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Hamilton flows and Lebesgue integral related to first order nonlinear hyperbolic equations

Sadao MIYATAKE
Department of Mathematics, Nara Women's University
Kitauoya-Nishimachi, Nara 630-8506, Japan

1. Introduction

Let us consider the Cauchy problem for general nonlinear hyperbolic equations of first order:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}+H(t,x,u,\dfrac{\partial u}{\partial x})=0 & \text{in } (0,T)\times \mathrm{R}^n, \\ u(0,y)=u_0(y) & \text{on } \mathrm{R}^n . \end{cases}$$

For given initial data $u_0 \in L_2^{\infty}$ we want to construct a continuous solution $u(t,\cdot)$ in t with values in $L_2^{\infty}(\mathrm{R}^n)$, which is called an evolutional solution in $L_2^{\infty}(\mathrm{R}^n)$ whose topology is given in Definition 1,(see also Lemma 3.) Here a measurable function $v(x)$ is said to belong to $L_k^{\infty}(\mathrm{R}^n)$, $k=1,2,\cdot,n$, if we can find a set e of measure zero so that $v(x)$ is uniformly bounded for $x \notin e$, up to $k-$th order partial derivatives in the sense of distribution. We assume that, roughly speaking, $H(t,x,u,p)$ is essentially bounded up to second derivatives in any compact region in $(t,u,p) \in [0,T) \times \mathrm{R} \times \mathrm{R}^n$. Precise conditions on H are stated in each Theorem. We simply call the above solution non-smooth classical solution, since derivatives of first order are verified to be Lipschitz continuous, (see Lemma 1) but second order derivatives may have discontinuities. In order to discuss that the problem has the property of evolution equation, we shall show that the problem (1) is equivalent to the following Cauchy problem for the Hamilton system of ordinary differential equations:

$$(1')\qquad \begin{cases} \dfrac{dX_k}{dt}=H_{p_k}(t,X(t),U(t),P(t)), \\ \dfrac{dP_k}{dt}=-H_{x_k}(t,X(t),U(t),P(t))-H_u(t,X(t),U(t),P(t))\,P_k(t), \\ \dfrac{dU}{dt}=-H(t,X(t),U(t),P(t))+\displaystyle\sum_{j=1}^{n}H_{p_j}(t,X(t),U(t),P(t))\,P_j(t),\ t\in(0,T), \\ X_k(0)=y_k\,,\ P_k(0)=\dfrac{\partial u_0}{\partial x_k}(y)\,,\ U(0)=u_0(y)\,,\ y=(y_1,\ldots,y_n)\in \mathrm{R}^n,\ k=1,2,\ldots,n, \end{cases}$$

where the system of solutions $(X(t;y),U(t;y),P(t;y))$, which we call the Hamilton flow, has a

n parameter $y \in \mathrm{R}^n$. To show (1′) from (1), we solve the solution $X(t) = X(t;y)$ to the problem

$$\begin{cases} \dfrac{dX_k}{dt} = H_{p_k}(t, X(t), u(t, X(t)), \frac{\partial u}{\partial x}(t, X(t))), & t \in (0,T), \\ X_k(0) = y_k \in \mathrm{R}^n, & k = 1, 2, \ldots, n. \end{cases}$$

Then we can verify that $U(t;y) = u(t, X(t;y))$ and $P(t;y) = \frac{\partial u}{\partial x}(t, X(t;y))$ satisfy other equations in (1′), (see Theorem 1 and Remark 3.) For this purpose, we rely upon the fact that the curve integral in (t,x) space can be also represented by the integral along the related curve in R^{2n+1} with values (t,x,p). This is one of the universal lifting principle, which we will explain a little more in Introduction. By virtue of this principle we are in a position to apply the Stokes theorem to the integral of simple monomials p_k and the Hamilton function $H(t,x,u(t,x),p)$ along the closed curve in $[0,T) \times \mathrm{R}^{2n}$. Especially we consider the curve located on the two dimmensional surfaces spanned by the flow $(X(t;y), P(t;y))$. Then we employ Lebesgue's density point theorem about each point on the surfaces to obtain the Hamilton system of equations in (1′). In order to obtain (1) from (1′), we need to show the fundamental relation $\frac{\partial U}{\partial y_j} = \sum_{k=1}^n P_k \frac{\partial X_k}{\partial y_j}$, together with the regularity of the solution (X, U, P). This argument requires us long calculus. However we can say that at the same time this process enables us to encounter the potential mechacism of the Hamilton flow. Then we proceed to understand that the Hamilton flow is an inevitable notion, which is intimately related to the infinitesimal fundamentanl formula for $v \in L_1^\infty(\mathrm{R}^n)$. Note that for the local argument for smooth data it suffices to follow the traditional formalism of the Hamilton flow. However we need a global analysis for the treatment in the space $L_1^\infty(\mathrm{R}^n)$, and occationally the lifting principle takes out the cover of some mysterious parts of the Hamilton mechanism. In order to obtain the estimate of evolution type, conversely we need to collect the Hamilton flow using the homeomorphism $y \to x = X(t;y)$ for each $t \in (0,T)$. After these consideration we can arrive at the evolutional apriori estimate of the solution in function spaces $C^0([0,T); L_1^\infty(\mathrm{R}^n))$ for $u \in L_2^\infty([0,T) \times \mathrm{R}^n)$. A suitable comparison theorem gives us an estimate of the lifespan T, (see Theorem 2.) After obtaing these results we can exhibit the existence theorem for the evolutional solution $u \in C^0([0,T); L_2^\infty(\mathrm{R}^n))$, (see Theorm 3.)

Now we mention to our motivations. Theorems in this paper will be applied elsewhere to construct the solution to the nonlinear diagonalized system of hyperbolic equations. The typical example is the following problem for the system of equations in the domain $(0,T) \times \mathrm{R}$:

$$\begin{cases} \dfrac{\partial r}{\partial t} + c_1(r,s)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + c_2(r,s)\dfrac{\partial s}{\partial x} = 0, \end{cases} \tag{2}$$

with the initaial data

$$r(0,x) = r_0(x), \quad s(0,x) = s_0(x).$$

B. Riemann[1] derived this diagonarized system (2) in 1860 from equations of gas dynamics with unknown variables the density ρ and the velocity u, introducing new variables $\{r.s\}$ which are called Riemann invariants nowadays. Using the relations

$$r = (u + f(\rho))/2, \ s = (-u + f(\rho))/2, \ \ f(\rho) = \int_1^\rho \frac{1}{\sqrt{\rho}} d\rho,$$

he obtained (2) if the pressure $\varphi(\rho)$ depends only on ρ. we can expect that the solution (r, s) of this system are constructed in the space L_1^∞ or L_2^∞ depending on the initial data. The problem (1) for the larger class of general nonlinear equation, will find much more applications. For this purpose, as we describe precisely later, the Hamiltonian H in (1) should have minimum regularity, so that we can use the successive approximation method for the construction of solutions to various problems for nonlinear systems of equations.

As for Cauchy problems for nonlinear equations of first order, many special cases were studied. For example equations of conservation law and Hamilton Jacobi equations are treated in the framework in L^2 or L^1. The local uniqueness and existence theorems in C^2 class for the quasi-linear equation of one space dimension are considered by the Hamilton flow. To the knowledge of author, the global theory for general nonlinear equations of first order was thought to be too complicate to state, in a systematic way, some results which concern with both equations of consevation law and Hamilton Jacobi equations simultaneously. Here in order to make this statement possible, we employ, besides the fundamental Hamilton flow, a device concerning the comparison method. This enable us to obtain also a reasonable concrete estimate of the lifespan T. Since the lifespan of the solution must be estimated uniformly in $x \in \mathrm{R}^n$, in this paper we try to compare the lifespan with that of a system of simple ordinary differential equations as in Theorem 3.

It is rather well-known See references M.Tsuji and Li Ta-tsien [2] T.Warzewski [3] R. Courant - D. Hilbert [5], F. John [6] and R. Courant - P. Lax [7], we may understand that the difficulty for global theorem exists not only in developing the technique but also in constructing the theoretical foundation. It is also necessary for us to detemine a suitable function space in R^n for the global problem (1), which is related to the decomposition of the non-smooth solution $u(t, x)$ into the n parameter family of solutions to the problem (1′). Consequently we are led to the above space stated in the beginning of Introduction. Moreover the detailed proof requires us to prepare various mathematical devices for the analysis. Especially we use the infinitesimal formula of the Lebesgue integral developed in the space L_1^∞, which is extended also to the Stokes theorem in Lemma 1. by virtue of this lemma theoretical difficulties in the detailed demonstration will be overcome.

There is always a serious question why the Hamilton flow play a special role even in nonlinear cases. It is a problem to answer this question. If we intend to attack to this problem in front, we are oblized to introduce the definition of the notion of general flows, (see Definition 2.) This notion corresponds to the variable z in the identity given below. Let us consider how to look for all solution of $x^2 + 3x - 10 = 0$ in the situation that we know only one solution $x = 2$ but do not knnow the formual of solutions. Then we can introduce a variable $z \in \mathrm{C}$ to consider the identity $(2+z)^2+3(2+z)-10=z(z+7)$, which gives us another solution $x=2-7=-5$.

Here we give some comments concerning the lifting principle which plays an important role in many parts of the proofs.

Lifting principle for partial differential equations. Let $f_j(x)$, $j=1,2,\dots,m$ be a continuous function defined in R^m. The integral of the one form $\omega=\sum_{j=1}^{m} f_j(x)dx_j$ along a given pieacewise smooth curve $C=\{x(s)=(x_1(s),\dots,x_m(s));0\le s\le 1\}$ in R^m can be represented also by the curve integral in the higher dimensional space R^{m+k} with variable $(x_1,\dots,x_m,p_1,\dots,p_k)$, where k is a natural number less than m. Namely

$$\int_C \sum_{j=1}^{m} f_j(x)dx_j=\int_{C^{(k)}}\sum_{i=1}^{k} p_i dx_i+\int_{C^{(k)}}\sum_{j=k+1}^{m}\tilde{f}_j(x,p)dx_j$$

holds, where the curve $C^{(k)}$ is defined by

$$C^{(k)}=\{(x(s),p_1(s),\dots,p_k(s))\equiv(x_1(s),\dots,x_m(s),f_1(x(s)),\dots,f_k(x(s)));0\le s\le 1\},$$

and $\tilde{f}_j$, $j=k+1,\dots,m$ is an arbitrary continuous function satisfying

$$\tilde{f}_j(x_1(s),\dots,x_m(s),p_1(s),\dots,p_k(s))=f_j(x_1(s),\dots,x_m(s))\,,\ s\in[0,1].$$

We can verify this fact by virtue of very definitions of curve integrals in R^m and R^{m+k}. Note that p_i, $i=1,2,\dots,k$, is a simple monomial and $\tilde{f}_j$, $j=k+1,\dots,m$, is an extension function of f_j into the domain R^{m+k}. In each actual problem, $\tilde{f}_j$ may be naturally determined in R^{m+k} as an analytic continuation as we see in the remark below. We consider mainly the case where ω is closed and C is also a closed curve, which implies that $C^{(k)}$ is also closed. Apply the Stokes formula to the one form $\sum_{i=1}^{k} p_i dx_i+\sum_{j=k+1}^{m}\tilde{f}_j(x,p)dx_j$ on a 2-dimensional surface in R^{m+k} whose boundary is the curve $C^{(k)}$, then we can obtain some relations described by simple differential equations, since we can take the closed curve C in R^m arbitrarily as we see later in the proofs of Theorems. In this paper we consider the case where $m=n+1$, $k=n$ and

$$(x,t)=(x_1,\dots,x_n,t)=(x_1,\dots,x_n,x_{n+1}).$$

Let $u(t,x)$ be a solution to $\frac{\partial u}{\partial t}(t,x)+H(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x))=0$. Then putting

$$f_j=\frac{\partial u}{\partial x_j},\ j=1,\dots,n,\ f_{n+1}=\frac{\partial u}{\partial t}(t,x),$$

we can take

$$\tilde{f}_{n+1}(t,x_1,\dots,x_n,p_1,\dots,p_n)=-H(t,x,u(t,x),p).$$

As above we can expect to arrive at a sytem of differential equations, which lead us to the Hamilton system. Besides the Hamilton flow, the terminology "lifting principle" is quated from K.Oka's work. The lifting principle is his main thema in the theory of functions with several complex variables, (see T.Nishino [8].)

Note that the regularity loss of the solution does not permit us to rely on the usual traditional method as in Carathéodory [9]. Therefore, by the above lifting principle, we are led into the study of a potential mechanism concerning the Hamilton flow. Then, we are able to see that two kinds of important tools in mathematical analysis, the Hamilton flow and the Lebesgue integral, are combined intimately in the proof of the estimate of evolution type for the solution in the function space $L_2^\infty(\mathrm{R}^n)$. Formerly it seemed to the author that these two mathematical theories are too different from each other to work jointly.
Remark that the function space L_k^∞ is suitable from the following viewpoint. The product of two L^∞ functions belongs to L^∞, and the composite function $f(g(x))$ of bounded continuous function $f(y)$ and L^∞ function is also in L^∞. On the other hand we note that, in H^k space, it seem difficult for us to construct the evolution solutions, since Sobolev's lemma, which we use in H^k space, does not work well if we consider the product operation of functions. We can refer M.E.Taylor [10] and literatures in Mathematical Reviews [11] in 1993-2000.

2. Statement of results

2.1. Statement of Definitions and Lemma 1

First we give Notations, Definitions and their remarks concerning the function spaces related to L_k^∞ $k = 1, 2, \cdots, n$.

Notation 1.
1) $v \in \mathcal{B}^k(\mathrm{R}^n)$, $k = 0, 1, 2, \ldots$, means that v is continuous and bounded in R^n up to k-th order derivatves. $\mathcal{B}$ stands for $\mathcal{B}^0$. The space $\mathcal{B}^k(\mathrm{R}^n)$ is a Banach space with norm defined by $|v|_{k,\Omega} \equiv \sum_{|\alpha|\le k} \sup_{x\in\Omega} |\dfrac{\partial^{|\alpha|} v}{\partial x^\alpha}(x)|$.
2) Let B be a function space with topology. $u(t,\cdot) \in C^k([0,T); B)$ means that $u(t,\cdot)$ is k times continuously differentiable in $t \in [0,T)$ with values in B.
3) We denote sometimes the space of the vector valued function by the same notation, for example $\frac{\partial v}{\partial x} \in L^\infty(\mathrm{R}^n)$ means $\frac{\partial v}{\partial x_j} \in L^\infty(\mathrm{R}^n)$ for all $j = 1, 2, \ldots, n$.
4) For $v \in L^1_{\mathrm{loc}}$, $[v]_\epsilon(x)$ stands for the mean value of in the ϵ neighborhood of $x \in \mathrm{R}^n$ given by

$$[v]_\epsilon(x) = \frac{1}{|D(\epsilon)|} \int_{D(x,\epsilon)} v(y)dy\,,\ \epsilon > 0,$$

where $|D(\epsilon)|$ is the volume of $D(x,\epsilon) = \{y; ||y-x|| \equiv (\sum(y_i - x_i)^2)^{1/2} \le \epsilon\}$. The density theorem of Lebesgue says that $\lim_{\epsilon\to 0}[v]_\epsilon(x)$ equals the given $v(x)$ almost everywhere and $\lim_{\epsilon\to 0}[v]_\epsilon = v$ in the

topology of L^∞.
5) For convenience we call $\lim_{\epsilon\to 0}[v]_\epsilon(x)$ the standart pointwise representation and denote it by $[v](x) = \lim_{\epsilon\to 0}\frac{1}{|D(\epsilon)|}\int_{D(x,\epsilon)} v(y)dy$.

Definition 1. $v \in L^\infty(\mathrm{R}^n)$ means that the function v is an essentially bounded measurable function defined in R^n, namely there exists a set $e \subset \mathrm{R}^n$ whose measure is zero, such that v is bounded in $\mathrm{R}^n - e$. If v belongs to $L^\infty(\Omega)$ up to k-th derivatives in the sense of distribution for non-negative integer k, we note $v \in L_k^\infty(\Omega)$ and the norm of $L_k^\infty(\Omega)$ is defined by $\|v\|_{L_k^\infty(\Omega)} = (\sum_{|\alpha|\le k} \|\frac{\partial^{|\alpha|}u}{\partial x^\alpha}\|^2_{L^\infty(\Omega)})^{1/2}$. L^∞ means L_0^∞.

Definition 1'. Let us say that $u(t,\cdot)$ is cotinuous in the space $L_k^\infty(\mathrm{R}^n)$, if the following two conditions are satisfied.
1) the value $||u(t,\cdot)||_{L_k^\infty}$ is continuous in t, i.e. $||u(t,\cdot)||_{L_k^\infty(\mathrm{R}^n)} \in C^0([0,T);R)$.
2) $u(t,\cdot)$ is continuous in local L_k^1. More precisely $u \in C^0([0,T);L_k^1(D_m))$, for any $m \in \mathrm{N}$, where $D_m = \{x \in \mathrm{R}^n; |x| \le m\}$.

Later in Lemma 3 we see the meaning of the topology in more general case.

Remark 1. We do not regard that $L_k^\infty(\mathrm{R}^n)$, $k = 1,2,\dots$, is a Banach space with usual norm. The $L^\infty(\mathrm{R}^n)$ norm of v is given also by

$$||v||_{L^\infty(\mathrm{R}^n)} = \sup_{x\in\mathrm{R}^n} |\lim_{\epsilon\to 0}[v]_\epsilon(x)|.$$

Here $[v]_\epsilon(x)$ is the mean value of v in the ϵ neighbourhood of $x \in \mathrm{R}^n$ given by

$$[v]_\epsilon(x) = \frac{1}{|D(\epsilon)|}\int_{D(x,\epsilon)} v(y)dy\,,\ \epsilon > 0,$$

where $|D(\epsilon)|$ is the volume of $D(x,\epsilon) = \{y; ||y-x|| \equiv (\sum(y_i-x_i)^2)^{1/2} \le \epsilon\}$. For any $\epsilon > 0$, $[v]_\epsilon(x)$ is Lipshitz continuous in x. The theorem of Lebesgue says that $\lim_{\epsilon\to 0}[v]_\epsilon(x)$ equals the given $v(x)$ almost everywhere and $\lim_{\epsilon\to 0}[v]_\epsilon = v$ in the topology of L^∞. For convenience we call $\lim_{\epsilon\to 0}[v]_\epsilon(x)$ the standart pointwise representation and denote it by $[v](x) = \lim_{\epsilon\to 0}\frac{1}{|D(\epsilon)|}\int_{D(x,\epsilon)} v(y)dy$. Note that even if the limit $[v](x)$ does not exist for $x \in e$, where $m(e) = 0$, we have

$$-||v||_{L^\infty(\mathrm{R}^n)} \le \liminf_{\epsilon\to 0}[v]_\epsilon(x) \le \limsup_{\epsilon\to 0}[v]_\epsilon(x) \le ||v||_{L^\infty(\mathrm{R}^n)},\ v \in L^\infty(\mathrm{R}^n),$$

for all x. Note also that for $v, w \in L^\infty(\mathrm{R}^n)$

$$[vw](x) = v(x)w(x) = [v](x)[w](x)$$

holds almost everywhere in R^n. For $v \in L_1^\infty(\mathrm{R}^n)$ and $\epsilon > 0$, Lebesgue's bounded convergence theorem gives $\frac{\partial[v]_\epsilon}{\partial x_k}(x) = [\frac{\partial v}{\partial x_k}]_\epsilon(x)$, $k = 1,2,\dots,n$,. Hence making ϵ tend to zero we have, for

$v \in L_1^\infty(\mathrm{R}^n)$, $\frac{\partial [v]}{\partial x_k}(x) = [\frac{\partial v}{\partial x_k}](x)$ in the sense of distribution, in other word, for almost everywhere in R^n. For $k \in \mathrm{N}$, the $L_k^\infty(\mathrm{R}^n)$ norm of v is given by $\|v\|_{L_k^\infty(\mathrm{R}^n)} \equiv \sum_{|\alpha|\leq k} \|\frac{\partial^\alpha v}{\partial x^\alpha}\|_{L^\infty(\mathrm{R}^n)}$. Therefore $[v]_\epsilon$ converges to v in L_k^∞ if v in L_k^∞. However $\|[v]_\epsilon - v\|_{L^\infty}$ does not converges to zero as we see easily for the Heaviside function $v = H(x)$. Note also that $[v]_\epsilon$ converges to v in $L_k^\infty(\mathrm{R}^n)$ space , if v belongs to $L_k^\infty(\mathrm{R}^n)$. Remark also that $v \in L_1^\infty(\mathrm{R}^n)$ implies that $[v](x)$ is Lipschitz continuous, (cf. Lemma 1 below.)

Now we state the key lemma in this paper.

Lemma 1. *Let v belong to $L_1^\infty(\mathrm{R}^n)$. Then we have the following three assertions.*
1) The standart pointwise representation $[v](x) = \lim_{\epsilon\to 0} v_\epsilon(x)$ is a Lipschitz continuous function defined on R^n, where the Lipschitz constant is less than $\sum_{j=1}^n \|\frac{\partial v}{\partial x_j}\|_{L^\infty(\mathrm{R}^n)}$.
2) Let $C = \{(x_1(r), x_2(r), \dots, x_n(r));\ 0 \leq r \leq 1\}$ be a curve defined by $x_j(r) \in L_1^\infty([0,1])$, $j = 1, 2, \dots, n$, satisfying $\|1/\sum_{j=1}^n |\frac{dx_j}{dr}|\ \|_{L^\infty} < \infty$. Then we have the formula

$$(3) \quad [v](x(1)) - [v](x(0)) = \lim_{\epsilon\to 0} \int_0^1 \sum_{j=1}^n \left[\frac{\partial v}{\partial x_j}\right]_\epsilon (x(r)) \frac{dx_j}{dr}(r) dr \equiv \lim_{\epsilon\to 0} \int_C \sum_{j=1}^n \frac{\partial [v]_\epsilon}{\partial x_j}(x) dx_j.$$

3) Let S be a two dimensional surface $S = \{x(s,t) = (x_1(s,t), \dots, x_n(s,t)); (s,t) \in D\}$ satisfying $x_k(s,t) \in L_1^\infty(D)$ and $\|1/\sum_{j,k} |\frac{\partial(x_j,x_k)}{\partial(s,t)}|\|_{L^\infty} < \infty$, where ∂D is piecewise smooth. C is a closed curve defined by $C = \{x(r) = x(s(r), t(r)); r \in [0,1],\ (s(0), t(0)) = (s(1), t(1))\}$. , which satisfies $\partial S = C$, $s(r) \in L_1^\infty([0,1]), t(r) \in L_1^\infty([0,1])$ and $\|1/(|\frac{ds}{dr}| + |\frac{dt}{dr}|)\|_{L^\infty} < \infty$. Then we have, for any $j \in \{1, 2, \dots, n\}$,
$(3')$

$$\int_C [v](x(s)) dx_j = \lim_{\epsilon\to 0} \int\int_S d[v]_\epsilon \wedge dx_j \equiv \lim_{\epsilon\to 0} \int\int_D \left(\frac{\partial [v]_\epsilon(x(s,t))}{\partial s} \frac{\partial x_j}{\partial t} - \frac{\partial [v]_\epsilon(x(s,t))}{\partial t} \frac{\partial x_j}{\partial s} \right) dsdt,$$

Note that, in (3) and (3′), we can replace $[v]_\epsilon$ by $v_\epsilon = v([x]_\epsilon(t,s))$ if v, which is continuously differentiable.

Definition 2. A continuous function $\Phi(t,y)$ defined on $[0,T) \times \mathrm{R}^n$ is said to be a (general) flow in R^n, if the following conditions are satisfied :
(1) $\Phi(0,y) \equiv y$ for all $y \in \mathrm{R}^n$.
(2) $\frac{\partial}{\partial t}\Phi(t,\cdot) \in L_1^\infty([0,T) \times (\mathrm{R}^n)) \cap C^0([0,T) : L_1^\infty(\mathrm{R}^n))$.
(3) $x = \Phi(t,y)$ is homeomorgh from $y \in \mathrm{R}^n$ to $x \in \mathrm{R}^n$ for any $t \in [0,T)$.
(4) The inverse $y = \Phi^{(-1)}(t,x)$ satisfies also (2) replaced y by x.

Lemma 2. *Suppose that $\Phi(t,y)$ is a flow satisfying the conditions in Definition 2. Then we* have

$$\left(\frac{\partial \Phi}{\partial y}(t,y)\right)\left(\frac{\partial \Phi^{(-1)}}{\partial x}(t,\Phi(t,y))\right) = \mathrm{I},$$

where I *is the identity matrix almost everywhere in* R^n *for* $t \in [0,T)$.

Lemma 3.

1) *Suppose that* $U(t;y)$ *is a continuous function of* $t \in [0,T)$ *with values in* $L^\infty(\mathrm{R}^n)$, *namely* $U(t;y) \in C^0([0,T);L^\infty(\mathrm{R}^n))$, *where the topology of* $L^\infty(\mathrm{R}^n)$ *is given in Definition 1. Assume that* $\Phi(t,y)$ *is a general flow given by Definition 2. Then* $u(t,x)$ *defined by* $U(t;\Phi^{(-1)}(t,x))$ *is a contimuous function of* t *in the space* $L^\infty(\mathrm{R}^n)$.

2) The space $L_k^\infty(\mathrm{R}^n)$, $k = 1,2,\ldots$, is complete in the following sense. If a sequence $\{v_m\}_{m=1}^\infty$ in L_k^∞ satisfy two conditions:

1) $\{||v_m||_{L_k^\infty(\mathrm{R}^n)}\}_{m=1}^\infty$ is a uniformly bounded sequence in R.

2) $\{v_m\}_{m=1}^\infty$ is a Cauchy sequence in $L_k^1(D_m)$ for any $m \in \mathrm{N}$.

Then there exists a limit $v_0 \in L_k^\infty(\mathrm{R}^n)$ of $\{v_n\}$ in the above topologyin the above topology, more precisely we have $||v_0||_{L_k^\infty} \leq \underline{\lim}||v_k||_{L_k^\infty}$ and $\lim ||v_0 - v_k||_{L_k^1(D_m)}$ for all $m \in \mathrm{N}$.

2.2. Statement of Theorems

Now we introduce a notation of the domain:

$$G(\nu,\mu) = \mathrm{R}^n \times (-\nu,\nu) \times (-\mu,\mu)^n\,,\ \mu > 0,\ \nu > 0,$$

and note also

$$G(N) = G(N,N),\ N \in \mathrm{N}\,,\ G(\infty) = \mathrm{R}^n \times \mathrm{R} \times \mathrm{R}^n\,.$$

Here we point out two examples which serve our convenience of explanation.

(Example 1)
$$H(t,x,u,p) = \sum_{k=1}^n p_k^2\,,$$

(Example 2)
$$H(t,x,u,p) = \sum_{k=1}^n a_k(x)p_k^2 + b(x)u^2\,,$$

where $a_k(x)$ and $b(x)$ are bounded and smooth.

Now we state

Theorem 1.

Let $H(t,x,u,p)$ *be a real valued function defined in* $\Omega \equiv [0,T) \times G(\infty)$. *And suppose that* $H(t,x,u,p)$, $\frac{\partial H}{\partial x}(t,x,u,p)$, $\frac{\partial H}{\partial u}(t,x,u,p)$ *and* $\frac{\partial H}{\partial p}(t,x,u,p)$ *belong to* $L_1^\infty((0,T) \times G(N))$ *for any natural number* N. *Assume that* $u(t,x) \in L_2^\infty((0,T) \times \mathrm{R}^n)$ *is a solution to the Cauchy problem*

Let $\Phi(t,y) = (\Phi_1(t,y),\dots,\Phi_n(t,y))$ be an arbitrary general flow introduced in Definition 2. We denote this flow also by

$$X^{(\Phi)}(t;y) = (X_1^{(\Phi)}(t;y),\dots,X_n^{(\Phi)}(t;y)) \equiv \Phi(t,y).$$

Now put

$$P^{(\Phi)}(t;y) = (P_1^{(\Phi)}(t;y),\dots,P_n^{(\Phi)}(t;y)) \equiv \left(\frac{\partial u}{\partial x_1}(t,X^{(\Phi)}(t;y)),\dots,\frac{\partial u}{\partial x_n}(t,X^{(\Phi)}(t;y))\right).$$

Then we can verify that $X^{(\Phi)}(t) = X^{(\Phi)}(t;y)$ and $P^{(\Phi)}(t) = P^{(\Phi)}(t;y)$ satisfy the following relations: For almost everywhere $y \in \mathrm{R}^n$, there exists a subset e_y of $[0,T)$ with measure zero so that, if $t \notin [0,T) - e_y$, we have for $k = 1,2,\dots,n$

$$\frac{dP_k^{(\Phi)}}{dt} = -\left(\frac{\partial H}{\partial x_k} + \frac{\partial H}{\partial u}P_k^{(\Phi)}\right) + \sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_k \partial x_j}(t,X^{(\Phi)}(t;y))\left(\frac{dX_j^{(\Phi)}}{dt} - \frac{\partial H}{\partial p_j}\right), \tag{4}$$

where $\frac{dP_k^{(\Phi)}}{dt} = \frac{\partial P_k^{(\Phi)}}{\partial t}(t;y)$, $\frac{dX_k^{(\Phi)}}{dt} = \frac{\partial X_k^{(\Phi)}}{\partial t}(t;y)$ and $\frac{\partial^2 u}{\partial x_k \partial x_j}(t,X^{(\Phi)}(t;y))$ are bounded measurable in t, and other functions in the right side $\frac{\partial H}{\partial x_k} = \frac{\partial H}{\partial x_k}(t,X^{(\Phi)}(t;y),U^{(\Phi)}(t;y),P^{(\Phi)}(t;y))$, etc. are continuous in t.

Now we give a special flow which is intimately related to the problem (1).

Proposition 1. *Suppose that $H(t,x,u,p)$ satisfies the conditions in Theorem 1. Let $u(t,x) \in L_2^\infty((0,T)\times\mathrm{R}^n)$ be a solution to the problem (1). Assume that $X(t;y)$ solves the initial value problem with parameters $(y_1,y_2,\dots,y_n) = y \in \mathrm{R}^n$ for the following system of ordinary differential equations :*

$$\begin{cases} \dfrac{dX_k}{dt} = H_{p_k}\left(t,X(t),u(t,X(t)),\dfrac{\partial u}{\partial x}(t,X(t))\right), \quad t \in (0,T), \\ X_k(0) = y_k, \quad y_k \in \mathrm{R}, \quad k = 1,2,\dots,n. \end{cases} \tag{5}$$

Then $\Phi(t,y) \equiv X(t;y)$ satisfies the conditions of the general flow given in Definition 2.

In order to state Theorem 2 concerning the apriori estimate, we introduce the following notations.

$$\nu(t) = \|u(t,\cdot)\|_{L_0^\infty(\mathrm{R}^n)}, \quad \mu(t) = \max_k \|\frac{\partial u}{\partial x_k}(t,\cdot)\|_{L_0^\infty(\mathrm{R}^n)}, \tag{6}$$

which means the quantity of the solution in L_1^∞ norm, and

$$\begin{cases} M_1(t,\nu,\mu) = \| - H(t,x,u,p) + \sum_j p_j H_{p_j}(t,x,u,p)\|_{L_0^\infty(G(\nu,\mu))}, \\ M_2(t,\nu,\mu) = \sum_j \|H_{x_j}(t,x,u,p) + p_j H_u(t,x,u,p)\|_{L_0^\infty(G(\nu,\mu))}, \end{cases} \tag{6'}$$

that stand for the maximum of the rate of increase of $||u(t,\cdot)||_{L_1^\infty(\mathrm{R}^n)}$. Under the assumption on H in Theorem 2 below, we can verify, by virtue of Lemma 1, that the function $M_i(t,\mu,\nu)$, $i = 1,2$, is verified to be continuous in t and Lipschitz continuous in μ and ν.

Theorem 2.

Suppose that $u \in L_2^\infty([0,T) \times \mathrm{R}^n)$ is a solution to the Cauchy problem for nonlinear hyperbolic equation in (1). The real valued function $H(t,x,u,p)$ and its derivatives $\frac{\partial H}{\partial u}(t,x,u,p)$, $\frac{\partial H}{\partial x_j}(t,x,u,p)$ and $\frac{\partial H}{\partial p_j}(t,x,u,p)$, $j = 1,2,\cdots,n$, are assumed to belong to $L_1^\infty([0,T) \times G(N))$ for any natural number N. Then the solution $u(t,\cdot)$ and its first order derivatives $\frac{\partial u}{\partial x_j}(t,\cdot)$, $j = 1,2,\ldots,n$, are estimated as follows:

$$(E_1) \qquad \nu(t) \le \mathcal{U}(t)\,,\ \mu(t) \le \mathcal{P}(t)\,,\ 0 \le t < T',$$

where $(\mathcal{U}(t),\mathcal{P}(t))$ is the unique solution to the comparison Cauchy problem for the following system of ordinary differential equations:

$$(C-P) \qquad \begin{cases} \dfrac{d\mathcal{U}(t)}{dt} = M_1(t,\mathcal{U}(t),\mathcal{P}(t)) \\ \dfrac{d\mathcal{P}(t)}{dt} = M_2(t,\mathcal{U}(t),\mathcal{P}(t)) \\ \mathcal{U}(0) = \nu(0)\,,\ \mathcal{P}(0) = \mu(0), \end{cases}$$

Here T' stands for the estimate of the lifespan of the solution $(\mathcal{U}(t),\mathcal{P}(t))$. $M_1 = M_1(t,\nu,\mu)$ and $M_2 = M_2(t,\nu,\mu)$ are given in (6'). We have also the precise estimate

$$(E'_1) \quad |\nu(t) - \nu(t')| \le |\mathcal{U}(t,t') - \nu(t')|\,,\ |\mu(t) - \mu(t')| \le |\mathcal{P}(t,t') - \mu(t')|\,,\ 0 < t' < t < T',$$

where $(\mathcal{U}(t,t'),\mathcal{P}(t,t'))$ is the solution to the problem $(C-P)$ replaced the intial plain $t = 0$ and the initial data $(\nu(0),\mu(0)$ respectively by $t = t'$ and $(\mathcal{U}(t',t'),\mathcal{P}(t',t')) = (\nu(t'),\mu(t'))$. We have the similar result even if we replace M_1 and M_2 by larger functions $\overline{M}_1$ and $\overline{M}_2$.

Corollary. *Note that the above apriori estimate involves the uniqueness of the solution u belonging to $L_2^\infty((0,T) \times \mathrm{R}^n)$ for the problem (1). The continuous dependence of the solution on the intial data is verified also at the same time.*

For the statement of the existence theorem, we introduce the notation

$$(S) \qquad \mathcal{S}(t) = \sum_{k,j=1}^{n} ||\frac{\partial^2 u}{\partial x_k \partial x_j}(t,\cdot)||_{L_0^\infty}.$$

Theorem 3.

Assume that the real valued function $H(t,x,u,p)$ belongs to $L_2^\infty([0,T) \times G(N))$ for any natural number N. Assume that the initial data $u_0(x)$ belongs to $L_2^\infty(\mathrm{R}^n)$. Then there exists a *positive constant T depending on $||u_0||_{L_1^\infty}$ and $H(t,x,u,p)$, such that the Cauchy problem* (1) *has the unique solution $u(t,x)$ belonging to $\bigcap_{k=0}^{2} C^k([0,T]; L^\infty_{2-k}(\mathrm{R}^n))$. Moreover we can estimate the second derivatives with respect to x as follows. For any T' smaller than T, there exists a number M such that*

$$(E_2) \qquad e^{-M(t-s)}\,\mathcal{S}(s) \le \mathcal{S}(t) \le e^{M(t-s)}\,\mathcal{S}(s)\,, \ \ 0 \le s < t \le T'.$$

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