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MODULI OF SHEAVES ON BLOWN-UP SURFACES

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INTRODUCTION

This paper is a joint work with K. Yoshioka in Kobe University.

Let $p\colon \widehat{X} \to X$ be the blow-up of a nonsingular complex projective surface X at a point $P \in X$ and C the exceptional divisor on $\widehat{X}$.

In this paper, we study relations between moduli spaces of coherent torsion-free sheaves on X and $\widehat{X}$, and then use them to compare "invariants" associated with moduli spaces for X and $\widehat{X}$. Here we consider the Betti numbers of moduli spaces, which have been studied in connection with the so-called S-duality conjecture of Vafa-Witten.

In fact, it is already known that there are explicit "universal" relations between invariants which are independent of the surface X. It is due to Yoshioka [9], under the assumption that moduli spaces are nonsingular projective varieties. His proof uses an ingenious trick, unstable sheaves. Although the proof give us the universal relations among invariants, the relations between moduli spaces are still obscure. For example, the universal relation for Euler numbers of moduli spaces coincides with the charcter formula for the basic representation of the affine Lie algebra $\widehat{\mathfrak{gl}}_r$. But the representation itself was not seen in Yoshioka's proof. To understand $\widehat{\mathfrak{gl}}_r$ is one of the motivation of this paper.

Our strategy is the following: since the "universal" relations are independent of a surface X, we choose a "good" X which has a torus action, so that we can calculate invariants via the localization technique. There might be many such X's, we take $X = \mathbb{C}^2$, a simplest complex surface in this paper. Since this X is noncompact, so we consider the framed moduli spaces instead of genuine moduli spaces. More precisely, we consider the framed moduli space $M(r,n)$ which parametrizes the pair (E,Φ) such that

(1) E is a torsion free sheaf of rank r, $\langle c_2(E), [\mathbb{P}^2]\rangle = n$ which is locally free in a neighbourhood of ℓ_∞,

(2) $\Phi\colon E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism called "framing at infinity".

Here $\ell_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\} \subset \mathbb{P}^2$ is the line at infinity. We also consider the framed moduli space $\widehat{M}(r,d,n)$ of sheaves on $\widehat{\mathbb{P}}^2$, where $d = \langle c_1(E), C\rangle \in \mathbb{Z}$. Then the moduli spaces $M(r,n)$ and $\widehat{M}(r,d,n)$ are smooth and have torus actions with isolated fixed points, which are explicitly described (see below). This result enables us to compute Hodge polynomials of $M(r,n)$ and $\widehat{M}(r,d,n)$.

There are another invariants related to moduli spaces, i.e., Donaldson invariants. Fintushel-Stern [3] showed that there exists an explicit universal relation between Donaldson invariants on a 4-manifold and its blowup. It seemed that Bott's formula gives us the intersection pairing on $\widehat{M}(r,d,n)$ at first sight. However, due to the noncompactness of $\widehat{M}(r,d,n)$, we could reproduce only a very small portion of Fintushel-Stern's formula.

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A similar approach to the blowup formula for Donaldson invariants was proposed by Bryan [1]. He used the framed moduli spaces of locally-free sheaves on $\mathbb{P}^2$, which is an open subset of $\widehat{M}(r,d,n)$. He used an integration instead of the localization, and obtained a portion of Fintushel-Stern's formula.

We have completed this work (except the failed trial of the derivation of Fintushel-Stern's formula) in July 1997, and the first author gave talks at Workshop on Complex Differential Geometry, 14-25 July 1997, Warwick and at Verallgemeinerte Kac-Moody-Algebren, 19-25 July 1998, Oberwolfach. We then noticed that W-P.Li and Z.Qin obtained results closely related to our results [4, 5, 6]. By a technical reason, they treat only rank 2-case, while we treat arbitary rank case. Thus most of techniques we used in this paper are independent of their results, but some parts of this paper is influenced by their papers, e.g., the universal formula using virtual Hodge polynomials. Thus it is probably fair to say that this paper is not independent of theirs.

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1. Universal relation

Let H be an ample line bundle over X. For $c_1 \in H^2(X,\mathbb{Z})$, $\Delta \in \mathbb{Q}$, let $M_H(r,c_1,\Delta)$ be the moduli space of H-stable sheaves E on X with $c_1(E) = c_1$, $c_2(E) - \frac{r-1}{2r}c_1(E)^2 = \Delta$.

We assume $\mathrm{GCD}(r, \langle c_1, H\rangle) = 1$.

Let $\widehat{M}(r, c_1 + dC, \Delta)$ be the moduli space of $(H - \varepsilon C)$-stable sheaves E on $\widehat{X}$ with $c_1(E) = p^*c_1 + dC$, $c_2(E) - \frac{r-1}{2r}c_1(E)^2 = \Delta$, where c_1, Δ is as above, and $d \in \mathbb{Z}$.

Let $e(Y; x, y)$ denote the virtual Hodge polynomial of Y introduced in [2].

Theorem 1.1. (1) *There exists a universal function $Z_{d,r}(x,y;q)$ (independent of the surface X) such that*

$$\sum_{\Delta} e(\widehat{M}(r, c_1 + dC, \Delta); x, y)q^{\Delta} = Z_{d,r}(x,y;q) \sum_{\Delta} e(M(r,c_1,\Delta); x,y)q^{\Delta}.$$

(2) *We also have*

$$\sum_{n} e(\widehat{M}(r,d,n); x,y)q^n = Z_{d,r}(x,y;q)\sum_{n} e(M(r,n);x,y)q^n.$$

2. Poincaré polynomials

Let $M(r,n)$, $\widehat{M}(r,d,n)$ as in the introduction. We have T^r-actions on both by the change of the framing. We also have extra T^2-actions induced from the action on $\mathbb{P}^2$ and $\widehat{\mathbb{P}}^2$.

Theorem 2.1. (1) *Both $M(r,n)$, $\widehat{M}(r,d,n)$ are nonsingular quasi-projective varieties.*

(2) *Both $M(r,n)$, $\widehat{M}(r,d,n)$ admit T^{r+2}-actions such that the fixed point sets are finite sets.*

(3) *$(E,\Phi) \in M(r,n)$ is fixed by the T^{r+2}-action if and only if E has a decomposition $E = I_1 \oplus \cdots \oplus I_r$ satisfying the following conditions for $\alpha = 1,\dots,r$:*

a) *I_α is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\mathbb{C}^2 = \mathbb{P}^2 \setminus \ell_\infty$.*

b) *Under Φ, $I_\alpha|_{\ell_\infty}$ is mapped to the α-th factor $\mathcal{O}_{\ell_\infty}$ of $\mathcal{O}_{\ell_\infty}^{\oplus r}$.*

c) *I_α is fixed by the action of $\mathbb{C}^* \times \mathbb{C}^*$, coming from that on $\mathbb{P}^2$.*

(4) *$(E,\Phi) \in \widehat{M}(r,d,n)$ is fixed by the T^{r+2}-action if and only if E has a decomposition $E = I_1(a_1C) \oplus \cdots \oplus I_r(a_rC)$ satisfying the following conditions for $\alpha = 1,\dots,r$:*

a) *$I_\alpha(a_\alpha C)$ is a tensor product $I_\alpha \otimes \mathcal{O}(a_\alpha C)$, where I_α is an ideal sheaf of* 0*-dimensional subscheme Z_α contained in $\widehat{\mathbb{C}}^2 = \widehat{\mathbb{P}}^2 \setminus \ell_\infty$.*

b) *Under Φ, $I_\alpha(a_\alpha C)|_{\ell_\infty}$ is mapped to the α-th factor $\mathcal{O}_{\ell_\infty}$ of $\mathcal{O}_{\ell_\infty}^{\oplus r}$.*

c) *I_α is fixed by the action of $\mathbb{C}^* \times \mathbb{C}^*$, coming from that on $\widehat{\mathbb{P}}^2$.*

By the Bialynicki-Birula decomposition, we get

Corollary 2.2. *$M(r,n)$ and $\widehat{M}(r,d,n)$ have the following properties:*
(1) *The homology groups have no torsion and vanish in odd degrees.*
(2) *The cycle maps from the Chow groups to the homology groups are isomorphisms.*

In order to compute Betti numbers, we determine the T^{r+2}-module structure of the tangent space at fixed points. Then we get the followings:

Theorem 2.3. *The generating function of the Poincaré polynomials of $M(r,n)$ is given by*

$$\sum_n P_t(M(r,n))q^n = \prod_{\alpha=1}^{r}\prod_{d=1}^{\infty} \frac{1}{1-t^{2(rd-\alpha)}q^d}.$$

Theorem 2.4. *The generating function of the Poincaré polynomials of $\widehat{M}(r,d,n)$ is given by*

$$\sum_n P_t(\widehat{M}(r,d,n))q^n = \prod_{d=1}^{\infty}\frac{1}{1-t^{2rd}q^d} \sum_{a\in M+d\lambda} t^{\langle a,b\rangle}(t^{2r}q)^{\langle a,a\rangle/2}\left(\sum_n P_t(M(r,n))q^n\right),$$

where $(M, \langle\ ,\ \rangle)$ is the A_{r-1}-lattice, $\lambda = (1-1/r, -1/r, \ldots, -1/r)$, $b = (r-1, r-3, \ldots, 1-r)$.

Thus we have

$$Z_{d,r}(x,y;q) = \prod_{d=1}^{\infty}\frac{1}{1-(xy)^{rd}q^d} \sum_{a\in M+d\lambda} (xy)^{\langle a,b\rangle/2}((xy)^r q)^{\langle a,a\rangle/2}.$$

3. $\widehat{\mathfrak{gl}}_r$-MODULE STRUCTURE

Let $\widehat{\mathfrak{gl}}_r$ be the affine Lie algebra of $\mathfrak{gl}_r$. Let $\mathfrak{s}$ be the infinite dimensional Heisenberg algebra. The generating function of Poincaré polynomials in Theorem 2.3, (resp. Theorem 2.4) coincides with the character of the "basic" modules of $s^r = s \times \cdots \times s$ (resp. $\widehat{\mathfrak{gl}}_r \times s^r$), after letting $t = 1$. Thus it is natural to expect that $\bigoplus_n H^*(M(r,n), \mathbb{C})$ (resp. $\bigoplus_{n,d} H^*(\widehat{M}(r,d,n), \mathbb{C})$) has such a module structure. When $r = 1$, then $M(1,n)$ is nothing but the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in $\mathbb{C}^2$. Then such a module structure was constructed by using correspondences (see [7, Chapter 8]). We also have a similar construction for $\widehat{M}(1,d,n)$ (see [7, Chapter 9]).

Here we construct module structures on (localized) equivariant cohomology groups

$$H^*_{T^r}(M(r,n), \mathbb{C}) \otimes \mathcal{R}, \quad H^*_{T^r}(\widehat{M}(r,d,n), \mathbb{C}) \otimes \mathcal{R},$$

where $\mathcal{R}$ is the quotient field of $\mathbb{C}[t_1, t_2, \ldots, t_r]$. We do not succeed to construct module structues on non-equivariant cohomology groups so far.

Let T^r be the r-torus acting on $M(r,n)$ and $\widehat{M}(r,d,n)$ by the change of the framing.

Let $H^*_{T^r}(M(r,n), \mathbb{C})$, $H^*_{T^r}(\widehat{M}(r,d,n), \mathbb{C})$ be the equivariant cohomology groups of $M(r,n)$, $\widehat{M}(r,d,n)$ with complex coefficients. These are modules over $H^*_{T^r}(\text{point}, \mathbb{C})$, the equivariant cohomology of a point, which is isomorphic to $\mathbb{C}[t_1, t_2, \ldots, t_r]$, the polynomial ring of r-variables. By Theorem 2.1 we have

Corollary 3.1. (1) *$H^*_{T^r}(M(r,n),\mathbb{C})$ and $H^*_{T^r}(\widehat{M}(r,d,n),\mathbb{C})$ are free $\mathbb{C}[t_1,t_2,\dots,t_r]$-modules and vanish in odd degrees.*

(2) *We have*

$$\operatorname{rank}_{\mathbb{C}[t_1,t_2,\dots,t_r]} H^*_{T^r}(M(r,n),\mathbb{C}) = \dim H^*(M(r,n),\mathbb{C})$$
$$\operatorname{rank}_{\mathbb{C}[t_1,t_2,\dots,t_r]} H^*_{T^r}(\widehat{M}(r,d,n),\mathbb{C}) = \dim H^*(\widehat{M}(r,d,n),\mathbb{C})$$

Let $\mathcal{R}$ be the quotient field of $\mathbb{C}[t_1,t_2,\dots,t_r]$. By the localization theorem for the equivariant cohomology, we have

$$H^*_{T^r}(M(r,n),\mathbb{C})\otimes_{\mathbb{C}[t_1,t_2,\dots,t_r]}\mathcal{R}\cong H^*(M(r,n)^{T^r},\mathbb{C})\otimes_{\mathbb{C}}\mathcal{R},$$
$$H^*_{T^r}(\widehat{M}(r,d,n),\mathbb{C})\otimes_{\mathbb{C}[t_1,t_2,\dots,t_r]}\mathcal{R}\cong H^*(\widehat{M}(r,d,n)^{T^r},\mathbb{C})\otimes_{\mathbb{C}}\mathcal{R},$$

where $M(r,n)^{T^r}$, $\widehat{M}(r,d,n)^{T^r}$ denote the fixed point set.

A fixed point (E,Φ) satisfies the conditions a),b) in Theorem 2.1. Then fixed point components are products of Hilbert schemes of points. More precisely, we have

$$M(r,n)^{T^r} = \bigsqcup_{\sum n_\alpha = n} (\mathbb{C}^2)^{[n_1]}\times\cdots\times(\mathbb{C}^2)^{[n_r]},$$

(3.2)
$$\widehat{M}(r,d,n)^{T^r} = \bigsqcup_{\substack{\sum a_\alpha = d\\ \sum n_\alpha+\frac{1}{2r}\sum|a_\alpha-a_\beta|^2=n}} (\widehat{\mathbb{C}^2})^{[n_1]}\times\cdots\times(\widehat{\mathbb{C}^2})^{[n_r]}.$$

By [7, Chapter 8], $\bigoplus_n H^*((\mathbb{C}^2)^{[n]},\mathbb{C})$ is the Fock space representation of $\mathfrak{s}$. Hence

$$\bigoplus_n H^*(M(r,n)^{T^r},\mathbb{C}) = \left(\bigoplus_n H^*((\mathbb{C}^2)^{[n]},\mathbb{C})\right)^{\otimes r}$$

is the Fock space representation of s^r.

The case of $\widehat{M}(r,d,n)$ is more interesting. By [7, Chapter 8], $\bigoplus_n H^*((\widehat{\mathbb{C}^2})^{[n]},\mathbb{C})$ is the Fock space representation of $\mathfrak{s}^2$. Then the structure of $\bigoplus_{d,n} H^*(\widehat{M}(r,d,n)^{T^r},\mathbb{C})$ given by (3.2) is the same as that of the so-called Frenkel-Kac construction (or the vertex algebra from a lattice) (see [7, Chapter 9]). Then we see that the above direct sum is the basic representation of $\widehat{\mathfrak{gl}}_r\times s^r$. Thus

Corollary 3.3. *The direct sum of the localized equivariant cohomology groups*

$$\bigoplus_{n,d} H^*_{T^r}(\widehat{M}(r,d,n),\mathbb{C})\otimes\mathcal{R}$$

has a structure of the basic representation of $\widehat{\mathfrak{gl}}_r\times s^r$.

4. Donaldson invariants

In this section, we consider the case $(r,d)=(2,0)$ case only.

Let $\mathcal{L}$ be the determinant line bundle over $\widehat{M}(2,0,n)$ where the fiber over (E,Φ) is

$$\left(\Lambda^{\max}H^1(\widehat{\mathbb{P}^2},E(-\ell_\infty))\right)^*\otimes\Lambda^{\max}H^1(\widehat{\mathbb{P}^2},E(C-\ell_\infty)).$$

Then we have $c_1(\mathcal{L})=\mu([C])$, where μ is the Donalson μ-map.

Let T^2 be the 2-torus acting on $\widehat{M}(2,d,n)$ by the change of the framing. We take a subgroup $\{(t,t^{-1}) \in T^2 \mid t \in S^1\}$. We consider the equivariant cohomology group $H^2_{S^1}(\widehat{M}(2,0,n),\mathbb{C})$. Then the Chern class $c_1(\mathcal{L})$ has a natural lift to $H^2_{S^1}(\widehat{M}(2,d,n),\mathbb{C})$. We denote it also by $c_1(\mathcal{L})$. Let us "define" $a_{n,m}$ by

$$a_{n,m} w^{2m-4n} = \int_{\widehat{M}(2,0,n)} c_1(\mathcal{L})^{2m} \in H^{4m-8n}_{S^1}(\text{point}), \tag{4.1}$$

where w is the generator of $H^*_{S^1}(\text{point})$. Then a naive consideration leads us to the following conjectures.

(1) Although $\widehat{M}(2,0,n)$ is noncompact, the above integration (4.1) is well-defined.
(2) $a_{n,m}$ can be computed by the localization technique. Namely, we consider an extra T^2 action coming from the action on $\widehat{\mathbb{P}^2}$, then $a_{n,m}$ is given by Bott's formula.
(3) $a_{n,m}$ gives us the blowup coefficients, i.e., if D_X and $D_{X\#\overline{\mathbb{P}^2}}$ denote Donaldson invariants for X and $X\#\overline{\mathbb{P}^2}$, then we have

$$D_{X\#\overline{\mathbb{P}^2}}(z[C]^{2m}) = \sum_{n=0}^{[m/2]} a_{n,m} D_X(zx^{m-2n}),$$

where x is the generator of $H_0(X,\mathbb{C})$, and $z \in \operatorname{Sym}^*(H_0(X,\mathbb{C}) \oplus H_2(X,\mathbb{C}))$.

Similar conjectures were made by Bryan [1] based on a unpublished work of Taubes.

By Bott's formula, the integration is replaced by a summation over the fixed point set, a finite set in our situation. The result is

$$a_{n,m}(t,s,w) \in \mathbb{C}(t,s,w)$$

which "should" be equal to $a_{n,m}w^{2m-4n}$ if we set $t = s = 0$.

Theorem 4.2. *The conjectures are true if $n = 1$. However the conjectures are false in general: $a_{n,m}(t,s,w)$ diverge if we set $t = s = 0$.*

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