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On global well-posedness of some nonlinear dispersive equations for rough data

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In this note, I consider the global well-posedness of dispersive equations for rough initial data. Most part of note concerns the results on the Cauchy problem of the KdV equation:

$$\begin{cases} u_t + u_{xxx} + (u^2)_x = 0, \\ u(0) = u_0. \end{cases}$$

The argument below has an application to another dispersive wave equations, but we shall not deal so here.

Anyway, my goal is to present the sharp global well-posedness for the KdV equation. To be clear, I would like to construct the content of note with three sections. Section one recalls the work of the local well-posedness through the Fourier restriction norm method. Next two sections are devoted to the extention of the local solution obtained in section one to global one. In traditionally, it is well-tried that the global well-posedness in the finite energy space. Recently, J. Bourgain introduced quite interesting story for the global well-posedness for much rough data. Section two will recall his method and try to apply this technique to the KdV equation. In finally, I want to develop the argument of Bourgain to lead the sharp global well-posedness of KdV equation, which is the main result in this note. The work in section three is a joint study with J. Colliander, M. Keel, G. Staffilani and T. Tao.

1 Local well-posedness by the Fourier restriction norm method

For the well-posedness for dispersive wave equations, such as the KdV equation and the nonlinear Schrödinger equation, there has been a remarkable progress in recent year. This section will recall the refined performance of the local well-posedness for the KdV equation induced by the Fourier restriction norm method. The Fourier restriction norm for the KdV equation is tailored with linear KdV equation as follows [1]:

$$\|u\|_{X_{s,b}} = \left(\int \langle \xi \rangle^{2s} \langle \tau - \xi^3 \rangle^{2b} |\widehat{u}(\tau, \xi)|^2 d\tau d\xi \right)^{\frac{1}{2}},$$

where we denote the Fourier transform in t and x of u by $\widehat{u}$. The interest aims not only the line case, but also the periodic boundary condition case through this note. Then the problem on the line case refers the measure $d\xi$ as the usual Lebesgue measure on $\mathbb{R}$. On the other hand, the counting measure on integer is taken to the periodic boundary condition case. Through this note, we often abbreviate $\|\cdot\|_{X_{s,b}}$ as $\|\cdot\|_{s,b}$.

In ordinary way, by Duhamel formula, we attempt to solve the integral equation associated with KdV equation:

$$u(t) = e^{-t\partial_{xxx}} u_0 - \int_0^t e^{-(t-t')\partial_{xxx}} (u^2)_x(t') dt'. \tag{1}$$

Now, we start by making a plan for the estimate of the right hand side of this integral equation (1). The point of the proof is how handle the nonlinear term. The standard computation estimates the Duhamel term of (1) by $\|(u^2)_x\|_{s,b-1}$ for $b > \frac{1}{2}$ [1], [13, 15]. Once the following bilinear estimate holds, which leads the local well-posedness of the Cauchy problem for the KdV equation by the usual contraction argument:

$$\|(uv)_x\|_{s,b-1} \lesssim \|u\|_{s,b} \|v\|_{s,b}. \tag{2}$$

Problem 1 *Can one have the bilinear estimate (2) ? Show when the estimate (2) holds.*

Problems of this kind has been introduced first by Bourgain [1] and by C. E. Kenig, G. Ponce and L. Vega [13, 15].

Proposition 1.1 ([1], [13, 15]) *For $s > -\frac{3}{4}$, there exists $b > \frac{1}{2}$ such that (2) for the line case holds.*

Remark 1.1 *Precisely, the paper of [1] focused in the case of $s = 0$. After this, the proof was improved by the paper of [13, 15], which reaches the value $s > -\frac{3}{4}$ of proposition 1.1.*

Periodic boundary condition case. Proposition 1.1 refers the evaluation of bilinear estimate (2) for the line case. It is also emphasized that the similar problem is developed by the same papers [1], [13, 15] for the periodic boundary condition problem.

Proposition 1.2 ([1], [13, 15]) *Let $b = \frac{1}{2}$. For $s \geq -\frac{1}{2}$, the estimate (2) for the periodic boundary condition case holds.*

As stated before, the bilinear estimate (2) leads the results of local well-posedness. However, in contrast to the line case, the proof for the periodic boundary condition case based on the estimate (2) needs the argument on some variant bilinear estimate together with (2), because, for instance, $H_t^b \not\subset L_t^\infty$ for $b = \frac{1}{2}$. Although, as far as we work for $s > -\frac{1}{2}$, not including the equal case, the paper [15] could show the local well-posedness in $H^s(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Remark 1.2 *The result of the local well-posedness in $H^s(\mathbb{T})$ for the end point $s = -\frac{1}{2}$ was recently solved by J. Colliander, M. Keel, G. Staffilani, T. Tao and the author [9].*

Now, we turn out attention to the opposite view on the failure of estimate (2). The same paper of [15] proved the following results, where they constructed the counterexample.

Proposition 1.3 ([15]) *For any $s < -\frac{3}{4}$ (resp. $s < -\frac{1}{2}$) and any $b \in \mathbb{R}$, the estimate (2) for the line case (resp. periodic boundary condition case) breaks down.*

It is remarked that their examples do not cover the end point $s = -\frac{3}{4}$. The problem whether $s = -\frac{3}{4}$ was left open. Recently, this problem is fixed by K. Nakanishi, Y. Tsutsumi and the author [23].

Proposition 1.4 ([23]) *The bilinear estimate (2) of the line case fails for $s = -\frac{3}{4}$ and any $b \in \mathbb{R}$.*

Of cause, it is not enough to conclude the ill-posedness results of KdV equation in $H^s(\mathbb{R})$ for $s < -\frac{3}{4}$, also for $s = -\frac{3}{4}$ by the aid of Propositions 1.3 and 1.4. We can find the results not only the failure of (2) but also the construction of the exact example for the ill-posedness of KdV equation.

Proposition 1.5 ([17]) *The Cauchy problem of KdV equation for the line case is ill-posed in H^s for $s < -\frac{3}{4}$.*

For this proposition, it is open whether the case of $s = -\frac{3}{4}$ is well-posed or ill-posed. It is noted that the paper [2] gave the explicit ill-posedness results of KdV equation for the periodic boundary condition case. Moreover, the papers [1, 2] and [14, 15, 17] covered the results for the modified KdV equation:

$$u_t + u_{xxx} \pm (u^3)_x = 0. \tag{3}$$

As one guesses, the trilinear estimate is treated there; for example,

$$\|(uvw)_x\|_{s,b-1} \lesssim \|u\|_{s,b}\|v\|_{s,b}\|w\|_{s,b}. \tag{4}$$

Their results are sketched in the following table:

	well-posedness	ill-posedness	
line KdV	$s > -\frac{3}{4}$	$s < -\frac{3}{4}$	(2) fails for $s \leqq -\frac{3}{4}$
periodic KdV	$s \geqq -\frac{1}{2}$	$s < -\frac{1}{2}$	(2) fails for $s < -\frac{1}{2}$
line mKdV	$s \geqq \frac{1}{4}$	$s < \frac{1}{4}$	(4) fails for $s < \frac{1}{4}$
periodic mKdV	$s \geqq \frac{1}{2}$	$s < \frac{1}{2}$	(4) fails for $s < \frac{1}{2}$

I complete this section by denoting the best known results for the local well-posedness of KdV equation by Kenig-Ponce-Vega [15].

Theorem 1.1 ([15]) *The Cauchy problem of KdV equation for the line case is locally well-posed in H^s for $s > -\frac{3}{4}$.*

2 Global well-posedness of KdV equation for the line case

It is natural to extend the local solution to global one, once the local solution is obtained. One may expect the global solution via the iteration of the proof of local well-posedness. However, iteration methods can not by themselves yield the global solution. In general, the proof of global well-posedness relies on the combination of the proof of local well-posedness and the a priori estimate of solution. It is known that the conserved quantities are available for providing the a priori estimate. Indeed, the use of L^2 conservation law for KdV equation leads the following theorem.

Theorem 2.1 ([1]) *The Cauchy problem of KdV equation for the line case is globally well-posed in H^s for $s \geq 0$.*

But, there is none that the global well-posedness below L^2, because of the lack of conservation law for the KdV equation. As for the results below L^2, we have the following theorem, which is a joint study with J. Colliander and G. Staffilani.

Theorem 2.2 ([7]) *Let $\frac{a}{12} < s < 0$ and $a \geq -\frac{3}{4}$. Then the Cauchy problem of KdV equation for the line case is globally well-posed in $H^s \cap \dot{H}^a$, where the space $\dot{H}^a$ denotes the usual homogeneous Sobolev space of order a.*

The proof of theorem is based on the new argument introduced by Bourgain [3]. He showed that the Cauchy problem of two dimensional nonlinear Schrödinger equation with the L^2 critical nonlinearity:

$$i\partial_t u + \Delta u = u|u|^2, \quad (t,x) \in \mathbb{R}^{1+2}, \tag{5}$$

was globally well-posed in H^s between L^2 and H^1 determined by the conservation laws. Let V_t and $V(t)$ denote the flow maps of the nonlinear and the linear Schrödinger equations, respectively. Let X and Y be Sobolev spaces such that

$$\begin{aligned} & X \subsetneq Y, \\ & X : \text{conservation law class}, \\ & Y : \text{initial data space}. \end{aligned}$$

His strategy is that if $(V_t - V(t))u_0 \in X$ for $\forall t \in \mathbb{R}$, then we have the global well-posedness in Y. Roughly speaking, his argument aims to estimate the high Sobolev norm of solution by the low Sobolev norm, which tells us the information of a spread of energy between the low frequency and the high frequency. He applies this argument to (5) with $X = H^1$ and $Y = H^s$. We follow his strategy and try to have

$$(S_t - S(t))u_0 \in L^2, \tag{6}$$

for $t \in \mathbb{R}$, where S_t denotes the flow map of KdV equation and $S(t) = e^{-t\partial_{xxx}}$, because of $X = L^2$ and $Y = H^s$ in our case.

There are some differences between [3] and our problem. In [3], the nonlinearity without the derivative was considered in the usual Sobolev space H^s with the positive index. For the case of the KdV equation, the difficulty of derivative loss stems from the derivative nonlinearity of KdV equation. Moreover in [7], we consider the well-posedness for the negative index s of H^s. To show the well-posedness, we use the Fourier restriction norm method, which was also applied to the local well-posedness for KdV. When we follow one and the same argument as Bourgain's with the Fourier restriction norm method, it is required to show the following bilinear estimate for some $\gamma < 0$:

$$\|(vw)_x\|_{0,b-1} \leq c\|v\|_{0,b}\|w\|_{\gamma,b}. \tag{7}$$

A glance of (7) recalls us the bilinear estimate (2) in section one. Unfortunately, unless $\gamma \geq 0$, the above estimate (7) fails for any $c > 0$ and any $b \in \mathbb{R}$, where there is a counter example. One derivative can be regained by the smoothing effect of KdV equation, but we have to gain the derivative of order more than one for the above bilinear estimate (7). In [3], the smoothing effect of Schrödinger equation was available, where there is no derivative nonlinearity in (5). But the above bilinear estimate (7) holds when we restrict the frequency of both v and w to high frequency. The failure occurs in the interaction between the low frequency and the high frequency. This observation motivates us to adopt the framework of homogeneous Sobolev space. Then we use the homogeneous Sobolev space $\dot{H}^s$ of negative index and we try to have

$$S_t u_0 \in \dot{H}^a, \tag{8}$$

for $t \in \mathbb{R}$. We note that the framework of homogeneous space is convenient to the gain of more regularity than in the inhomogeneous space. But the homogeneous Sobolev space of negative order is not conserved for the KdV equation. We have to evaluate precisely the growth order of this norm. This is an essential difference from [3]. A consequence of (6) and (8) is Theorem 2.2.

Roughly speaking, the advantage of [3] is related with the division of solution into three potions:

$$S_t u_0 = S_t(u_0^{\text{low}}) + S(t)u_0^{\text{high}} + (\text{error term}), \tag{9}$$

where u_0^{low} and u_0^{high} mean respectively the low frequency and the high frequency parts of u_0. The evaluation of low frequency associated with the nonlinear flow of KdV equation can be controlled in the conserved space L^2, thanks to the smoothness of u_0^{low}. On the other hand, the evolution of high frequency concerning the linear flow of KdV equation is globally well-posed. Bourgain has demonstrated in his proof that the error term, which stems from the interaction between the low frequency and the high frequency, is very small. Then we conclude that most part of solution is represented by the first two portions of (9), which dominates the behavior of solution. Note that the smallness of error term is derived by the gap of estimate between the high Sobolev norm and the low one of the estimate (6).

Remark 2.1 *The above argument is also applicable to another kind nonlinear dispersive equations and nonlinear wave equations, see [4, 3D-NLS], [12, mKdV], [16, semilinear NLW], [24, KP-2], [25, DNLS], [26, KP-2].*

3 Sharp global well-posedness for the KdV equation

We again come to the position for the global well-posedness of KdV equation below L^2. Concerning the local regularity weaker than L^2 space, we find the result of Theorem 2.2.

Problem 2 *In Theorem 2.2, to obtain a global solution, we have imposed an additional assumption in the homogeneous Sobolev space $\dot{H}^a(\mathbb{R})$. The proof controls the high regularity by low one, so that it seems difficult to cover all local solution in section one. Furthermore, from such a reason, it seems to be difficult to adapt the proof of section two to the periodic boundary condition problem, because this problem has no dispersive smoothing effect of solution.*

The main result in this note is the following theorem, which is a joint work with J. Colliander, M. Keel, G. Staffilani and T. Tao.

Theorem 3.1 (Colliander, Keel, Staffilani, T. and Tao [9]) *The Cauchy problem of KdV for the line case is globally well-posed in H^s for $s > -\frac{3}{4}$.*

This theorem allows us to succeed in solving the sharp global well-posedness for the line case. In the rest of this note, I want to outline of the proof. Our proof is also motivated by the argument of Bourgain in section two. However we do not evaluate the high Sobolev norm of solution by the low one, which is a significant difference from section two. Turn to the proof of theorem, my goal is the following:

Goal 1 *Give the a priori estimate for $E_I^2(t) = \|Iu(t)\|_{L^2}^2$, where I denotes the linear operator from H^s to L^2 defined with the Fourier multiplier $m(\xi)$ such that $m(\xi) = 1$ for $|\xi| \lesssim N$, $m(\xi) = \left|\frac{\xi}{N}\right|^s$ for $|\xi| \gg N$, $m(\xi) \in C^2$. We will mention of $N \gg 1$ soon later.*

It is noted that

$$E_I^2(t) \sim \begin{cases} \|u(t)\|_{L^2}, & \text{if support}\widehat{u} \subset \{|\xi| \lesssim N\}, \\ \|u(t)\|_{H^s}, & \text{if support}\widehat{u} \subset \{|\xi| \gg N\}, \end{cases}$$

where we transfer the Fourier transform in x of $u(t,x)$ by the same notion $\widehat{u}$ as before, for simplicity. Therefore, the quantity $E_I^2(t)$ seems to be looked like the combination of L^2 and H^s norms. The inverse operator I^{-1} leads the a priori estimate in H^s, once the a priori estimate for $E_I^2(t)$ is obtained. So the problem comes down to the time global estimate for $E_I^2(t)$. The standard calculation with the use of KdV equation yields the estimate.

Lemma 3.1 *Let $s > -\frac{3}{4}$. For the solution of KdV equation $u(t)$, we have*

$$\frac{d}{dt}E_I^2(t) = \int M_3(\xi_1,\xi_2,\xi_3)\widehat{u}(t,\xi_1)\widehat{u}(t,\xi_2)\widehat{u}(t,\xi_3)\delta(\xi_1+\xi_2+\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

where

$$M_3(\xi_1,\xi_2,\xi_3) = \xi_1 m(\xi_1)^2 + \xi_2 m(\xi_2)^2 + \xi_3 m(\xi_3)^2.$$

Remark 3.1 *A glance of the above expression shows us the cancellation of the interaction among the low frequencies, since* $M_3(\xi_1, \xi_2, \xi_3) = 0$ *for* $|\xi_i| \ll N$ *under* $\xi_1 + \xi_2 + \xi_3 = 0$. *Thus, the quantity* $E_I^2(t)$ *acts like almost conserved quantity.*

The energy transportation is observed by the following lemma.

Lemma 3.2 *Let* $s > -\frac{3}{4}$. *For the solution of KdV equation* $u(t)$, *we have*

$$\left| \int_0^{T_0} \frac{d}{ds} E_I^2(s) ds \right| \lesssim N^{-1+} \|Iu_0\|_{L^2}^3,$$

where T_0 *denotes the existence time assured by the proof of local well-posedness.*

We may assume that $\|Iu_0\|_{L^2}$ is very small. This request is carried out by the scaling of KdV equation:

$$u_\lambda(t, x) = \frac{1}{\lambda^2} u(\frac{t}{\lambda^3}, \frac{x}{\lambda}),$$

which leads the desired condition:

$$\|Iu_\lambda(0)\|_{L^2} \lesssim \lambda^{-s-\frac{3}{2}} N^{-s} \|Iu_0\|_{L^2} = O(1),$$

for $\lambda \sim N^{-\frac{2s}{3+2s}}$. Moreover, by the proof of local well-posedness, it is all right to take $T_0 = 1$ in Lemma 3.2. Fix $T > 0$, the goal of this section will be replaced as follows.

Goal 2 *Assume* $\|Iu_0\|_{L^2} = O(1)$. *Give the a priori estimate up to time* $t = \lambda^3 T$.

By Lemma 3.2, we have

$$E_I^2(1) - E_I^2(0) \lesssim N^{-1+}. \tag{10}$$

Some iteration method using Lemma 3.2 can controls the spread of energy as far as the correspoinding right hand side of (10) never be greater than $\|Iu_0\|_{L^2} = O(1)$. In order to reach up to $t = \lambda^3 T$, we have to iterate the above procedure at least $\lambda^3 T$ steps. Thus, if $\frac{\lambda^3 T}{N^{1-}} \lesssim 1$, we achieve the goal, which holds for $-\frac{6s}{3+2s} - 1 < 0$, that is, $s > -\frac{3}{8}$ and for large $N > 0$. Here we make a success of the removal of the additional assumption on data in $\dot{H}^a$ of Theorem 2.2. However the gap between the sharp local well-posedness in H^s for $s > -\frac{3}{4}$ and the result for $s > -\frac{3}{8}$ remains yet.

Observation. If $\frac{d}{dt}E_I^2(t)$ can be divided into two portions, $a_1'(t)$ and $b_1(t)$. The first term $a_1'(t)$ dominates the quantity of $\frac{d}{dt}E_I^2(t)$ but $a_1(t)$ is controls by $E_I^2(t)$ (not so big). Therefore the second term $b_1(t)$ will correspond to the the more explicit estimate for $\frac{d}{dt}E_I^2(t)$.

Following this observation, we have obtained the following lemma.

Lemma 3.3 *Let*

$$a_1(t) = \int \frac{M_3(\xi_1,\xi_2,\xi_3)}{\xi_1^3+\xi_2^3+\xi_3^3}\widehat{u}(t,\xi_1)\widehat{u}(t,\xi_2)\widehat{u}(t,\xi_3)\delta(\xi_1+\xi_2+\xi_3)d\xi_1 d\xi_2 d\xi_3,$$

$$b(t) = \frac{d}{dt}\left(E_I^2(t) - a_1(t)\right).$$

Let $s > -\frac{3}{4}$. Then for the solution of KdV equation $u(t)$, we have

$$|a_1(t)| \leq o(E_I^2(t)),$$

$$\left|\int_0^1 b(t)dt\right| \lesssim N^{-\frac{3}{2}+}\|Iu_0\|_{L^2}^4.$$

This lemma brings us the more explicit information of the energy transportation than Lemma 3.2. The analogous argument to above will reduce the relation $\frac{\lambda^3 T}{N^{1-}} \lesssim 1$ to $\frac{\lambda^3 T}{N^{\frac{3}{2}-}} \lesssim 1$, which holds for $s > -\frac{1}{2}$. However, unfortunately, there appears a gap again. It is remarkable that the quantity $E_I^2(t) - a_1(t)$ in Lemma 3.3 corresponds to the H^1 energy conservation law of KdV equation by regarding I as ∂_x. So we recognize that the above argument is based on the use of H^1 conservation law as compered to the proof in section two. As one guesses, the KdV equation has an infinite conservation laws, then we surely develop more computation by means of H^2 conservation law according to above observation.

Lemma 3.4 *Let*

$$\begin{aligned} & M_4(\xi_1,\xi_2,\xi_3,\xi_4) \\ = \; & \frac{M_3(\xi_1,\xi_2,\xi_3+\xi_4)(\xi_3+\xi_4) + (\textit{permutation among } \xi_1,\xi_2,\xi_3,\xi_4)}{6(\xi_1^3+\xi_2^3+\xi_3^3+\xi_4^3)}, \end{aligned}$$

$$a_2(t) = \int M_4(\xi_1,\xi_2,\xi_3,\xi_4)\widehat{u}(t,\xi_1)\widehat{u}(t,\xi_2)\widehat{u}(t,\xi_3)\widehat{u}(t,\xi_4)\delta(\xi_1+\xi_2+\xi_3+\xi_4)$$
$$d\xi_1 d\xi_2 d\xi_3 d\xi_4,$$

$$b_2(t) = \frac{d}{dt}\left(E_I^2(t) - a_1(t) - a_2(t)\right).$$

Let $s > -\frac{3}{4}$. For the solution of KdV equation $u(t)$, we have

$$|a_2(t)| \leq o(E_I^2(t)),$$

$$\left|\int_0^1 \frac{d}{dt} b_2(t)dt\right| \lesssim N^{-3+}\|Iu_0\|_{L^2}^5.$$

The order N^{-3+} of Lemma 3.4 is sufficient to win the sharp global well-posedness of KdV equation for $s > -\frac{3}{4}$, in accordance with the above scheme. O.K.

I would like to complete this note by noting further application of our argument.

Further applications. It is noted that the proof of section three does not need the dispersive smoothing effect of solution directly in contrast to section two. In fact, we have the sharp global well-posedness for the periodic KdV equation [9]. Moreover, the same paper knows the sharp results on the modified KdV equation (3) for both line and periodic boundary condition cases. More details and the application to another type equations are developed in the papers [8, 9, 10, 11].

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