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Locally connected tree-like invariant continua under Kleinian groups

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This note deals with invariant continua under Kleinian groups. Here, a continuum is a compact connected subset of the Riemann sphere S^2, and a Kleinian group is a discrete subgroup of Möbius transformations of S^2.

Let L be a continuum on S^2. By definition, L is locally connected at a point $y \in L$ if, for any neighborhood U of y, there exists a smaller neighborhood $V \subset U$ such that $L \cap V$ is connected. We say that L is *locally connected* if it is locally connected at any point $y \in L$. We say that a continuum L is *tree-like* if the complement $S^2 - L$ is connected and if the interior of L is empty. Locally connected, tree-like continua are characterized by the following property. See [1, Section 10].

Proposition. *Let $L \subset S^2$ be a locally connected, tree-like continuum. Then, for any points x and y in L, there exists a unique arc $\overline{xy}$ in L that connects x and y.*

A point ξ on a locally connected, tree-like continuum L is called an *endpoint* if there exists no arc λ in L such that ξ is an interior point of λ with respect to the relative topology on λ. This is equivalent to saying that $L - \{\xi\}$ is connected.

Let Γ be a Kleinian group. A loxodromic fixed point of Γ is a point that is fixed by a loxodromic element of Γ. The limit set $\Lambda(\Gamma)$ for Γ is the closure of the set of all loxodromic fixed points of Γ. We say that $\xi \in S^2$ is a *point of approximation* (or a conical limit point) for Γ if there exists a sequence of elements $\gamma_n \in \Gamma$ and distinct points x and y on S^2 such that $\gamma_n(\xi)$ converge to x and $\gamma_n(z)$ converge to y locally uniformly for $z \in S^2 - \{\xi\}$. See [3, p.22]. Points of approximation belong to the limit set. If all the points in the limit set $\Lambda(\Gamma)$ are points of approximation, then the Kleinian group Γ is *convex cocompact.* A Schottky group is a convex cocompact, free Kleinian group.

Abikoff [1, Lemma 1] proved that any loxodromic fixed point of a Kleinian group Γ with the locally connected, tree-like limit set $\Lambda(\Gamma)$ is its endpoint. In this note, we extend this result in the following form.

Theorem. *Let Γ be a Kleinian group and L a locally connected, tree-like continuum that is invariant under Γ. Then any point of approximation for Γ is an endpoint*

Proof. Suppose that a point ξ of approximation for Γ is not an endpoint of L. Then there exists an arc λ in L such that ξ is in its interior. Let z_1 and z_2 be the endpoints of λ. Since ξ is a point of approximation, there exists a sequence of elements $\gamma_n \in \Gamma$ and distinct points x and y on S^2 such that $\gamma_n(\xi)$ converge to x and $\gamma_n(z_i)$ converge to y for $i = 1, 2$. The Γ-invariance of L implies that $\gamma_n(\overline{z_i\xi}) = \overline{\gamma_n(z_i)\gamma_n(\xi)}$ lies in L as well as y belongs to L.

Let V be an open neighborhood of y such that x is not contained in the closure of V and that $L \cap V$ is connected. For a sufficiently large n, $\gamma_n(z_i)$ is contained in V but $\gamma_n(\xi)$ is not. Since $\gamma_n(z_i)$ can be connected with y in $L \cap V$, we take an arc $\overline{y\gamma_n(z_i)}$ there. Then $\overline{y\gamma_n(z_i)} \cup \overline{\gamma_n(z_i)\gamma_n(\xi)}$ for $i = 1, 2$ are distinct arcs in L connecting y and $\gamma_n(\xi)$. However, this contradicts the uniqueness of the arc in L as in the previous proposition. □

Corollary. *Let Γ be a convex cocompact Kleinian group and L a locally connected, tree-like continuum that is invariant under Γ. Then $L - \Lambda(\Gamma)$ is connected.*

Maskit [2] considered this problem for the case that L is the limit set for a degenerate Kleinian group G and Γ is a Schottky subgroup of G. His arguments did not involve any assumption on local connectivity for L, however, a certain property for L seems to have been necessary to complete the proof. It is conjectured that the limit set for a degenerate Kleinian group is locally connected (cf. [1]), however, only partial solutions have so far been obtained.

References

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2. B. Maskit, *A remark on degenerate groups*, Math. Scand. **36** (1975), 17–20.
3. B. Maskit, *Kleinian groups*, Springer, 1988.

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