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Binding through coupling to a radiation field

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1 Introduction

1.1 Definition

This is a joint work with H. Spohn[1]. We consider a system of one electron interacting with a quantized radiation field. In particular we investigate the so called *Pauli-Fierz* [13] model[2]. Although the Pauli-Fierz model is a nonrelativistic model, it correctly describes the interaction between low energy electrons and photons in a sense. Actually the Lamb shift and gyromagnetic ratio shift were described by using the Pauli-Fierz model. See [2, 14, 12].

In this paper we take the dipole approximation for simplicity. Moreover we suppose that the electron is spinless, moves in the d-dimensional space, and has the $d-1$ transverse degrees of freedom. Throughout this paper we assume

$$d \geq 3.$$

The Hamiltonian of the system is of the form

$$H(\alpha)=\frac{1}{2m}(p\otimes I-\alpha I\otimes A)^2+V\otimes I+I\otimes H_{\mathrm{f}} \tag{1.1}$$

acting on the Hilbert space

$$\mathcal{H}:=L^2(\mathbb{R}^d)\otimes\mathcal{F}_{\mathrm{EM}}.$$

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[2] See [10] for recent advances of the Pauli-Fierz model.

Here $\mathcal{F}_{\rm EM}$ denotes the Boson Fock space over $W := \oplus^{d-1} L^2(\mathbb{R}^d)$

$$\mathcal{F}_{\rm EM} := \oplus_{n=0}^{\infty} [\otimes_s^n W],$$

where $\otimes_s^n W$ denotes the n-fold symmetric tensor product of W with $\otimes_s^0 W := \mathbb{C}$. m is the bare mass of the electron and α a coupling constant. We adopt the unit $\hbar = 1 = c$. Then $\alpha \approx \sqrt{137}$. $p = -i\vec{\nabla}$ is the momentum operator canonically conjugate to the position operator x in $L^2(\mathbb{R}^d)$, and $V = V(x)$ an external potential for which precise conditions will be specified below. The smeared radiation field is defined by

$$A_\mu := \sum_{r=1}^{d-1} \frac{1}{\sqrt{2}} \int e_\mu^r(k) \left\{ \frac{\hat{\varphi}(-k)}{\sqrt{(2\pi)^d \omega(k)}} a^{\dagger r}(k) + \frac{\hat{\varphi}(k)}{\sqrt{(2\pi)^d \omega(k)}} a^r(k) \right\} d^d k,$$

and the free Hamiltonian by

$$H_{\rm f} := \sum_{r=1}^{d-1} \int \omega(k) a^{\dagger r}(k) a^r(k) dk,$$

where the dispersion relation is given by

$$\omega(k) := |k|.$$

$a^{\dagger r}(k)$ and $a^r(k)$ denote the annihilation and creation operators, respectively. They satisfy the canonical commutation relations,

$$[a^r(k), a^{\dagger s}(k')] = \delta_{rs}\delta(k-k'), \quad [a^r(k), a^s(k')] = [a^{\dagger r}(k), a^{\dagger s}(k')] = 0.$$

The vectors, $e^r(k) = (e_1^r(k), \cdots, e_d^r(k))$, $r = 1, ..., d-1$, denote polarization vectors satisfying

$$e^r(k) \cdot e^s(k) = \delta_{rs}, \quad k \cdot e^r(k) = 0.$$

Finally $\hat{\varphi}$ denotes a form factor serving as an ultraviolet cutoff. We assume that

$$\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d), \tag{1.2}$$

and

$$\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k). \tag{1.3}$$

(1.2) and (1.3) ensure that $H(\alpha)$ is a well defined symmetric operator in $\mathcal{H}$. It is known that

$$\mathrm{Spec}(H_{\mathrm{f}}) = [0, \infty)$$

and

$$\mathrm{Spec}_{\mathrm{p}}(H_{\mathrm{f}}) = \{0\}.$$

The multiplicity of $\{0\}$ is one, and

$$H_{\mathrm{f}}\Omega = 0,$$

where $\Omega := 1 \oplus 0 \oplus 0 \oplus \cdots$ is the Fock vacuum in $\mathcal{F}_{\mathrm{EM}}$.

1.2 Problems

Suppose that V is relatively bounded with respect to $-\Delta$ with a sufficiently small relative bound. Then it is proven [8] that $H(\alpha)$ is self-adjoint on $D(\Delta \otimes I) \cap D(I \otimes H_{\mathrm{f}})$ and bounded from below for arbitrary couplings. Moreover by investigating the integral kernel of $e^{-tH(\alpha)}$, $t \geq 0$, the uniqueness of the ground state, if it exists, is established in [6][3] .

In the case when $-\frac{1}{2m}\Delta + V$ has the positive spectral gap,

$$\inf \mathrm{Spec}_{\mathrm{ess}}(-\frac{1}{2m}\Delta + V) - \inf \mathrm{Spec}(-\frac{1}{2m}\Delta + V) > 0,$$

the existence of the ground state of the full Pauli-Fierz Hamiltonian is established in [3, 5, 9, 4]. In particular, Bach, Fröhlich and Sigal [3] proved it under *no* assumption of infrared cutoff condition[4] but sufficiently weak couplings. For arbitrary couplings, it is established in [4] due to Griesemer, Lieb and Loss.

The main purpose of this paper is to prove the existence of the ground state of $H(\alpha)$ under *no* assumption of the positive spectral gap. In the

[3] For the *full* Pauli-Fierz Hamiltonian, self-adjointness and the uniqueness of the ground state are established in [8] and [6], respectively.

[4] The condition $\int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2/\omega(k)^3 dk < \infty$ is called the *infrared cutoff* condition. In the case of $d = 3$ this condition implies $0 = \hat{\varphi}(0) = (2\pi)^{-3/2} \int \varphi(x)dx$, i.e., physically the electron charge turns out to be zero!

zero spectral gap case, $-\frac{1}{2m}\Delta+V$ may have no ground state. That is, we show that strong couplings produce the ground state. The physical reasoning behind such a result is as follows. As the electron binds photons it acquires the effective mass

$$m \to m+\delta m(\alpha^2)$$

which is increasing in $|\alpha|$. Roughly speaking $H(\alpha)$ may be replaced by

$$H(\alpha) \sim -\frac{1}{2(m+\delta m(\alpha^2))}\Delta + V, \tag{1.4}$$

and, for the sufficiently large $|\alpha|$, the right hand side of (1.4) may have ground states. Needless to say (1.4) has no sharp mathematical meaning, we show, however, the associated phenomena in this paper.

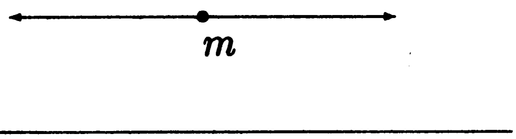

Figure 1: $H(0)$

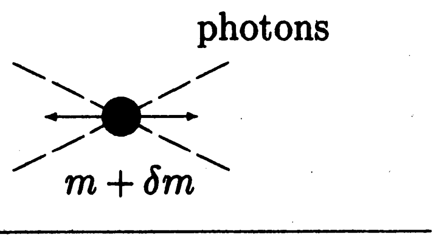

Figure 2: $H(\alpha)$

This paper is organized as follows. In Section 2 we prove the binding. In Section 3 we give some examples of the external potentials. Finally in Section 4 we give some remarks.

2 Binding

We suppose the following assumptions on V.

(1) $\|Vf\| \le a\|\Delta f\| + b\|f\|$ for $f \in D(\Delta)$ with sufficiently small $a \ge 0$, and positive $b \ge 0$.

(2) $V \in C^1(\mathbb{R}^d)$ and $\partial_\mu V \in L^\infty(\mathbb{R}^d)$, $\mu = 1, ..., d$.

(3) There exist $\mu_0 \ge 1$ and $r_0 > 0$ such that for all $\eta > \mu_0$

$$\inf \mathrm{Spec}(-\frac{1}{2m}\Delta + \eta V) \le -r_0,$$

and

$$\mathrm{Spec}_{\mathrm{ess}}(-\frac{1}{2m}\Delta + \eta V) = [0, \infty).$$

It is of interest to investigate sufficiently shallow external potentials. Since $d \ge 3$, for such a shallow V, $-\frac{1}{2m}\Delta + V$ may have no ground state. If $-\frac{1}{2m}\Delta + V$ has no ground state, then the decoupled Hamiltonian

$$H(\alpha = 0) = \left(-\frac{1}{2m}\Delta + V\right) \otimes I + I \otimes H_{\mathrm{f}}$$

also has no ground state.

For later use we define the dilatation unitary of $L^2(\mathbb{R}^d)$ by

$$D(\kappa)f(k) := \kappa^{d/2} f(k/\kappa),$$

where $\kappa > 0$ denotes the scaling parameter. The scaled Hamiltonian is defined by

$$H(\alpha, \kappa)$$
$$:= \kappa^2 D(\kappa)^{-1} \left\{ \frac{1}{2m}(p \otimes I - \alpha I \otimes A)^2 + I \otimes H_{\mathrm{f}} + \frac{1}{\kappa^2} V(x/\kappa) \otimes I \right\} D(\kappa)$$

$$= \frac{1}{2m}(p \otimes I - \kappa\alpha I \otimes A)^2 + V \otimes I + \kappa^2 I \otimes H_{\mathrm{f}}.$$

We suppose the following technical assumptions on $\hat{\varphi}$.

(1) $\hat{\varphi}(k) = \hat{\varphi}(|k|)$.

(2) $\omega^{n/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$ for $n = -5, -4, -3, -2, -1, 0, 1, 2$.

(3) $|\hat{\varphi}(\sqrt{s})|s^{(d-1)/2} \in L^{\epsilon}([0,\infty), ds)$, $0 < \epsilon < 1$, and is Lipschitz continuous of order strictly less than one.

(4) $\|\hat{\varphi}\omega^{(d-2)/2}\|_\infty < \infty$ and $\|\hat{\varphi}\omega^{(d-1)/2}\|_\infty < \infty$.

(5) $\hat{\varphi}(k) \neq 0$ for all $k \neq 0$.

Thus (1)–(5) ensure the following lemmas[5] .

Lemma 2.1 *There exist the unitary operator $U(\kappa)$ such that*

$$U(\kappa)^{-1}H(\alpha,\kappa)U(\kappa) = H_{\mathrm{eff}} + \kappa^2 H_{\mathrm{f}} + \kappa^2\alpha^2 g + \delta V,$$

where

$$H_{\mathrm{eff}} := -\frac{1}{2m_{\mathrm{eff}}}\Delta + V,$$

$$m_{\mathrm{eff}} = m_{\mathrm{eff}}(\alpha^2) := m + \alpha^2\left(\frac{d-1}{d}\right)\|\hat{\varphi}/\omega\|^2,$$

and

$$g := \frac{d-1}{2\pi}\int_{-\infty}^{\infty}\frac{t^2\|\hat{\varphi}/(t^2+\omega^2)\|^2}{m + \alpha^2(\frac{d-1}{d})\|\hat{\varphi}/\sqrt{t^2+\omega^2}\|^2}dt.$$

Moreover

$$\delta V = \delta V(\alpha,\kappa) := U(\kappa)^{-1}(V \otimes I)U(\kappa) - V \otimes I.$$

Lemma 2.2 *We have*

$$-\frac{D(\alpha)}{\kappa}(H_{\mathrm{f}} + I) \leq \delta V \leq \frac{D(\alpha)}{\kappa}(H_{\mathrm{f}} + I)$$

in the sense of form, where $D(\alpha)$ is a real number satisfying

$$\lim_{|\alpha|\to\infty} D(\alpha) = 0.$$

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[5] See [1, 11] for details.

Let

$$\alpha_{\text{critical}} := \sqrt{m(\mu_0 - 1)}\sqrt{\frac{d}{d-1}}\|\hat{\varphi}/\omega\|^{-1}.$$

We see

$$H_{\text{eff}} = \frac{m}{m_{\text{eff}}}\left(-\frac{1}{2m}\Delta + \frac{m_{\text{eff}}}{m}V\right).$$

Then, in the case of $|\alpha| > \alpha_{\text{critical}}$, it follows that

$$\inf_x V(x) \leq \inf \text{Spec}(H_{\text{eff}}) < -r_0 \frac{m}{m_{\text{eff}}}.$$

In particular the ground states of H_{eff} exist.

Theorem 2.3 *Let $\kappa = 1$. There exists $\alpha_* > \alpha_{\text{critical}}$ such that for all $|\alpha| \geq \alpha_*$ the ground state of $H(\alpha)$ exists and it is unique.*

Proof: Let N be the number operator in $\mathcal{F}_{\text{EM}}$ and $0 < \nu$. By a momentum lattice approximation we see that $H(\alpha) + \nu N$ has the normalized ground state Φ_ν. Let E_I denote the spectral projection of H_{eff} to $I \subset \mathbb{R}$ and P_Ω the projection to Ω. Let $P = E_{(-\infty, -r_0 m/m_{\text{eff}})} \otimes P_\Omega$ and $\Sigma := \inf \text{Spec}(H_{\text{eff}})$. Then we can see that

$$(\Phi_\nu, P\Phi_\nu) \geq 1 - \left(\frac{|\alpha|\epsilon}{m_{\text{eff}}}\right)^2 - \frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2} \tag{2.1}$$

with some constant ϵ. Note that

$$\lim_{|\alpha|\to 0} \frac{|\alpha|}{m_{\text{eff}}(\alpha^2)} = 0$$

and

$$\lim_{|\alpha|\to 0} \Sigma = \inf_x V(x).$$

Thus for sufficiently large $|\alpha|$ the right hand side of (2.1) is strictly positive. Take a subsequence ν' such that $\Phi_{\nu'} \to \Phi$ as $\nu \to 0$ weakly. Since P is a finite rank operator, $P\Phi_\nu$ strongly converges to $P\Phi$ and

$$(\Phi, P\Phi) \geq 1 - \left(\frac{|\alpha|\epsilon}{m_{\text{eff}}}\right)^2 - \frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2}$$

holds. In particular $\Phi \neq 0$. Hence Φ is the ground state. □

By the assumptions H_{eff} has ground states for $|\alpha| > \alpha_{\text{critical}}$. We have to make sure that $H(\alpha)$ has the same properties.

Theorem 2.4 *We suppose that κ is sufficiently large. Set $V_\kappa(x) := \kappa^{-2}V(x/\kappa)$. Then the ground state of*

$$H_\kappa(\alpha) = \frac{1}{2m}(p \otimes I - \alpha I \otimes A)^2 + V_\kappa \otimes I + I \otimes H_{\text{f}}$$

exists for all $|\alpha| > \alpha_{\text{critical}}$ and it is unique.

Proof: We have

$$H_\kappa(\alpha) = \frac{1}{\kappa^2} D(\kappa) H(\alpha, \kappa) D(\kappa)^{-1}.$$

Thus it is enough to prove the existence of the ground states of $H(\alpha, \kappa)$. From the momentum lattice approximation we see that $H(\alpha, \kappa) + \nu N$ has the ground state Φ_ν. Moreover we have the inequality

$$(\Phi_\nu, P\Phi_\nu) \geq 1 - \frac{1}{\kappa^6}\left(\frac{|\alpha|\epsilon}{m_{\text{eff}}}\right)^2 - \frac{1}{\kappa^2}\left(\frac{D(\alpha)/2}{|\Sigma| - D(\alpha)/2}\right).$$

Then the theorem follows in the same way as in Theorem 2.3. □

3 Example

Suppose that

$$V(x) \leq 0.$$

Let

$$N(V) := a_d \int_{\mathbb{R}^d} |mV(x)|^{d/2} dx,$$

where a_d is a universal constant. The following is known as the Lieb-Thirring equality

$$N(V) = \#\{\text{the nonnegative eigenvalues of} - \frac{1}{2m}\Delta + V\}.$$

Suppose that

$$N(V) < 1.$$

Then $H(0)$ has no ground state, $H(\alpha)$ for sufficiently large $|\alpha|$, however, has the ground state and it is unique by Theorem 2.3.

Remark 3.1 *If $-\frac{1}{2m}\Delta + V$ has the ground state with a positive spectral gap, then $H(\alpha)$ has the ground state for arbitrary $\alpha \in \mathbb{R}$.*

4 Concluding remarks

(1) The full Pauli-Fierz Hamiltonian is defined by

$$H(\alpha) = \frac{1}{2m}(p \otimes I - \alpha A)^2 + V \otimes I + I \otimes H_{\mathrm{f}}.$$

Here under the identification $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F}_{\mathrm{EM}} dx$

$$A_\mu := \int_{\mathbb{R}^d}^{\oplus} A_\mu(x) dx,$$

and

$$A_\mu(x) := \sum_{r=1}^{d-1} \frac{1}{\sqrt{2}} \times$$

$$\times \int e_\mu^r(k) \left\{ \frac{\hat{\varphi}(-k)}{\sqrt{(2\pi)^d \omega(k)}} e^{-ikx} a^{\dagger r}(k) + \frac{\hat{\varphi}(k)}{\sqrt{(2\pi)^d \omega(k)}} e^{ikx} a^r(k) \right\} d^d k.$$

For the full Pauli-Fierz Hamiltonian, it seems to be unknown the binding.

(2) For α such that $0 < |\alpha| < \alpha_{\text{critical}}$, *no* existence of the ground state is not known.

(3) The Nelson Hamiltonian with two charged particles is defined by

$$H_{\text{Nelson}}(\alpha) := \left(-\frac{1}{2m}\Delta + V\right) \otimes I + I \otimes H_{\mathrm{N}} + \alpha\phi$$

acting on

$$\mathcal{H} := L^2(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathcal{F},$$

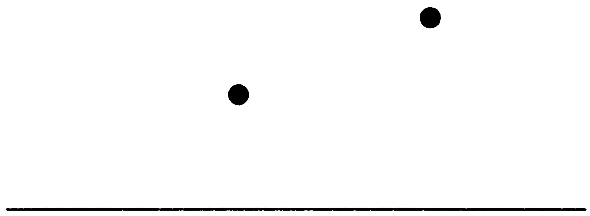

Figure 3: $H_{\text{Nelson}}(0)$

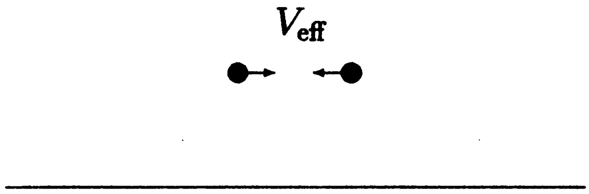

Figure 4: $H_{\text{Nelson}}(\alpha)$

where $\mathcal{F}$ denotes the Boson Fock space over $L^2(\mathbf{R}^d)$. The free Hamiltonian is defined by

$$H_{\text{N}} := \int \omega(k) a^\dagger(k) a(k) dk$$

and the scalar field by

$$\phi := \int_{\mathbf{R}^d \times \mathbf{R}^d}^{\oplus} \phi(x) dx,$$

$$\phi(x) := \sum_{j=1}^{2} \frac{1}{\sqrt{2}} \int \hat{\lambda}(-k) e^{-ikx^j} a^\dagger(k) + \hat{\lambda}(k) e^{ikx^j} a(k) dk.$$

Roughly speaking $H_{\text{Nelson}}(\alpha)$ may be replaced by

$$H_{\text{Nelson}}(\alpha) \sim -\frac{1}{2m}\Delta + V + V_{\text{eff}},$$

$$V_{\text{eff}}(x^1, x^2) = -\frac{\alpha^2}{2} \int_{\mathbf{R}^d} \frac{\hat{\lambda}(k)^2}{\omega(k)} e^{-ik(x^1 - x^2)} dk.$$

Then we can also prove the binding of the Nelson Hamiltonian under certain conditions. We omit details.

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