数理解析研究所講究録 1208 巻 2001 年 135-139

Nonrelativistic limit of scattering theory for nonlinear Klein-Gordon equations

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We study the scattering theory in the nonrelativistic limit for the nonlinear Klein-Gordon equation:

$$\ddot{v}/c^2 - \Delta v + c^2 v + f(v) = 0, \tag{1}$$

where $v = v(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}$ is the unknown function, $c \gg 1$ denotes the propagation speed, namely the speed of light, and $f(u) = |u|^p u$ is a given nonlinearity with $p > 0$. Actually we can deal with the power $p \in (4/n, 4/(n-2))$ (without the upper bound when $n \leq 2$). We can easily anticipate from the simpler equation

$$\ddot{v}/c^2 + c^2 v = 0, \tag{2}$$

that the nonrelativistic limit $c \to \infty$ causes time oscillation of the form $e^{\pm ic^2 t}$. So we first eliminate this time oscillation by putting $u := e^{-ic^2 t} v$, which obeys the following modulated equation:

$$\ddot{u}/c^2 + 2i\dot{u} - \Delta u + f(u) = 0. \tag{3}$$

Then we can take the singular limit as $c \to \infty$ to the nonlinear Schrödinger equation:

$$2i\dot{v} - \Delta v + f(v) = 0. \tag{4}$$

Our main goal is to see if we can describe the asymptotic behavior of solutions to (3) via nonrelativistic approximation by (4). It was proved in [4] that every finite energy solution to the Cauchy problem for (3) converges to the corresponding solution of (4) in the energy space, locally uniformly in time. The nonrelativistic limit can not approximate solutions globally in time for the free equation, neither for the nonlinear one in case every solution behaves asymptotically free. Nevertheless, we can show that the wave operators, their inverses and the scattering operator for (3) converge to those for (4). This means that the time-asymptotic behavior can be approximated through the nonrelativistic equation.

Now we breifly recall the most important conserved quantities, namely the energy and the charge, for (3) and (4). The energy for (3) and (4) is given respectively by

$$\begin{aligned} E^c(u) &= \int_{\mathbb{R}^n} |\dot{u}/c|^2 + |\nabla u|^2 + F(u)dx = \text{const.}, \\ E(v) &= \int_{\mathbb{R}^n} |\nabla v|^2 + F(v)dx = \text{const.}, \end{aligned} \tag{5}$$

where $F(u) := 2|u|^{p+2}/(p+2)$. The charge for (3) and (4) is given by

$$\begin{aligned} Q^c(u) &= \int_{\mathbb{R}^n} |u|^2 + \Im \dot{u}\bar{u}/c^2 dx = \text{const.}, \\ Q(v) &= \int_{\mathbb{R}^n} |v|^2 dx = \text{const.} \end{aligned} \tag{6}$$

For any space-time function u, we denote

$$\vec{u} := (u, \dot{u}/c), \tag{7}$$

and define $E := H^1 \oplus L^2$. Then the above conservation laws ensure global bound of solution u to (3) in $\vec{u} \in E$.

Next we define the wave operators for (3) and (4).

Definition 1. The wave operators $W^c_\pm$ for (3) are maps from E into itself which map the initial data $\vec{u}_0(0)$ of any finite energy solution u_0 of the free modulated Klein-Gordon:

$$\ddot{u}_0/c^2 + 2i\dot{u}_0 - \Delta u_0 = 0, \tag{8}$$

into the initial data $(u_\pm(0), \dot{u}_\pm(0))$ of the solution of (3) satisfying

$$\lim_{t\to\pm\infty} \|\vec{u}_\pm(t) - \vec{u}_0(t)\|_E = 0, \tag{9}$$

respectively. Similarly, the wave operators $W_\pm$ for (4) are defined as maps from H^1 into itself which map the initial data of any finite energy solution of the free Schrödinger:

$$2i\dot{v} - \Delta v = 0, \tag{10}$$

into that of (4) asymptotic as $t \to \pm\infty$. We denote $M^*_* := (W^*_*)^{-1}$, $S^c := M^c_+ W^c_-$ and $S := M_+ W_-$.

We review the known results about these wave operators. If $4/n < p < 4/(n-2)$, then $W^c_\pm$ and $W_\pm$ are well defined as bijections, which was proved in [2, 3] for $n \geq 3$ and in [6] for $n \leq 2$. In the lower critical case $p = 4/n$, it is known that $W^c_\pm$ and $W_\pm$ are well defined as injections (see [3]). In the upper critical case $p = 4/(n-2)$,

$W^c_\pm$ is well defined as bijections [5], while $W_\pm$ is known to exist only for radially symmetric data [1] (the nonsymmetric case is an open problem).

Now we can state our main result.

Theorem 2. *Assume $n \in \mathbb{N}$ and $4/n < p < 4/(n-2)$. Let $\Phi^c \in E$ and $\varphi \in H^1$. Suppose*

$$\Phi^c \to (\varphi, 0) \quad \text{in } E, \tag{11}$$

as $c \to \infty$. Then we have

$$\begin{aligned} W^c_\pm \Phi^c &\to (W_\pm \varphi, 0) \quad \text{in } E, \\ M^c_\pm \Phi^c &\to (M_\pm \varphi, 0) \quad \text{in } E, \\ S^c \Phi^c &\to (S\varphi, 0) \quad \text{in } E. \end{aligned} \tag{12}$$

Key ingredients in our proof are a uniform decay estimate in the sense of space-time norms for (8), compactness argument combined with the conservation laws and the uniform Strichartz estimate derived in [4]. We use the space-time norms of the following form:

$$\|u\|_{S\,|\,W \cap K} := \|\chi^c * u\|_S + \|\chi_c * u\|_{W \cap K}, \tag{13}$$

where χ^c smoothly cuts off the higher frequency part $|\xi| \gtrsim c$, which is carried by the latter term $\chi_c * u = u - \chi^c * u$. S, W and K denote the space-time norms of Strichartz type for Schrödinger, wave and Klein-Gordon equations, respectively. The following linear estimate of Strichartz type plays a crucial role.

Lemma 3 ([4]). *Let $U^c(t) := e^{\pm ic\langle\nabla\rangle_c t}$. For any $c > 0$, we have*

$$\|U^c(t)\varphi\|_{S_0\,|\,(W_0 \cap K_0)} \le C\|\varphi\|_{L^2}, \tag{14}$$

$$\left\| \int_0^t U^c(t-s) f(s) ds \right\|_{S_0\,|\,(W_0 \cap K_0)} \le C\|f\|_{S_1'\,|\,(W_1' + K_1')}, \tag{15}$$

where C is a positive constant independent of c, φ and f. S_i, W_i, and K_i denote arbitrary spaces of the form $c^{-\mu} L^p(\mathbb{R}; \dot{B}^\sigma_{q,2})$ satisfying the following conditions. Here $\dot{B}^*_{*,*}$ denotes the homogeneous Besov space. Let $b := 1/p$ and $\alpha := 1/2 - 1/q$. All the spaces S_i, W_i and K_i must obey

$$-2b + n\alpha + \sigma + \mu = 0, \quad 0 \le 2b < 1, \quad 0 \le 2\alpha \le 1, \tag{16}$$

and each space should satisfy

$$S_i: \quad \mu = 0, \qquad 2b \le n\alpha, \tag{17}$$

$$W_i: \quad \mu = b, \qquad 2b \le (n-1)\alpha, \tag{18}$$

$$K_i: \quad \mu = (1+2/n)b, \quad 2b \le n\alpha, \tag{19}$$

respectively. X' denotes the dual space of X.

(16) shows that we have the same scaling both for the lower and the higher frequency parts. A part of regularity is transferred to the weight of c^{-1} in the higher frequency, compared with the lower part. We can recover this lost regularity because we have a kind of regularization property for the higher frequency in the associated integral equation:

$$u = u_0 - \int_{\infty}^{t} e^{ic^2(t-s)} \sin\{c\langle\nabla\rangle_c(t-s)\} \frac{c}{\langle\nabla\rangle_c} f(u(s))ds, \tag{20}$$

where u_0 is the free solution asymptotic to u as $t \to \infty$ and the regularization is caused by the operator $c/\langle\nabla\rangle_c$, where $\langle\xi\rangle_c := \sqrt{|\xi|^2 + c^2}$ and $\varphi(\nabla) := \mathcal{F}^{-1}\varphi(i\xi)\mathcal{F}$ denotes the Fourier multiplier.

We demonstrate the outline of the main estimate in a simple case where $n = 3$ and $p = 2$ (for the general case, see [7]). Then we can choose the norm S for the lower frequency and K for the higher frequency as

$$S := L_t^4(W^{1,3}), \quad K := c^{-5/12} L_t^4(B^{7/12}_{3,2}), \tag{21}$$

where $W^{*,*}$ and $B^{*}_{*,*}$ denotes the inhomogeneous Sobolev and Besov spaces, respectively. We do not need the space of wave type, since the cubic nonlinearity is quite regular for H^1 solution when $n = 3$. Using the Sobolev embedding $W^{1,3} \subset L^6$ and $B^{7/12}_{3,2} \subset L^6$, we can estimate the nonlinearity as

$$\| |u|^2 u \|_{L^{4/3}(W^{1,3/2} + c^{-5/12} B^{7/12}_{3,2})} \lesssim \|u\|^2_{L^4(L^6)} \|u\|_{S|K} \lesssim \|u\|^3_{S|K}. \tag{22}$$

Then we can use the regularizing property of $c/\langle\nabla\rangle_c$ as

$$\|c\langle\nabla\rangle_c^{-1} f(u)\|_{L^{4/3}(W^{1,3/2} | c^{-5/12} B^{7/12}_{3,2})} \lesssim \|f(u)\|_{L^{4/3}(W^{1,3/2} + c^{-5/12} B^{7/12}_{3,2})}. \tag{23}$$

Finally, we obtain by the Strichartz estimate

$$\|u - u_0\|_{S|K(T,\infty)} \lesssim \|u\|^3_{S|K(T,\infty)}. \tag{24}$$

Thus, we deduce uniform estimates for u and large T from the estimate for the free solution u_0, which can be derived from linear decay estimates.

Then the convergence of M_+^c can be proved as follows. Let u be the solution of (3) with $\vec{u}(0) = \Phi^c$, u_0 be the free solution of (8) asymptotic to u as $t \to \infty$, v be the solution of (4) with $v(0) = \varphi$ and v_0 be the free solution of (10) asymptotic to v as $t \to \infty$. What we want to prove is $\vec{u}_0(0) \to (v_0(0), 0)$ in E as $c \to \infty$. From the above argument, we can uniformly approximate u by u_0 and v by v_0 in the energy spaces for $t > T$ and $c > c_0$ with some T and c_0. If we take $c > c_0$ sufficiently large, then, by the time-local convergence result in [4], $\vec{u}(T)$ is very close to $(v(T), 0)$, so is $\vec{u}_0(T)$ to $(v_0(T), 0)$. Taking c large again if necessary, we can approximate $\vec{u}_0(0)$ by $(v_0(0), 0)$, as desired.

For the proof of W^c, we use the compactness argument for the sequence $K^c(-t)\vec{u}(t)$, where $K^c(t)$ denotes the matrix valued free propagator for $\vec{u}_0$.

Our result reflects that the nonrelativistic limit converges globally in space-time norms and that high-frequency modification does not effect the nonlinearity so much. It is also possible to retrace the argument in [6] to derive a uniform global estimate for space-time norms in terms of the energy only. It would be interesting if we could get the same result in the Sobolev critical case $p = 4/(n-2)$, where we know only the estimate dependent on c and global wellposedness for (4) with general data is still open (see [1] for radial data).

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