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Generalized fractional integrals

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1. Introduction

The fractional integral I_{α} $(0<\alpha<n)$ is defined by

$$I_{\alpha}f(x)=\int_{\mathbb{R}^{n}}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

It is known that I_{α} is bounded from $L^{p}(\mathbb{R}^{n})$ to $L^{q}(\mathbb{R}^{n})$ when $p>1$ and $n/p-\alpha=n/q>0$ as the Hardy-Littlewood-Sobolev theorem. The fractional integral was studied by many authors (see, for example, Rubin [10] or Chapter 5 in Stein [11]). The Hardy-Littlewood-Sobolev theorem is an important result in the fractional integral theory and the potential theory. We introduce generalized fractional integrals and extend the Hardy-Littlewood-Sobolev theorem to the Orlicz spaces. We show that, for example, a generalized fractional integral is bounded from $\exp L^{p}$ to $\exp L^{q}$.

Let $B(a,r)$ be the ball $\{x\in \mathbb{R}^{n} : |x-a|<r\}$ with center a and of radius $r>0$, and $B_{0}=B(\mathrm{O},1)$ with center the origin and of radius 1. The modified fractional integral $\tilde{I}_{\alpha}$ $(0<\alpha<n+1)$ is defined by

$$\tilde{I}_{\alpha}f(x)=\int_{\mathbb{R}^{n}}f(y)\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1-\chi_{B_{0}}(y)}{|y|^{n-\alpha}}\right)dy,$$

where $\chi_{B_{0}}$ is the characteristic function of B_{0}. It is known that the modified fractional integral $\tilde{I}_{\alpha}$ is bounded from $L^{p}(\mathbb{R}^{n})$ to $\mathrm{BMO}(\mathbb{R}^{n})$ when $p>1$ and $n/p-\alpha=0$, from $L^{p}(\mathbb{R}^{n})$ to $\mathrm{Lip}_{\beta}(\mathbb{R}^{n})$ when $p>1$ and $-1<n/p-\alpha=-\beta<0$, from $\mathrm{BMO}(\mathbb{R}^{n})$ to $\mathrm{Lip}_{\alpha}(\mathbb{R}^{n})$ when $0<\alpha<1$, and from $\mathrm{Lip}_{\beta}(\mathbb{R}^{n})$ to $\mathrm{Lip}_{\gamma}(\mathbb{R}^{n})$ when $0<\alpha+\beta=\gamma<1$.

We investigate the boundedness of generalized fractional integrals from the Orlicz space $L^{\Phi}(\mathbb{R}^{n})$ to $\mathrm{BMO}_{\phi}(\mathbb{R}^{n})$ and from $\mathrm{BMO}_{\phi}(\mathbb{R}^{n})$ to $\mathrm{BMO}_{\psi}(\mathbb{R}^{n})$. If $\phi(r)\equiv 1$, then $\mathrm{BMO}_{\phi}(\mathbb{R}^{n})=\mathrm{BMO}(\mathbb{R}^{n})$. If $\phi(r)=r^{\alpha}$ $(0<\alpha\leq 1)$, then $\mathrm{BMO}_{\phi}(\mathbb{R}^{n})=\mathrm{Lip}_{\alpha}(\mathbb{R}^{n})$. We also investigate the boundedness of generalized fractional integrals on the Morrey and Campanato spaces.

2. Notations and definitions

For a function $\rho : (0,+\infty) \to (0,+\infty)$, let

$$I_\rho f(x) = \int_{\mathbb{R}^n} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\,dy.$$

We consider the following conditions on ρ:

$$\int_0^1 \frac{\rho(t)}{t}\,dt < +\infty, \tag{2.1}$$

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le A_1 \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.2}$$

$$\frac{\rho(r)}{r^n} \le A_2\frac{\rho(s)}{s^n} \quad \text{for} \quad s \le r, \tag{2.3}$$

where $A_1, A_2 > 0$ are independent of $r, s > 0$. If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the fractional integral or the Riesz potential denoted by I_α.

We define the modified version of I_ρ as follows:

$$\tilde{I}_\rho f(x) = \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right)dy.$$

We consider the following conditions on ρ: (2.1), (2.2) and

$$\frac{\rho(r)}{r^{n+1}} \le A_2'\frac{\rho(s)}{s^{n+1}} \quad \text{for} \quad s \le r, \tag{2.4}$$

$$\int_r^{+\infty} \frac{\rho(t)}{t^2}\,dt \le A_2''\frac{\rho(r)}{r}, \tag{2.5}$$

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le A_3|r-s|\frac{\rho(r)}{r^{n+1}} \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.6}$$

where $A_2', A_2'', A_3 > 0$ are independent of $r, s > 0$. If $\rho(r)r^\alpha$ is increasing for some $\alpha \ge 0$ and $\rho(r)/r^\beta$ is decreasing for some $\beta \ge 0$, then ρ satisfies (2.2) and (2.6). If $\rho(r) = r^\alpha$, $0 < \alpha < n+1$, then $\tilde{I}_\rho = \tilde{I}_\alpha$. If $\tilde{I}_\rho f$ and $I_\rho f$ are well defined, then $\tilde{I}_\rho f - I_\rho f$ is a constant.

A function $\Phi : [0,+\infty) \to [0,+\infty]$ is called a Young function if Φ is convex, $\lim_{r\to+0}\Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty}\Phi(r) = +\infty$. Any Young function is increasing. For a Young function Φ, the complementary function is defined by

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \ge 0\}, \quad r \ge 0.$$

For example, if $\Phi(r) = r^p/p$, $1 < p < \infty$, then $\widetilde{\Phi}(r) = r^{p'}/p'$, $1/p + 1/p' = 1$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0 (0 \le r \le 1), = +\infty (r > 1)$.

For a Young function Φ, let

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)\, dx < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\},$$

$$L^{\Phi}_{weak}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_{r>0} \Phi(r)\, m(r, \epsilon f) < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\Phi,weak} = \inf \left\{ \lambda > 0 : \sup_{r>0} \Phi(r)\, m\left(r, \frac{f}{\lambda}\right) \le 1 \right\},$$

$$\text{where} \quad m(r, f) = |\{x \in \mathbb{R}^n : |f(x)| > r\}|.$$

Then

$$L^{\Phi}(\mathbb{R}^n) \subset L^{\Phi}_{weak}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{\Phi,weak} \le \|f\|_{\Phi}.$$

If a Young function Φ satisfies

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty, \tag{2.7}$$

then Φ is continuous and bijective from $[0, +\infty)$ to itself. The inverse function Φ^{-1} is also increasing and continuous.

A function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \quad r \ge 0,$$

for some $k > 1$.

Let $Mf(x)$ be the maximal function, i.e.

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x.

We assume that Φ satisfies (2.7). Then M is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Phi}_{weak}(\mathbb{R}^n)$ and

$$\|Mf\|_{\Phi,weak} \le C_0 \|f\|_{\Phi}. \tag{2.8}$$

If $\Phi \in \nabla_2$, then M is bounded on $L^{\Phi}(\mathbb{R}^n)$ and

$$\|Mf\|_{\Phi} \le C_0 \|f\|_{\Phi}. \tag{2.9}$$

For functions $\theta, \kappa : (0, +\infty) \to (0, +\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r) \quad \text{for} \quad r > 0.$$

A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\theta(r) \leq C\theta(s) \ (\theta(r) \geq C\theta(s)) \quad \text{for} \quad r \leq s.$$

A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{\theta(r)}{\theta(s)} \leq C \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2.$$

For $1 \leq p < \infty$ and a function $\phi : (0, +\infty) \to (0, +\infty)$, let

$$\|f\|_{L_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(x)|^p \, dx \right)^{1/p},$$
$$L_{p,\phi}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L_{p,\phi}} < +\infty \right\}.$$

We assume that ϕ satisfies the doubling condition and that $\phi(r)r^{n/p}$ is almost increasing. If $\phi(r) = r^{(\lambda-n)/p}$ $(0 \leq \lambda \leq n)$, then $L_{p,\phi}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ which is the classical Morrey space. If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\lambda = n$, then $L^{p,\lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

For $1 \leq p < \infty$ and a function $\phi : (0, +\infty) \to (0, +\infty)$, let

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx \right)^{1/p},$$
$$\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}_{p,\phi}} < +\infty \right\},$$
$$\text{where} \qquad f_B = \frac{1}{|B|} \int_B f(x) \, dx.$$

We assume that ϕ satisfies the doubling condition and that $\phi(r)r^{n/p}$ is almost increasing. If $\phi(r) = r^{(\lambda-n)/p}$ $(0 \leq \lambda \leq n+1)$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ which is the classical Companato space.

If ϕ is almost increasing, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ for all $p > 1$. Let $\text{BMO}_\phi(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$. If $\phi \equiv 1$, then $\text{BMO}_\phi(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha$, $0 < \alpha \leq 1$, then it is known that $\text{BMO}_\phi(\mathbb{R}^n) = \text{Lip}_\alpha(\mathbb{R}^n)$.

The letter C shall always denote a constant, not necessarily the same one.

3. Main results

Our main results are as follows:

Theorem 3.1. *Let ρ satisfy* (2.1)–(2.3). *Let Φ and Ψ be Young functions with* (2.7). *Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,*

$$\int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{A\int_0^r (\rho(s)/s)\,ds\;\Phi^{-1}(1/r^n)t^n}\right) t^{n-1}\,dt \le A', \tag{3.1}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}\left(\frac{1}{r^n}\right) \le A''\,\Psi^{-1}\left(\frac{1}{r^n}\right), \tag{3.2}$$

where $\widetilde{\Phi}$ is the complementary function with respect to Φ. Then, for any $C_0 > 0$, there exists a constant $C_1 > 0$ such that, for $f \in L^\Phi(\mathbb{R}^n)$,

$$\Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right). \tag{3.3}$$

Therefore I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi_{weak}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Remark 3.1. From (2.2) it follows that

$$\rho(r) \le C\int_0^r \frac{\rho(t)}{t}\,dt. \tag{3.4}$$

If $\rho(r)/r^\varepsilon$ is almost increasing for some $\varepsilon > 0$ and $\rho(t)/t^n$ is almost decreasing, then ρ satisfies (2.1)–(2.3) and $\int_0^r (\rho(t)/t)\,dt \sim \rho(r)$. Let, for example, $\rho(r) = r^\alpha(\log(1/r))^{-\beta}$ for small r. If $\alpha = 0$ and $\beta > 1$, then $\int_0^r (\rho(t)/t)\,dt \sim (\log(1/r))^{-\beta+1}$. If $\alpha > 0$ and $-\infty < \beta < +\infty$, then $\int_0^r (\rho(t)/t)\,dt \sim \rho(r)$.

Remark 3.2. In the case $\Phi(r) = r$, (3.1) is equivalent to

$$\frac{\rho(t)}{t^n} \le \frac{A\int_0^r (\rho(s)/s)\,ds}{r^n}, \quad 0 < r \le t.$$

This inequality follows from (2.3) and (3.4).

The following corollary is stated without the complementary function.

Corollary 3.2. *Let ρ satisfy* (2.1)–(2.3). *Let Φ and Ψ be Young functions with* (2.7). *Assume that*

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}\left(\frac{1}{r^n}\right)$$

is almost decreasing and that there exist constants $A, A' > 0$ such that, for all $r > 0$,

$$\int_r^{+\infty} \frac{\rho(t)}{t} \Phi^{-1}\left(\frac{1}{t^n}\right) dt \le A \int_0^r \frac{\rho(t)}{t}\, dt\, \Phi^{-1}\left(\frac{1}{r^n}\right), \tag{3.5}$$

$$\int_0^r \frac{\rho(t)}{t}\, dt\, \Phi^{-1}\left(\frac{1}{r^n}\right) \le A'\, \Psi^{-1}\left(\frac{1}{r^n}\right). \tag{3.6}$$

Then (3.3) *holds. Therefore I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi_{weak}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Remark 3.3. If $r^\varepsilon \rho(r)\Phi^{-1}(1/r^n)$ is almost decreasing for some $\varepsilon > 0$, then

$$\int_r^{+\infty} \frac{\rho(t)}{t} \Phi^{-1}\left(\frac{1}{t^n}\right) dt \le C\rho(r)\, \Phi^{-1}\left(\frac{1}{r^n}\right).$$

This inequality and (3.4) yield (3.5).

Remark 3.4. We cannot replace (3.2) or (3.6) by

$$\rho(r)\, \Phi^{-1}\left(\frac{1}{r^n}\right) \le A\, \Psi^{-1}\left(\frac{1}{r^n}\right) \quad \text{for all } r > 0$$

(see Section 5 in [6]).

O'Neil [7] showed the boundedness for convolution operators on the Orlicz spaces. Cianchi [1] gave a necessary and sufficient condition on Φ and Ψ so that the fractional integral I_α is bounded from L^Φ to L^Ψ.

Theorem 3.3. *Let ρ satisfy* (2.1), (2.2), (2.4) *and* (2.6). *Let Φ be Young function with* (2.7), *ϕ satisfy the doubling condition and be almost increasing. Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,*

$$\int_r^{+\infty} \widetilde{\Phi}\left(\frac{r\rho(t)}{A\int_0^r (\rho(s)/s)\, ds\, \Phi^{-1}(1/r^n) t^{n+1}}\right) t^{n-1}\, dt \le A', \tag{3.7}$$

$$\int_0^r \frac{\rho(t)}{t}\, dt\, \Phi^{-1}\left(\frac{1}{r^n}\right) \le A''\phi(r), \tag{3.8}$$

where $\widetilde{\Phi}$ is the complementary function with respect to Φ. Then $\tilde{I}_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{BMO}_\phi(\mathbb{R}^n)$.

Theorem 3.4. *Let ρ satisfy* (2.1), (2.2), (2.4) *and* (2.6). *Let ϕ and ψ satisfy the doubling condition, and $\phi(r)r^n$ and $\psi(r)r^n$ be almost increasing. Assume* *that there exist constants* $A, A' > 0$ *such that, for all* $r > 0$,

$$\int_r^{+\infty} \frac{\rho(t)\phi(t)}{t^2}\,dt \le A \int_0^r \frac{\rho(t)}{t}\,dt\,\frac{\phi(r)}{r}, \tag{3.9}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\,\phi(r) \le A'\psi(r). \tag{3.10}$$

Then $\tilde{I}_\rho$ *is bounded from* $L_{1,\phi}(\mathbb{R}^n)$ *to* $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$.

If $\Phi \in \nabla_2$ and $\Phi^{-1}(1/r^n) = \phi(r)$, then we can show

$$L^\Phi_{weak}(\mathbb{R}^n) \subset L^{1,\phi}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^{1,\phi}} \le C\|f\|_{\Phi,weak}.$$

Then we have the following.

Corollary 3.5. *Let* ρ *satisfy* (2.1), (2.2), (2.4) *and* (2.6). *Let* Φ *be Young function with* (2.7), $\Phi \in \nabla_2$, ϕ *satisfy the doubling condition and be almost increasing. Assume that there exist constants* $A, A' > 0$ *such that, for all* $r > 0$,

$$\int_r^{+\infty} \frac{\rho(t)\Phi^{-1}(1/t^n)}{t^2}\,dt \le A \int_0^r \frac{\rho(t)}{t}\,dt\,\frac{\Phi^{-1}(1/r^n)}{r}, \tag{3.11}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\,\Phi^{-1}\left(\frac{1}{r^n}\right) \le A'\phi(r). \tag{3.12}$$

Then $\tilde{I}_\rho$ *is bounded from* $L^\Phi_{weak}(\mathbb{R}^n)$ *to* $\mathrm{BMO}_\phi(\mathbb{R}^n)$.

Theorem 3.6. *Let* ρ *satisfy* (2.1), (2.2), (2.5) *and* (2.6). *Let* ϕ *and* ψ *satisfy the doubling condition, and* $\phi(r)r^n$ *and* $\psi(r)r^n$ *are almost increasing. Assume that there exist constants* $A, A' > 0$ *such that, for all* $r > 0$,

$$\int_r^{+\infty} \frac{\rho(t)\phi(t)}{t^2}\,dt \le A \int_0^r \frac{\rho(t)}{t}\,dt\,\frac{\phi(r)}{r}, \tag{3.13}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\,\phi(r) \le A'\psi(r). \tag{3.14}$$

Then $\tilde{I}_\rho$ *is bounded from* $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ *to* $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$.

Remark 3.5. From Lemma 4.3 it follows that $\tilde{I}_\rho 1$ is a constant. Hence $\tilde{I}_\rho$ is well defined as an operator from $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ to $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$.

The boundedness of the fractional integral I_α on the Campanato space is known (Peetre [8]).

Corollary 3.7. *Let* ρ *satisfy* (2.1), (2.2), (2.5) *and* (2.6). *Let* ϕ *and* ψ *satisfy the doubling condition and be almost increasing. Assume that there exist* *constants* $A, A' > 0$ *such that, for all* $r > 0$,

$$\int_r^{+\infty} \frac{\rho(t)\phi(t)}{t^2}\, dt \le A \int_0^r \frac{\rho(t)}{t}\, dt \frac{\phi(r)}{r}, \tag{3.15}$$

$$\int_0^r \frac{\rho(t)}{t}\, dt\, \phi(r) \le A'\psi(r). \tag{3.16}$$

Then $\tilde{I}_\rho$ *is bounded from* $\mathrm{BMO}_\phi(\mathbb{R}^n)$ *to* $\mathrm{BMO}_\psi(\mathbb{R}^n)$.

This Corollary is a generalization of the well-known result that $\tilde{I}_\alpha$ is bounded from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\mathrm{Lip}_\beta(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha+\beta}(\mathbb{R}^n)$ when $\alpha > 0$, $\beta > 0$ and $0 < \alpha + \beta < 1$.

The results in Figure 1 are known. Our results contain these. Moreover, we have the results in Figure 2.

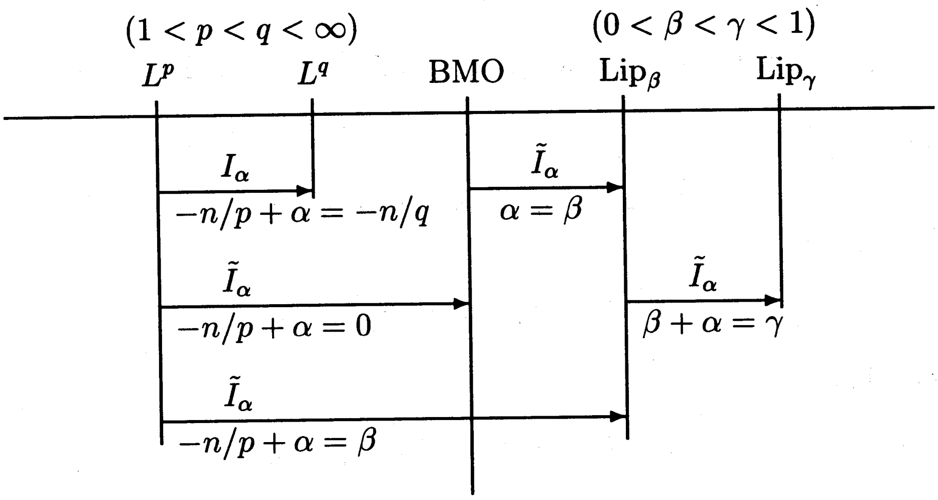

FIGURE 1. Boundedness of fractional integrals

We can also state our results on spaces of homogeneous type with appropriate conditions.

4. PROOFS

Let Φ be a Young function. By the convexity and $\Phi(0) = 0$, we have

$$\Phi(r) \le \frac{r}{s}\Phi(s) \quad \text{for } r \le s. \tag{4.1}$$

Let $\tilde{\Phi}$ be the complementary function with respect to Φ. Then

$$\tilde{\Phi}\left(\frac{\Phi(r)}{r}\right) \le \Phi(r), \quad r > 0. \tag{4.2}$$

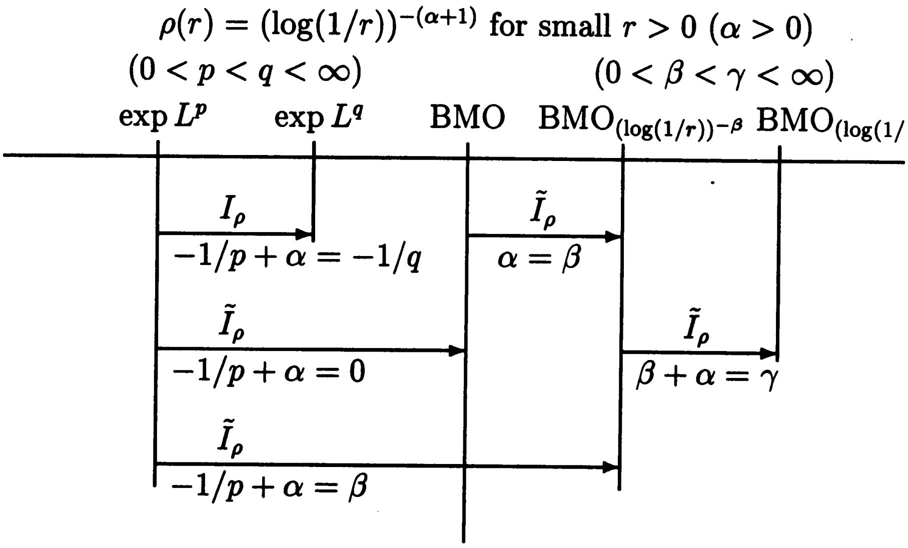

FIGURE 2. Boundedness of generalized fractional integrals

Actually,

$$\frac{\Phi(r)}{r}s - \Phi(s) \le \Phi(r) \quad \text{for } s < r$$

and

$$\frac{\Phi(r)}{r}s - \Phi(s) \le 0 \quad \text{for } s \ge r.$$

We note that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le 2\|f\|_\Phi \|g\|_{\tilde{\Phi}} \tag{4.3}$$

(see for example [9]).

Proof of Theorem 3.1. Let

$$\begin{aligned} J_1 &= \int_{|x-y|<r} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\, dy \quad \text{and} \\ J_2 &= \int_{|x-y|\ge r} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\, dy. \end{aligned}$$

Let

$$h(r) = \inf\left\{\frac{\rho(s)}{s^n} : s \le r\right\}, \quad r > 0.$$

Then h is nonincreasing. It follows that

$$\int_{|x-y|<r} |f(y)| h(|x-y|)\, dy \le Mf(x) \int_{|x-y|<r} h(|x-y|)\, dy$$

(see Stein[12, p.57]). Since $h(r) \sim \rho(r)/r^n$,

$$|J_1| \le CMf(x) \int_{|x-y|<r} \frac{\rho(|x-y|)}{|x-y|^n}\, dy \le CMf(x) \int_0^r \frac{\rho(t)}{t}\, dt. \tag{4.4}$$

Next we estimate J_2. By (4.3) we have

$$|J_2| \le 2 \left\| \frac{\rho(|x-\cdot|)}{|x-\cdot|^n} \chi_{B(x,r)^{\mathrm{C}}}(\cdot) \right\|_{\widetilde{\Phi}} \|f\|_{\Phi}, \tag{4.5}$$

where $\chi_{B(x,r)^{\mathrm{C}}}$ is the characteristic function of the complement of $B(x,r)$. Let

$$F(r) = \int_0^r \frac{\rho(s)}{s}\, ds\; \Phi^{-1}\left(\frac{1}{r^n}\right). \tag{4.6}$$

We show

$$\left\| \frac{\rho(|x-\cdot|)}{|x-\cdot|^n} \chi_{B(x,r)^{\mathrm{C}}}(\cdot) \right\|_{\widetilde{\Phi}} \le CF(r). \tag{4.7}$$

From (2.2) and the increasingness of $\widetilde{\Phi}$ it follows that

$$\int_{|x-y|\ge r} \widetilde{\Phi}\left(\frac{\rho(|x-y|)}{\lambda |x-y|^n}\right) dy \le C_2 \int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{\lambda t^n}\right) t^{n-1}\, dt, \tag{4.8}$$

where C_2 is independent of $\lambda > 0$, $r > 0$ and $x \in \mathbb{R}^n$. We may assume that $C_2 A' \ge 1$. By (4.1) and (3.1) we have

$$\begin{aligned} &\int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{C_2 A A' F(r) t^n}\right) t^{n-1}\, dt \\ &\qquad \le \frac{1}{C_2 A'} \int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{A F(r) t^n}\right) t^{n-1}\, dt \le \frac{1}{C_2}. \end{aligned} \tag{4.9}$$

Let $\lambda = C_2 A A' F(r)$. Then, by (4.8) and (4.9) we have

$$\int_{|x-y|\ge r} \widetilde{\Phi}\left(\frac{\rho(|x-y|)}{\lambda |x-y|^n}\right) dy \le 1,$$

and so (4.7). By (4.4), (4.5) and (4.7) we have

$$|I_\rho f(x)| = |J_1 + J_2| \le C\left(Mf(x) + \|f\|_\Phi \Phi^{-1}\left(\frac{1}{r^n}\right)\right) \int_0^r \frac{\rho(t)}{t}\, dt. \tag{4.10}$$

Choose $r > 0$ so that

$$\Phi^{-1}\left(\frac{1}{r^n}\right) = \frac{Mf(x)}{C_0 \|f\|_\Phi}. \tag{4.11}$$

Then

$$\int_0^r \frac{\rho(t)}{t}\, dt \le A'' \frac{\Psi^{-1}\left(\frac{1}{r^n}\right)}{\Phi^{-1}\left(\frac{1}{r^n}\right)} = A'' \frac{\Psi^{-1} \circ \Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right)}{\frac{Mf(x)}{C_0\|f\|_\Phi}}. \tag{4.12}$$

By (4.10), (4.11) and (4.12) we have

$$|I_\rho f(x)| \le C_1 \|f\|_\Phi \Psi^{-1} \circ \Phi \left(\frac{Mf(x)}{C_0 \|f\|_\Phi} \right).$$

Therefore we have (3.3).

Let C_0 be as in (2.8). Then

$$\begin{aligned}\sup_{r>0} \Psi(r)\, m \left(r, \frac{|I_\rho f(x)|}{C_1 \|f\|_\Phi} \right) &= \sup_{r>0} r\, m \left(r, \Psi \left(\frac{|I_\rho f(x)|}{C_1 \|f\|_\Phi} \right) \right) \\ &\le \sup_{r>0} r\, m \left(r, \Phi \left(\frac{Mf(x)}{C_0 \|f\|_\Phi} \right) \right) = \sup_{r>0} \Phi(r)\, m \left(r, \frac{Mf(x)}{C_0 \|f\|_\Phi} \right) \le 1,\end{aligned}$$

i.e.

$$\|I_\rho f\|_{\Psi, weak} \le C_1 \|f\|_\Phi.$$

Let C_0 be as in (2.9). Then

$$\int_{\mathbb{R}^n} \Psi \left(\frac{|I_\rho f(x)|}{C_1 \|f\|_\Phi} \right) dx \le \int_{\mathbb{R}^n} \Phi \left(\frac{Mf(x)}{C_0 \|f\|_\Phi} \right) dx \le 1,$$

i.e.

$$\|I_\rho f\|_\Psi \le C_1 \|f\|_\Phi. \quad \square$$

Proof of Corollary 3.2. Let $F(r)$ be as (4.6). By the almost decreasingness of $F(r)$ we have

$$F(t) \le CF(r) \quad \text{for } 0 < r \le t < +\infty.$$

By (3.4) we have

$$\frac{1}{t^n} \ge \frac{\rho(t)}{C' \int_0^t (\rho(s)/s)\, ds\, t^n}.$$

From (4.1) and (4.2) it follows that

$$\begin{aligned}
\widetilde{\Phi}\left(\frac{\rho(t)}{CC'F(r)t^n}\right) &\le \frac{F(t)}{CF(r)}\widetilde{\Phi}\left(\frac{\rho(t)}{C'F(t)t^n}\right) \\
&= \frac{F(t)}{CF(r)}\widetilde{\Phi}\left(\frac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,\Phi^{-1}(1/t^n)\,t^n}\right) \\
&\le \frac{F(t)}{CF(r)}\widetilde{\Phi}\left(\frac{\dfrac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,t^n}}{\Phi^{-1}\left(\dfrac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,t^n}\right)}\right) \\
&\le \frac{F(t)}{CF(r)}\frac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,t^n} = \frac{1}{CC'F(r)}\frac{\rho(t)}{t^n}\Phi^{-1}\left(\frac{1}{t^n}\right).
\end{aligned}$$

By (3.5) we have (3.1). Therefore this corollary follows from Theorem 3.1. □

Lemma 4.1. *Let Φ be a Young function with* (2.7) *and $\widetilde{\Phi}$ be the complementary function with respect to Φ. Then there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,*

$$\|\chi_{B(a,r)}\|_{\widetilde{\Phi}} \le C\Phi^{-1}\left(\frac{1}{r^n}\right)r^n.$$

Proof. Let $\lambda = \Phi^{-1}(1/|B(a,r)|)|B(a,r)|$. Then we have by (4.2)

$$\begin{aligned}
\int_{\mathbb{R}^n}\widetilde{\Phi}\left(\frac{\chi_{B(a,r)}(x)}{\lambda}\right)dx &= \int_{B(a,r)}\widetilde{\Phi}\left(\frac{1}{\lambda}\right)dx \\
&= |B(a,r)|\widetilde{\Phi}\left(\frac{\frac{1}{|B(a,r)|}}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)}\right) \le 1. \quad \square
\end{aligned}$$

Proof of Theorem 3.3. First we note that there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,

$$\left\|\frac{\rho(|a-\cdot|)}{|a-\cdot|^{n+1}}\chi_{B(a,r)^C}(\cdot)\right\|_{\widetilde{\Phi}} \le C\frac{1}{r}\int_0^r\frac{\rho(t)}{t}\,dt\,\Phi^{-1}\left(\frac{1}{r^n}\right). \tag{4.13}$$

We have this inequality (4.13) by (3.7) in a way similar to the proof of (4.7), For any ball $B = B(a,r)$, let $\tilde{B} = B(a,2r)$ and

$$E_B(x) = \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n}\right) dy,$$

$$C_B = \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right) dy,$$

$$E_B{}^1(x) = \int_{\tilde{B}} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\, dy,$$

$$E_B{}^2(x) = \int_{\tilde{B}^C} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}\right) dy.$$

Then

$$\tilde{I}_\rho f(x) - C_B = E_B(x) = E_B{}^1(x) + E_B{}^2(x) \quad \text{for } x \in B.$$

By (2.6) we have

$$\left|\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right| \le \begin{cases} C, & |y| \le 2|a|, \\ C|a|\dfrac{\rho(|y|)}{|y|^{n+1}}, & |y| \ge 2|a|. \end{cases}$$

From (4.3) and (4.13) it follows that C_B is well defined. By (4.3), Lemma 4.1 and (3.8) we have

$$\begin{aligned}
\int_{\tilde{B}}\left(\int_B |f(y)|\frac{\rho(|x-y|)}{|x-y|^n}\, dx\right) dy &\le \int_{\tilde{B}} |f(y)| \left(\int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n}\, dx\right) dy \\
&\le \int_{\tilde{B}} |f(y)|\, dy \int_0^{3r} \frac{\rho(t)}{t}\, dt \le C\|f\|_\Phi \|\chi_{\tilde{B}}\|_{\tilde{\Phi}} \int_0^r \frac{\rho(t)}{t}\, dt \\
&\le C\|f\|_\Phi \Phi^{-1}\left(\frac{1}{r^n}\right) r^n \int_0^r \frac{\rho(t)}{t}\, dt \le C\phi(r) r^n \|f\|_\Phi.
\end{aligned}$$

From Fubini's theorem it follows that $E_B{}^1$ is well defined and that

$$\int_B |E_B{}^1(x)|\, dx \le C\phi(r) r^n \|f\|_\Phi. \tag{4.14}$$

By (2.6) we have

$$\left|\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}\right| \le C\frac{|a-x|\rho(|a-y|)}{|a-y|^{n+1}}, \quad x \in B \text{ and } y \in \tilde{B}^C.$$

From (4.3), (4.13) and (3.8) it follows that $E_B{}^2$ is well defined and

$$|E_B{}^2(x)| \le C\phi(r)\|f\|_\Phi. \tag{4.15}$$

By (4.14) and (4.14) we have

$$\frac{1}{|B|}\int_B |\tilde{I}_\rho f(x) - C_B|\,dx \le C\phi(r)\|f\|_\Phi,$$

and

$$\|\tilde{I}_\rho f\|_{\mathrm{BMO}_\phi} \le C\|f\|_\Phi. \quad \square$$

Lemma 4.2. *Under the assumption in Theorem 3.4, there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,*

$$\int_{B(a,r)^C} \frac{\rho(|a-y|)}{|a-y|^{n+1}}|f(y)|\,dy \le C\frac{\psi(r)}{r}\|f\|_{L_{1,\phi}}.$$

Proof. By (3.9) and (3.10), we have

$$\begin{aligned}
\int_{B(a,r)^C} \frac{\rho(|a-y|)}{|a-y|^{n+1}}|f(y)|\,dy &= \sum_{j=1}^{+\infty}\int_{2^{j-1}r\le|a-y|\le 2^j r} \frac{\rho(|a-y|)}{|a-y|^{n+1}}|f(y)|\,dy \\
&\le C\sum_{j=1}^{+\infty}\frac{\rho(2^j r)}{(2^j r)^{n+1}}\int_{B(a,2^j r)}|f(y)|\,dy \le C\sum_{j=1}^{+\infty}\frac{\rho(2^j r)\phi(2^j r)}{2^j r}\|f\|_{L_{1,\phi}} \\
&\sim \int_r^{+\infty}\frac{\rho(t)\phi(t)}{t^2}\|f\|_{L_{1,\phi}} \le C\int_0^r \frac{\rho(t)}{t}\,dt\frac{\phi(r)}{r}\|f\|_{L_{1,\phi}} \\
&\le C\frac{\psi(r)}{r}\|f\|_{L_{1,\phi}}. \quad \square
\end{aligned}$$

Proof of Theorem 3.4. For any ball $B = B(a,r)$, let $\tilde{B} = B(a,2r)$ and

$$\begin{aligned}
E_B(x) &= \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n}\right)dy, \\
C_B &= \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right)dy, \\
{E_B}^1(x) &= \int_{\tilde{B}} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\,dy, \\
{E_B}^2(x) &= \int_{\tilde{B}^C} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}\right)dy.
\end{aligned}$$

Then

$$\tilde{I}_\rho f(x) - C_B = E_B(x) = {E_B}^1(x) + {E_B}^2(x) \quad \text{for } x \in B.$$

By (2.6) we have

$$\left|\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right| \le \begin{cases} C, & |a-y| \le \max(2|a|, 2r) \\ C|a|\dfrac{\rho(|a-y|)}{|a-y|^{n+1}}, & |a-y| \ge \max(2|a|, 2r). \end{cases}$$

From Lemma 4.2 it follows that C_B is well defined. By (3.10) we have

$$\begin{aligned} &\int_{\tilde{B}} \left(\int_B |f(y)| \frac{\rho(|x-y|)}{|x-y|^n}\, dx \right) dy \\ &\quad \le \int_{\tilde{B}} |f(y)| \left(\int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n}\, dx \right) dy \\ &\quad \le \int_{\tilde{B}} |f(y)|\, dy \int_0^{3r} \frac{\rho(t)}{t}\, dt \le C\|f\|_{L_{1,\phi}} r^n \phi(r) \int_0^r \frac{\rho(t)}{t}\, dt \\ &\quad \le C\|f\|_{L_{1,\phi}} r^n \psi(r). \end{aligned}$$

From Fubini's theorem it follows that ${E_B}^1$ is well defined and that

$$\int_B |{E_B}^1(x)|\, dx \le C\psi(r) r^n \|f\|_{L_{1,\phi}}. \tag{4.16}$$

From (2.6) and Lemma 4.2 it follows that ${E_B}^2$ is well defined and

$$|{E_B}^2(x)| \le C\psi(r)\|f\|_{L_{1,\phi}}. \tag{4.17}$$

By (4.16) and (4.17) we have

$$\frac{1}{|B|}\int_B \left| \tilde{I}_\rho f(x) - C_B \right| dx \le C\psi(r)\|f\|_{L_{1,\phi}},$$

and

$$\|\tilde{I}_\rho f\|_{\mathcal{L}_{1,\psi}} \le C\|f\|_{L_{1,\phi}}. \quad \square$$

Lemma 4.3. *If ρ satisfies* (2.1), (2.2), (2.5) *and* (2.6), *then*

$$\frac{\rho(|x_1-y|)}{|x_1-y|^n} - \frac{\rho(|x_2-y|)}{|x_2-y|^n} \tag{4.18}$$

is integrable on $\mathbb{R}^n$ as a function of y and the value is equal to 0 *for every choice of x_1 and x_2.*

Proof. Let $r = |x_1 - x_2|$. For large $R > 0$, let

$$J_1 = \int_{B(x_1,R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n}\,dy - \int_{B(x_2,R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n}\,dy,$$

$$J_2 = \int_{B(x_1,R+r)\setminus B(x_1,R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n}\,dy - \int_{B(x_1,R+r)\setminus B(x_2,R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n}\,dy,$$

$$J_3 = \int_{B(x_1,R+r)^C} \left(\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n}\right) dy.$$

Then

$$J_1 + J_2 + J_3 = \int_{\mathbb{R}^n} \left(\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n}\right) dy.$$

From (2.1) it follows that $\dfrac{\rho(|x_i - y|)}{|x_i - y|^n}$ $(i = 1, 2)$ are in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that $J_1 = 0$. By (2.6) we have

$$\int_{B(x_1,R+r)^C} \left|\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n}\right| dy.$$

$$\leq \int_{B(x_1,R+r)^C} A_3 r \frac{\rho(|x_1 - y|)}{|x_1 - y|^{n+1}}\,dy = Cr \int_{R+r}^{+\infty} \frac{\rho(t)}{t^2}\,dt.$$

From (2.5) it follows that (4.18) is integrable and that $|J_3| \to 0$ as $R \to +\infty$. By (2.2) and (2.5) we have

$$|J_2| \leq \int_{B(x_1,R+r)\setminus B(x_1,R-r)} \left(\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} + \frac{\rho(|x_2 - y|)}{|x_2 - y|^n}\right) dy$$

$$\sim ((R + r)^n - (R - r)^n)\frac{\rho(R)}{R^n} \leq Cr\frac{\rho(R)}{R} \to 0 \quad \text{as } R \to +\infty. \qquad \square$$

Lemma 4.4. *Under the assumption in Theorem 3.6, there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,*

$$\int_{B(a,r)^C} \frac{\rho(|a - y|)}{|a - y|^{n+1}} \left|f(y) - f_{B(a,r)}\right| dy \leq C\frac{\psi(r)}{r}\|f\|_{\mathcal{L}_{1,\phi}}.$$

Proof. By the doubling condition of ϕ we have

$$|f_{B(a,2^k r)} - f_{B(a,2^{k+1}r)}| \leq \frac{1}{|B(a, 2^k r)|} \int_{B(a,2^k r)} |f(y) - f_{B(a,2^{k+1}r)}|\,dy$$

$$\leq \frac{1}{|B(a, 2^k r)|} \int_{B(a,2^{k+1}r)} |f(y) - f_{B(a,2^{k+1}r)}|\,dy$$

$$\leq 2^n \phi(2^{k+1}r)\|f\|_{\mathcal{L}_{1,\phi}} \leq C \int_{2^k r}^{2^{k+1}r} \frac{\phi(s)}{s}\,ds\|f\|_{\mathcal{L}_{1,\phi}},$$

for $k = 0, 1, \cdots, j-1$, and so

$$\begin{aligned}
&\frac{1}{|B(a,2^j r)|}\int_{B(a,2^j r)} |f(y) - f_{B(a,r)}|\,dy \\
&\quad\le \frac{1}{|B(a,2^j r)|}\int_{B(a,2^j r)} |f(y) - f_{B(a,2^j r)}|\,dy + |f_{B(a,r)} - f_{B(a,2^j r)}| \\
&\qquad\qquad\qquad\qquad \le C\int_r^{2^j r} \frac{\phi(s)}{s}\,ds\,\|f\|_{\mathcal{L}_{1,\phi}}.
\end{aligned}$$

Hence, using (2.5) and (3.15), we have

$$\begin{aligned}
&\int_{B(a,r)^C} \frac{\rho(|a-y|)}{|a-y|^{n+1}}\left|f(y) - f_{B(a,r)}\right| dy \\
&\quad= \sum_{j=1}^{+\infty}\int_{2^{j-1}r\le|a-y|\le 2^j r} \frac{\rho(|a-y|)}{|a-y|^{n+1}}\left|f(y) - f_{B(a,r)}\right| dy \\
&\quad\le C\sum_{j=1}^{+\infty}\frac{\rho(2^j r)}{(2^j r)^{n+1}}\int_{B(a,2^j r)} \left|f(y) - f_{B(a,r)}\right| dy \\
&\le C\sum_{j=1}^{+\infty}\frac{\rho(2^j r)}{2^j r}\int_r^{2^j r}\frac{\phi(s)}{s}\,ds\,\|f\|_{\mathcal{L}_{1,\phi}} \sim \int_r^{+\infty}\frac{\rho(t)}{t^2}\left(\int_r^{2t}\frac{\phi(s)}{s}\,ds\right)dt\,\|f\|_{\mathcal{L}_{1,\phi}} \\
&= \int_r^{+\infty}\left(\int_{s/2}^{+\infty}\frac{\rho(t)}{t^2}\,dt\right)\frac{\phi(s)}{s}\,ds\,\|f\|_{\mathcal{L}_{1,\phi}} \le C\int_r^{+\infty}\frac{\rho(s)}{s}\frac{\phi(s)}{s}\,ds\,\|f\|_{\mathcal{L}_{1,\phi}} \\
&\qquad\qquad \le C\int_0^r \frac{\rho(t)}{t}\,dt\,\frac{\phi(r)}{r}\,\|f\|_{\mathcal{L}_{1,\phi}} \le C\frac{\psi(r)}{r}\,\|f\|_{\mathcal{L}_{1,\phi}}. \qquad \square
\end{aligned}$$

Proof of Theorem 3.6. For any ball $B = B(a,r)$, let $\tilde{B} = B(a,2r)$ and

$$\begin{aligned}
E_B(x) &= \int_{\mathbb{R}^n} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n}\right) dy, \\
{C_B}^1 &= \int_{\mathbb{R}^n} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right) dy, \\
{C_B}^2 &= \int_{\mathbb{R}^n} f_{\tilde{B}}\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right) dy, \\
{E_B}^1(x) &= \int_{\tilde{B}} (f(y) - f_{\tilde{B}})\frac{\rho(|x-y|)}{|x-y|^n}\,dy, \\
{E_B}^2(x) &= \int_{\tilde{B}^C} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}\right) dy.
\end{aligned}$$

$$\tilde{I}_\rho f(x) - (C_B{}^1 + C_B{}^2) = E_B(x) = E_B{}^1(x) + E_B{}^2(x) \quad \text{for } x \in B.$$

By (2.6) we have

$$\left| \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right| \leq \begin{cases} C, & |a-y| \leq \max(2|a|, 2r) \\ C|a| \dfrac{\rho(|a-y|)}{|a-y|^{n+1}}, & |a-y| \geq \max(2|a|, 2r). \end{cases}$$

From Lemma 4.4 it follows that $C_B{}^1$ is well defined. By Lemma 4.3 and (2.1) we have

$$\int_{\mathbb{R}^n} \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right) dy = \int_{\mathbb{R}^n} \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)}{|y|^n} \right) dy + \int_{B_0} \frac{\rho(|y|)}{|y|^n} dy = C.$$

By (3.16) we have

$$\begin{aligned} \int_{\tilde{B}} \left(\int_B |f(y) - f_{\tilde{B}}| \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy &\leq \int_{\tilde{B}} |f(y) - f_{\tilde{B}}| \left(\int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy \\ &\leq \int_{\tilde{B}} |f(y) - f_{\tilde{B}}| \, dy \int_0^{3r} \frac{\rho(t)}{t} dt \leq C\|f\|_{\mathcal{L}_{1,\phi}} r^n \phi(r) \int_0^r \frac{\rho(t)}{t} dt \\ &\leq C\|f\|_{\mathcal{L}_{1,\phi}} r^n \psi(r). \end{aligned}$$

From Fubini's theorem it follows that $E_B{}^1$ is well defined and that

$$\int_B |E_B{}^1(x)| \, dx \leq C\psi(r) r^n \|f\|_{\mathcal{L}_{1,\phi}}. \tag{4.19}$$

From (2.6), Lemma 4.4 and (3.16) it follows that $E_B{}^2$ is well defined and

$$|E_B{}^2(x)| \leq C\psi(r)\|f\|_{\mathcal{L}_{1,\phi}}. \tag{4.20}$$

By (4.19) and (4.20) we have

$$\frac{1}{|B|} \int_B |\tilde{I}_\rho f(x) - (C_B{}^1 + C_B{}^2)| \, dx \leq C\psi(r)\|f\|_{\mathcal{L}_{1,\phi}},$$

and

$$\|\tilde{I}_\rho f\|_{\mathcal{L}_{1,\psi}} \leq C\|f\|_{\mathcal{L}_{1,\phi}}. \quad \square$$

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