}, \frac{-x_1}{|x|^2}, 0\right).$$

Then $a(x) \in L^{3,\infty}_\sigma(\mathbb{R}^3)$. Hence Kozono and Yamazaki [24] implies that, if $|c|$ is sufficiently small, there exists a solution $u(t,x) \in BC\left((0,+\infty), L^{3,\infty}_\sigma(\mathbb{R}^3)\right)$ of the evolution equation

$$\begin{aligned}\frac{du}{dt}(t,x) &= -A_{u_\infty}u(t,x) - P\left[(w\cdot\nabla)u(t,\cdot)\right](x) \\ &\quad - P\left[(u(t,\cdot)\cdot\nabla)w\right](x) - P\left[(u(t,\cdot)\cdot\nabla)u(t,\cdot)\right](x) + g(t,x) \quad (3.7)\end{aligned}$$

with $f(t,x) \equiv 0$ on $(0,+\infty)$, satisfying a kind of boundedness property and the initial condition $u(0,x) = a$ in a suitable sense. Since the initial data $a(x)$ is homogeneous of order -1, it follows that the solution $u(t,x)$ is forward self-similar; namely, $u(t,x)$ enjoys the scaling property $u(\lambda^2 t, \lambda x) = \lambda^{-1}u(t,x)$. for every λ, $t > 0$ and $x \in \mathbb{R}^n$. From this fact we see that $\|u(\lambda^2 t,\cdot)\|_{3,\infty} = \|u(t,\cdot)\|_{3,\infty}$. It follows that $\|u(t,\cdot)\|_{3,\infty}$ is independent of $t > 0$. This implies that even the trivial solution 0 is not asymptotically stable in the space $L^{3,\infty}_\sigma(\mathbb{R}^3)$, in contrast to the space $L^3_\sigma(\mathbb{R}^3)$.

4 Estimates of L^p-L^q type.

In this section we assume either $n \geq 4$ and $u_\infty = 0$, or $n = 3$ and $|u_\infty|$ is sufficiently small.

We first observe the following version in the Lorentz spaces of the L^p-L^q inequality for the Oseen semigroup.

Theorem 4.1. *For every $p \in (1,\infty)$, the operator $-A_{u_\infty}$ generates a bounded analytic semigroup $T_{u_\infty}(t) = \exp(-tA_{u_\infty})$ on $L^{p,1}(\Omega)$, and this semigroup enjoys the following estimates for p, q such that $1 < p \leq q < \infty$:*

(1) *There exists a positive constant C such that the estimate*

$$\|T_{u_\infty}(t)a\|_{q,1} \leq Ct^{n/2q-n/2p}\|a\|_{p,1} \tag{4.1}$$

holds for every $a \in L^{p,1}_\sigma(\Omega)$ and every $t > 0$.

(2) *Suppose that $q \leq n$. Then there exists a positive constant C such that the estimate*

$$\|\nabla T(t)a\|_{q,1} \leq Ct^{n/2q-n/2p-1/2}\|a\|_{p,1} \tag{4.2}$$

holds for every $a \in L^{p,1}_\sigma(\Omega)$ and every $t > 0$.

Remark 4.1. For $q < n$, the estimate (4.2) follows immediately from the results of Iwashita [16] and Kobayashi and Shibata [18], together with real interpolation. In order to include the case $q = n$ we need some more effort.

In the case $u_\infty = 0$, this theorem coincides with [45, Theorem 2.2], and in the case $n = 3$ and $u_\infty \neq 0$, this theorem is an immediate consequence of the following theorem.

Theorem 4.2. *For every p, q such that $1 < p < \infty$ and that $p \leq q \leq \infty$ and every $r \in [1,\infty]$ and for sufficiently small $\varepsilon_0 > 0$, there exists a positive constant C such that the estimates*

$$\|T_{u_\infty}(t)a\|_{q,r} \leq Ct^{-3/2p+3/2q}\|a\|_{p,r} \quad \text{for every } t \in (0,\infty), \tag{4.3}$$

$$\|\nabla T_{u_\infty}(t)a\|_{q,r} \leq Ct^{-3/2p+3/2q-1/2}\|a\|_{p,r} \quad \text{for every } t \in (0,1], \tag{4.4}$$

and

$$\|\nabla T_{u_\infty}(t)a\|_{q,r} \leq Ct^{-3/2p+\rho}\|a\|_{p,r} \quad \text{for every } t \in [1,\infty) \tag{4.5}$$

hold for every $u_0 \in \mathbb{R}^3$ satisfying $|u_0| \le \varepsilon_0$, where ρ is defined as

$$\rho = \begin{cases} \dfrac{3}{2q} - \dfrac{1}{2} & \textit{for } 1 < q \le 3, \\ 0 & \textit{for } 3 \le q \le \infty. \end{cases}$$

We prove this theorem by following the calculation in [18] and making use of real interpolation.

For $q \in [p, \infty)$, the estimates (4.3) for $t \in (0,1]$ and (4.4) are immediate consequences of the fact that $T_{u_\infty}(t)$ is an analytic semigroup on $L^p_\sigma(\Omega)$, together with the fact that the inequality

$$\|\nabla u\|_q \le C\left(\left\|A^{1/2}_{u_\infty} u\right\|_q + \|u\|_q\right).$$

For $q = \infty$, let $r \in (p, \infty)$ and $\varepsilon \in (0, 3/r)$. Then we have

$$\begin{aligned}
&\|T_{u_\infty}(t)a\|_\infty \\
&\le C\left\|T_{u_\infty}(t)a \,\middle|\, B^{n/r}_{r,1}\right\| \\
&\le C\left\|T_{u_\infty}(t)a \,\middle|\, H^{n/r-\varepsilon}_r\right\|^{1/2}\left\|T_{u_\infty}(t)a \,\middle|\, H^{n/r+\varepsilon}_r\right\|^{1/2} \\
&\le C\left(\|T_{u_\infty}(t)a\|_r + \left\|A^{3/2r-\varepsilon/2}_{u_\infty} T_{u_\infty}(t)a\right\|_r\right)^{1/2} \\
&\qquad \left(\|T_{u_\infty}(t)a\|_r + \left\|A^{3/2r+\varepsilon/2}_{u_\infty} T_{u_\infty}(t)a\right\|_r\right)^{1/2} \\
&\le Ct^{1/2\{-3/2(1/p-1/r)-(3/2r-\varepsilon/2)\}}t^{1/2\{-3/2(1/p-1/r)-(3/2r+\varepsilon/2)\}}\|a\|_p \\
&\le Ct^{-3/2p}\|a\|_p
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla T_{u_\infty}(t)a\|_\infty \\
&\le C\left\|T_{u_\infty}(t)a \,\middle|\, B^{n/r+1}_{r,1}\right\| \\
&\le C\left\|T_{u_\infty}(t)a \,\middle|\, H^{n/r+1-\varepsilon}_r\right\|^{1/2}\left\|T_{u_\infty}(t)a \,\middle|\, H^{n/r+1-\varepsilon}_r\right\|^{1/2} \\
&\le C\left(\|T_{u_\infty}(t)a\|_r + \left\|A^{3/2r+1/2-\varepsilon/2}_{u_\infty} T_{u_\infty}(t)a\right\|_r\right)^{1/2} \\
&\qquad \left(\|T_{u_\infty}(t)a\|_r + \left\|A^{3/2r+1/2+\varepsilon/2}_{u_\infty} T_{u_\infty}(t)a\right\|_r\right)^{1/2} \\
&\le Ct^{1/2\{-3/2(1/p-1/r)-1/2-(3/2r-\varepsilon/2)\}}t^{1/2\{-3/2(1/p-1/r)-1/2-(3/2r+\varepsilon/2)\}}\|a\|_p \\
&\le Ct^{-3/2p-1/2}\|a\|_p
\end{aligned}$$

for $t \in (0,1]$.

The estimates (4.3) and (4.4) holds also for $0 < t \leq 2$, possibly with different constants. Hence the main problem is to prove (4.3) and (4.5) for $t \geq 2$. For this purpose we recall the following proposition, which is proved in Kobayashi and Shibata [18, p. 37, (6.18), p. 39, (6.27) and Theorem 1.1].

Proposition 4.3. *For every positive number ε_0, every non-negative integer m, every positive number b and every $p \in (1, \infty)$, there exists a positive number C such that, for every $u_\infty \in \mathbb{R}^3$ such that $|u_\infty| \leq \varepsilon_0$ and every $a(x) \in L^p_\sigma(\Omega)$, the function $u(t,x) = T_{u_\infty}(t+1)a(x)$ and the associated pressure function $\pi(t,x)$ normalized so as to satisfy the identity*

$$\int_{\Omega_b} \pi(t,x)\, dx = 0,$$

where $\Omega_b = \{x \in \Omega \mid |x| < b\}$, enjoy the estimate

$$\begin{aligned} \|u(t,\cdot)\,|W^{2m}_p(\Omega_b)\| + \left\| \frac{\partial u}{\partial t}(t,\cdot) \right| W^{2m}_p(\Omega_b) \bigg\| + \|\pi(t,\cdot)\,|W^{2m}_p(\Omega_b)\| \\ \leq C(1+t)^{-3/2p}\|a\|_p \quad (4.6) \end{aligned}$$

for every $t > 0$.

Moreover, if $a(x)$ satisfies $\operatorname{supp} a \subset \Omega_b$ *as well, then the following sharper estimate*

$$\begin{aligned} \|u(t,\cdot)\,|W^{2m}_p(\Omega_b)\| + \left\| \frac{\partial u}{\partial t}(t,\cdot) \right| W^{2m}_p(\Omega_b) \bigg\| + \|\pi(t,\cdot)\,|W^{2m}_p(\Omega_b)\| \\ \leq C(1+t)^{-3/2}\|a\|_p \quad (4.7) \end{aligned}$$

holds for every $t > 0$.

From the proposition above, real interpolation and the Sobolev embedding theorem, we have the following corollary.

Corollary 4.4. *Let ε, m, b and p as in Proposition 4.3, and let $q \in [p, \infty]$. Then there exists a positive constant C such that, for every u_∞ as in Proposition 4.3 and every $a(x) \in L^{p,r}_\sigma(\Omega)$, the functions $u(t,x)$ and $\pi(t,x)$ enjoy the estimate*

$$\begin{aligned} \bigg\{ \int_0^\infty \bigg(\|u(t,\cdot)\,|W^{2m}_q(\Omega_b)\| + \left\| \frac{\partial u}{\partial t}(t,\cdot) \right| W^{2m}_q(\Omega_b) \bigg\| + \|\pi(t,\cdot)\,|W^{2m}_q(\Omega_b)\| \bigg)^r \\ (1+t)^{3r/2p-1}\, dt \bigg\}^{1/r} \leq C\|a\|_{p,r} \quad (4.8) \end{aligned}$$

for every $t > 0$.

Moreover, if $\operatorname{supp} a \subset \Omega_b$ *holds, then we have the estimate*

$$\|u(t,\cdot)\,|W_q^{2m}(\Omega_b)\| + \left\|\frac{\partial u}{\partial t}(t,\cdot)\right| W_q^{2m}(\Omega_b)\bigg\| + \|\pi(t,\cdot)\,|W_q^{2m}(\Omega_b)\| \\ \leq C(1+t)^{-3/2}\|a\|_{p,r} \quad (4.9)$$

for every $t > 0$.

We next assume that $\Omega \subset \{x \in \mathbb{R}^3 \mid |x| < b-2\}$, and choose a function $\psi(x) \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \psi(x) \leq 1$, $\psi(x) \equiv 1$ if $|x| \leq b-2$ and $\psi(x) \equiv 0$ if $|x| \geq b-1$. Then we have

$$\begin{aligned} &\int_{|x|\leq b-1} (\nabla\psi)\cdot u(t,\cdot)\,dx \\ &= \int_{|x|\leq b-1} \nabla\cdot\big(\psi u(t,\cdot)\big)\,dx \\ &= \int_{|x|=b-1} n(x)\cdot\psi(x)u(t,x)\,dS(x) + \int_{\partial\Omega} n(x)\cdot\psi(x)u(t,x)\,dS(x) = 0. \end{aligned}$$

On the other hand, we have the following proposition and definition.

Proposition 4.5. *Let* $p \in (1,\infty)$ *and* $r \in [1,\infty]$, *and let* m *be a nonnegative integer. Then there exists a positive constant* $C = C_{p,r,D,m}$ *such that, for every* $f(x) \in H^m_{p,r,0}(D)$ *such that* $\int_D f(x)dx = 0$, *there uniquely exists a function* $w(x) \in \big(H^{m+1}_{p,r,0}(D)\big)^n$ *such that* $\operatorname{div} w = f$ *in* D *and that*

$$\|w\,|H^{m+1}_{p,r}(D)\| \leq C\|f\,|H^m_{p,r}(D)\|.$$

Definition 4. Let B denote the Bogovskii operator on D which maps the function $f(x) \in H^m_{p,r,0}(D)$ in Proposition 4.5 to the unique function $w(x) \in \big(H^{m+1}_{p,r,0}(D)\big)$ given by Proposition 4.5.

Proof of Theorem 4.2. In view of Proposition 4.5 applied to the bounded domain $D = \Omega \cup \{x \in \mathbb{R}^3 \mid |x| < b-1\}$, we put

$$z(t,x) = \big(1-\psi(x)\big)u(t,x) + B\big[(\nabla\psi)\cdot u(t,\cdot)\big].$$

Then we have $\|z(0,\cdot)\|_{p,r} \leq C\|a\|_{p,r}$ and

$$\frac{\partial z}{\partial t} - \Delta z + (u_\infty\cdot\nabla)z + \nabla\big\{\big(1-\psi(x)\big)\pi\big\} = h \quad (4.10)$$

$$\nabla\cdot z = 0 \quad (4.11)$$

in $(0,\infty)\times\mathbb{R}^3$, where

$$h(t,\cdot)=(\nabla\psi)\cdot\pi(t,\cdot)+\{2(\nabla\psi)\cdot\nabla u(t,\cdot)\}$$
$$-\big((u_\infty\cdot\nabla)\psi\big)u(t,\cdot)+\left(\frac{\partial}{\partial t}-\Delta+(u_\infty\cdot\nabla)\right)B\big[(\nabla\psi)\cdot u(t,\cdot)\big].$$

Then we have $\operatorname{supp} h(t,\cdot)\subset\Omega_b$ and $\|h(t,\cdot)\,|W^m_\infty\|\le C(1+t)^{-3/2p}\|a\|_{p,r}$ for every positive integer m in view of Corollary 4.4. Now let $S_{u_\infty}(t)$ denote the Oseen semigroup on $\mathbb{R}^3$. Then (4.10) yields that we can write

$$z(t,\cdot)=S_{u_\infty}(t)z(0,\cdot)+\int_0^t S_{u_\infty}(t-s)h(s,\cdot)\,ds.$$

Here we remark that $\big(S_{u_\infty}(t)f\big)(x)=\big(S_0(t)f\big)(x-tu_\infty)$. It follows that $S_{u_\infty}(t)$ enjoys the same unweighted estimates of L^p-L^q type as $S_0(t)$, the Stokes semigroup on the whole space $\mathbb{R}^3$, does. It follows that

$$\|S_{u_\infty}(t)z(0,\cdot)\|_{q,r}\le Ct^{3/2(1/q-1/p)}\|a\|_{p,r} \tag{4.12}$$

and

$$\|\nabla S_{u_\infty}(t)z(0,\cdot)\|_{q,r}\le Ct^{3/2(1/q-1/p)-1/2}\|a\|_{p,r} \tag{4.13}$$

hold for p and q such that $1<p<\infty$ and that $p\le q\le\infty$. On the other hand, since $\operatorname{supp} h(t,\cdot)\subset\Omega_b$, we can apply Corollary 4.4 to obtain

$$\begin{aligned}
&\left\|\int_0^t S_{u_\infty}(s)h(t-s,\cdot)\,ds\right\|_{q,r}\\
&\le\int_0^t\|S_{u_\infty}(s)h(t-s,\cdot)\|_{q,r}\,ds\\
&\le C\left(\int_0^1\|h(t-s,\cdot)\|_{q,r}\,ds+\int_1^t s^{-3/2}\|h(t-s,\cdot)\|_{3/2,r}\,ds\right)\\
&\le C\|a\|_{p,r}\left(\int_0^1(1+t-s)^{-3/2p}ds+\int_1^t s^{-3/2}(1+t-s)^{-3/2p}\,ds\right)\\
&\le C\|a\|_{p,r}t^{-3/2p}
\end{aligned}$$

and

$$\left\|\nabla\int_0^t S_{u_\infty}(t-s)h(s,\cdot)\,ds\right\|_{q,r}$$

$$
\begin{aligned}
&\le \int_0^t \|\nabla S_{u_\infty}(t-s)h(s,\cdot)\|_{q,r}\,ds \\
&\le C\left(\int_0^1 s^{-1/2}\|h(t-s,\cdot)\|_{q,r}\,ds + \int_1^t s^{-3/2}\|h(t-s,\cdot)\|_{3/2,r}ds\right) \\
&\le C\|a\|_{p,r}\left(\int_0^1 s^{-1/2}(1+t-s)^{-3/2p}\,ds + \int_1^t s^{-3/2}(1+t-s)^{-3/2p}\,ds\right) \\
&\le C\|a\|_{p,r}t^{-3/2p}
\end{aligned}
$$

for $t \ge 1$. These estimates, together with (4.12) and (4.13), complete the proof of Theorem 4.2. □

Remark 4.2. In the case $u_\infty = 0$, the estimate

$$\left\|A_{u_\infty}^{1/2}T_{u_\infty}(t)a\right\|_q \le Ct^{-n/2p+n/2q-1/2}\|a\|_p$$

holds for $q > n$ as well, and we derive Theorem 4.3 from this fact. However, in the case $n = 3$ and $u_\infty \ne 0$, the author does not know whether the estimate above holds for $q > n$ as well.

We next derive from Theorem 4.3 the following stronger estimate, which plays the most important role in the proof of the results in the preceding section.

Theorem 4.6. *Suppose that $1 < p < q \le \infty$, and suppose that u_∞ enjoys $|u_\infty| \le \varepsilon_0$ in the case $n = 3$, and $u_\infty = 0$ in the case $n = 4$. Then we have the following assertions:*

(1) *There exists a constant C independent of u_∞ such that the estimate*

$$\int_0^\infty t^{n/2(1/p-1/q)-1}\|T_{u_\infty}(t)a\|_{q,1}\,dt \le C\|a\|_{p,1} \tag{4.14}$$

holds for every $a \in L_\sigma^{p,1}(\Omega)$.

(2) *If $q \le n$, the constant C can be chosen so that the estimate*

$$\int_0^\infty t^{n/2(1/p-1/q)-1/2}\|\nabla T_{u_\infty}(t)a\|_{q,1}\,dt \le C\|a\|_{p,1} \tag{4.15}$$

holds for every $a \in L_\sigma^{p,1}(\Omega)$.

Proof. Although this theorem can be proved exactly in the same way as [45, Corollary 2.3], we give another proof, which does not rely on real interpolation between non-Banach spaces and seems to be more elementary. We shall prove (4.15) only, since (4.14) can be proved exactly in the same way.

Fix p and q, and choose p_0 and p_1 such that $1 < p_0 < p < p_1 < q$, and for every $j \in \mathbb{Z}$, put

$$c_j = \int_{2^j}^{2^{j+1}} t^{n/2(1/p-1/q)-1/2} \|\nabla T_{u_\infty}(t)a\|_{q,1}\, dt. \tag{4.16}$$

Suppose that $a \in L^{p_h,1}_\sigma(\Omega)$. Then Theorem 4.3 implies that

$$c_j \le C \int_{2^j}^{2^{j+1}} t^{n/2(1/p-1/p_h)-1} \|a\|_{p_h,1}\, dt \le C 2^{jn/2(1/p-1/p_h)} \|a\|_{p_h,1}.$$

In other words, the sequence $\{c_j\}_{j=-\infty}^{\infty}$ belongs to the function space $\ell^{n/2(1/p-1/p_h),\infty}$ and the estimate

$$\left\| \{c_j\}_{j=-\infty}^{\infty} \,|\ell^{n/2(1/p-1/p_h),\infty} \right\| \le C \|a\|_{p_h,1} \tag{4.17}$$

holds for $h = 0, 1$. Now choose $\theta \in (0,1)$ so that $1/p = (1-\theta)/p_0 + \theta/p_1$. Then we have the real interpolation relations

$$L^{p,1}_\sigma(\Omega) = \left(L^{p_0,1}_\sigma(\Omega), L^{p_1,1}_\sigma(\Omega)\right)_{\theta,1}, \quad \ell^1 = \left(\ell^{n/2(1/p-1/p_0),\infty}, \ell^{n/2(1/p-1/p_1),\infty}\right)_{\theta,1}. \tag{4.18}$$

From (4.17) and (4.18) we conclude that

$$\sum_{j=-\infty}^{\infty} c_j \le C \|a\|_{p,1}.$$

From this inequality and (4.16) we obtain the conclusion. □

Now the results in Section 3 follows from this theorem in the same way as in [45].

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