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HODGE INTEGRALS, TAUTOLOGICAL CLASSES AND GROMOV-WITTEN THEORY

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0. Introduction

In the symposium, I talked about the recent joint work with Rahul Pandharipande. See the papers [FP1, FP2, FP3] for more details and references; some results have not yet been written up.

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1. Results

Let $\overline{M}_{g,n}$ be the moduli stack of stable n-pointed curves of genus g. Denote by $\mathbb{E}$ the Hodge bundle on $\overline{M}_{g,n}$, the rank g vector bundle with fiber $H^0(C,\omega_C)$ over $[C,p_1,\ldots,p_n]$. Let $\lambda_j = c_j(\mathbb{E})$. For $1\leq i\leq n$, let $\mathbb{L}_i$ be the cotangent line bundle on $\overline{M}_{g,n}$ with fiber T^*_{C,p_i}, and let $\psi_i = c_1(\mathbb{L}_i)$. A Hodge integral over $\overline{M}_{g,n}$ is an integral (intersection number) of products of the λ and ψ classes.

More generally, for X a nonsingular complex projective variety and $\beta\in H_2(X,\mathbb{Z})$, one has the moduli stacks of stable maps $\overline{M}_{g,n}(X,\beta)$ with evaluation maps $e_i:\overline{M}_{g,n}(X,\beta)\to X$ (for $1\leq i\leq n$). Hodge integrals over $\overline{M}_{g,n}(X,\beta)$ are integrals against the virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$ of products of the λ and ψ classes (defined analogously) and classes of the form $e_i^*(\gamma_i)$, with $\gamma_i\in H^*(X)$. If no λ classes occur, these are the descendent Gromov-Witten invariants of X.

Here are some of our results (briefly stated).

1. The set of Hodge integrals over moduli stacks of maps to X may be uniquely reconstructed from the set of descendent integrals.

2. For $g\geq 2$,

$$\int_{\overline{M}_g}\lambda_g\lambda_{g-1}\lambda_{g-2}=\frac{|B_{2g}||B_{2g-2}|}{8g(g-1)\cdot(2g-2)!}\,. \tag{1}$$

Note that $(-1)^g\chi(Y)\int_{\overline{M}_g}\lambda_g\lambda_{g-1}\lambda_{g-2}$ is the genus g, degree 0 Gromov-Witten invariant of a Calabi-Yau threefold Y [BCOV, GeP].

3. For $d \geq 1$, let $\overline{M}_g(d) = \overline{M}_{g,0}(\mathbf{P}^1, d)$, with universal curve

$$\pi : U = \overline{M}_{g,1}(\mathbf{P}^1, d) \to \overline{M}_g(d)$$

and universal map $\mu : U \to \mathbf{P}^1$. Let $N = \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$. Put

$$C(g,d) = \int_{[\overline{M}_g(d)]^{\mathrm{vir}}} c_{\mathrm{top}}(R^1\pi_*\mu^* N).$$

In suitable circumstances, $C(g,d)$ is the contribution to the genus g, degree d Gromov-Witten invariant of a Calabi-Yau threefold Y coming from degree d covers of a fixed $\mathbf{P}^1 \subset Y$ with normal bundle N. We prove

$$C(g,d) = \frac{|(2g-1)B_{2g}|}{(2g)!}\, d^{2g-3} \left(= |\chi(M_g)| \cdot \frac{d^{2g-3}}{(2g-3)!} \right).$$

This was known for $g = 0$ and 1. The formula in parentheses makes sense for $g \geq 2$ only; $\chi(M_g) = B_{2g}/(2g(2g-2))$ is the Harer-Zagier formula for the orbifold Euler characteristic of M_g.

4. Define $F(t,k) \in \mathbb{Q}[k][[t]]$ by

$$F(t,k) = 1 + \sum_{g \geq 1} \sum_{i=0}^{g} t^{2g} k^i \int_{\overline{M}_{g,1}} \psi_1^{2g-2+i} \lambda_{g-i}\,.$$

Then

$$F(t,k) = \left(\frac{t/2}{\sin(t/2)} \right)^{k+1}. \tag{2}$$

Putting $b_0 = 1$ and $b_g = \int_{\overline{M}_{g,1}} \psi_1^{2g-2}\lambda_g$ for $g \geq 1$, we have for example

$$\sum_{g \geq 0} b_g t^{2g} = F(t,0) = \frac{t/2}{\sin(t/2)},$$

hence for $g \geq 1$

$$b_g = \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}.$$

Also, by repeated differentiation w.r.t. k of (2) one obtains in particular

$$\int_{\overline{M}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g g!}$$

for $g \geq 1$. This is an immediate consequence of Witten's conjecture [W] proved by Kontsevich [K1]. See [FP1,§3.1] for a direct proof of this identity.

5. The integrals over $\overline{M}_{g,n}$ of ψ classes against λ_g, the Euler class of the Hodge bundle, are completely known [FP2]:

(3)
$$\int_{\overline{M}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n}\lambda_g = \binom{2g-3+n}{a_1,\ldots,a_n}\int_{\overline{M}_{g,1}} \psi_1^{2g-2}\lambda_g = \binom{2g-3+n}{a_1,\ldots,a_n} b_g$$

for nonnegative integers $a_1,\ldots,a_n$ with sum $2g-3+n$.

6. For $g \geq 1$,

$$\int_{\overline{M}_{g,1}} \psi_1^{g-1}\lambda_g\lambda_{g-1} = \frac{1}{2^{2g-1}(2g-1)!!}\frac{|B_{2g}|}{2g}. \tag{4}$$

More generally, define polynomials $P_g(k)$ in k by

$$\frac{|B_{2g}|}{2g}P_g(k) = \sum_{j=0}^{g-1}(-1)^j k^{g-1-j}\int_{\overline{M}_{g,1}} \psi_1^{g-1-j}\lambda_j\lambda_g\lambda_{g-1} \tag{5}$$

and write $P_g(k) = \sum_{i=0}^{g-1} c_{g,i}k^i$. These polynomials show up in [FP3] and were studied in detail by Zagier [Z]. For example, Zagier proved the following. Define polynomials $Q_i(y)$ using the recursion

$$Q_0(y) = y, \qquad Q_{i+1}(y) = \int_0^y \frac{x-1}{x}Q_i(x)\,dx \qquad (i \geq 0)$$

(so $Q_1(y) = \frac{1}{2}y^2 - y$, $Q_2(y) = \frac{1}{6}y^3 - \frac{3}{4}y^2 + y$, $\ldots$). Then

$$c_{g,i} \text{ equals the coefficient of } x^{2g-1} \text{ in the power series } Q_i\left(\frac{x}{1-e^{-x}}\right).$$

For $i = 1$, this is equivalent to (1).

In order to prove the first result, one begins by interpreting Mumford's Grothendieck-Riemann-Roch calculation of the Chern character $\mathrm{ch}(\mathbb{E})$ of the Hodge bundle [M] in Gromov-Witten theory. Combining this with properties of the cotangent line classes and of the restriction of the Hodge bundle to virtual boundary divisors, one obtains differential equations for a natural generating function for Hodge integrals. The generating function is determined by these differential equations and initial conditions corresponding exactly to the descendent integrals. See [FP1,§1].

The main idea in the proofs of the remaining results is to use the virtual localization formula of Graber and Pandharipande [GrP] in reverse. The localization formula expresses an integral of suitable $\mathbb{C}^*$-equivariant cycle classes over (e.g.) $\overline{M}_{g,n}(\mathbf{P}^r, d)$ as a sum over the fixed point loci; the summands are products of Hodge integrals. Applying this procedure to a known integral yields a relation among Hodge integrals. Usually we take an integral that vanishes for obvious reasons. One can also obtain a relation among Hodge integrals by computing a not necessarily known integral via two different linearizations of the torus action.

So far, we have exclusively used spaces of maps to $\mathbf{P}^1$. Even in this case, it is important to have some control on the number of summands. By a careful choice of integrand and linearizations one can sometimes arrange that many fixed point loci don't contribute, so that one obtains a manageable relation among Hodge integrals that is hopefully non-trivial. An example is given in §3.

2. Remarks

The following remarks are intended to provide some explanation for the results above and to put them into context.

Recall that on $\overline{M}_{g,n}$ one defines the kappa classes κ_i by

$$\kappa_i = \pi_{n+1,n*}(\psi_{n+1}^{i+1})$$

where $\pi_{n+1,n} : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is the map obtained by forgetting the $(n+1)$st point and stabilizing. That these are the 'right' kappa classes in this context was demonstrated by Arbarello and Cornalba [AC]. For example, the pullback of κ_i to the product $\prod_j \overline{M}_{g_j,n_j}$ canonically covering a boundary stratum is the sum of the product classes consisting of κ_i on one factor and the identity on the other factors. Also,

$$\kappa_i = \pi_{n,n-1}^* \kappa_i + \psi_n^i \, .$$

To compare κ_i with $\pi_{n,n-1}^* \pi_{n-1,n-2}^* \kappa_i$, combine this with Witten's observation

$$\psi_j = \pi_{n,n-1}^* \psi_j + \delta_{0,\{j,n\}} \qquad (1 \le j \le n-1)$$

where $\delta_{0,\{j,n\}}$ is the boundary divisor of curves possessing a rational tail containing only the marked points j and n.

As a direct consequence of the results in §1, the integrals (for $g \ge 2$)

$$\int_{\overline{M}_g} \kappa_{3g-3} \, , \quad \int_{\overline{M}_g} \kappa_{2g-3} \lambda_g \, , \quad \int_{\overline{M}_g} \kappa_{g-2} \lambda_g \lambda_{g-1} \, , \quad \int_{\overline{M}_g} \lambda_g \lambda_{g-1} \lambda_{g-2}$$

are *nonzero*. As explained in [FP3,§0], this shows that the evaluations on the quotient rings of the tautological ring $R^*(\overline{M}_g)$ considered there are non-trivial.

We extend the perspective of [FP3] to pointed curves and consider the natural sequence of quotient rings of the tautological ring $R^*(\overline{M}_{g,n})$:

$$R^*(\overline{M}_{g,n}) \to R^*(M_{g,n}^{\mathrm{ct}}) \to R^*(M_{g,n}^{\mathrm{rt}}) \to R^*(X_{g,n}^{\mathrm{rt}})$$

associated to the filtration

$$\overline{M}_{g,n} \supset M^{\mathrm{ct}}_{g,n} \supset M^{\mathrm{rt}}_{g,n} \supset X^{\mathrm{rt}}_{g,n} \,.$$

Here $M^{\mathrm{ct}}_{g,n}$ is the moduli space of stable pointed curves of *compact type* (i.e., the dual graph is a tree). It is the complement of the boundary divisor Δ_{irr} (empty for $g = 0$) parameterizing irreducible singular curves and their degenerations. ($M^{\mathrm{ct}}_{g,0}$ was denoted M^{c}_{g} in [FP3].)

Next, $M^{\mathrm{rt}}_{g,n}$ is the moduli space of stable pointed curves with *rational tails* (i.e., one component has geometric genus g and the other components (if any) have genus 0). Note that $M^{\mathrm{rt}}_{1,n} = M^{\mathrm{ct}}_{1,n}$ while $M^{\mathrm{rt}}_{g,n}$ equals the inverse image of M_g under $\pi : \overline{M}_{g,n} \to \overline{M}_g$ for $g \geq 2$. (So $M^{\mathrm{rt}}_{g,n} = M_{g,n}$ for $n \leq 1$.)

Finally, let X_g be a general nonsingular curve of genus $g \geq 2$. With $X^{\mathrm{rt}}_{g,n}$ we denote the fiber of $M^{\mathrm{rt}}_{g,n}$ over $[X_g]$. This equals the Fulton-MacPherson compactification of $X^n_g - \Delta$, where Δ denotes the union of the diagonals in the cartesian product. (So $X^{\mathrm{rt}}_{g,n} = X^n_g$ for $n \leq 2$.)

Actually, X_g should perhaps be the generic curve of genus $g \geq 2$; it is not completely clear yet. In any case, we were naturally led to including the term $X^{\mathrm{rt}}_{g,n}$ in the filtration, as I will now try to explain.

The goal is to understand the tautological ring $R^*(\overline{M}_{g,n})$ (compare [FP3,§0.3]). From the start it is clear that the presence of a multitude of boundary strata makes this a complicated task, at least from the combinatorial point of view. Therefore it is desirable to find 'natural' ways of forgetting some of the boundary strata.

$M^{\mathrm{ct}}_{g,n}$ seems a first good choice. It parameterizes the (pointed) curves whose Jacobian is an abelian variety. Further, the class λ_g vanishes when restricted to the complement Δ_{irr}; just as in [FP3] this gives rise to an evaluation ϵ on $A^*(M^{\mathrm{ct}}_{g,n})$:

$$\xi \mapsto \epsilon(\xi) = \int_{\overline{M}_{g,n}} \xi \cdot \lambda_g \,.$$

Formula (3) shows that the evaluation of monomials in the ψ classes against λ_g is very well behaved. (This should be contrasted with the more complicated Witten-Kontsevich theory for the evaluation of ψ-monomials on $\overline{M}_{g,n}$.) The λ_g-evaluations of the ψ-monomials (for varying g and n) determine the λ_g-evaluations of all intersection products in the tautological rings $R^*(M^{\mathrm{ct}}_{g,n})$. One may hope that the rings $R^*(M^{\mathrm{ct}}_{g,n})$ will be easier to understand than the rings $R^*(\overline{M}_{g,n})$. It should be pointed out that the λ_g-formula (3) was found by Getzler and Pandharipande [GeP] as a very special corollary of the Virasoro conjecture of Eguchi, Hori and Xiong [EHX].

Forgetting the boundary strata of $\overline{M}_{g,n}$ lying over boundary strata of $\overline{M}_g$ leads to the next step in the filtration, $M^{\mathrm{rt}}_{g,n}$. This is the relative Fulton-MacPherson compactification of $M_{g,n}$ over M_g. Just as in [F], the class $\lambda_g\lambda_{g-1}$ vanishes when restricted to the complement. (The class is proportional to $\mathrm{ch}_{2g-1}(\mathbb{E})$, which is the top Chern character component in genus g, so it vanishes on any boundary stratum parameterizing curves that have no component of geometric genus g.) Again, this leads to an evaluation ϵ on $A^*(M^{\mathrm{rt}}_{g,n})$:

$$\xi \mapsto \epsilon(\xi) = \int_{\overline{M}_{g,n}} \xi \cdot \lambda_g\lambda_{g-1}\,.$$

In this case, there is a *conjectural* formula [F] for the evaluation of ψ-monomials against $\lambda_g\lambda_{g-1}$:

(6)

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n}\lambda_g\lambda_{g-1} = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{i=1}^n(2a_i-1)!!}\int_{\overline{M}_{g,1}} \psi_1^{g-1}\lambda_g\lambda_{g-1}$$

for *positive* integers $a_1,\ldots,a_n$ with sum $g-2+n$. In [GeP] this formula was shown to be yet another corollary of the Virasoro conjecture of [EHX]. The formula determines the $\lambda_g\lambda_{g-1}$-evaluations of all intersection products in the tautological rings $R^*(M^{\mathrm{rt}}_{g,n})$.

Extending the speculations of [FP3], one may hope that $R^*(\overline{M}_{g,n})$, $R^*(M^{\mathrm{ct}}_{g,n})$, and $R^*(M^{\mathrm{rt}}_{g,n})$ are all Gorenstein algebras, with socle in codimension $3g-3+n$ resp. $2g-3+n$ resp. $g-2+n$. But the evidence for such speculations remains scant.

Although $X^{\mathrm{rt}}_{g,n}$ doesn't arise as the result of forgetting even more boundary strata, there are nevertheless two reasons to include it in the filtration. The first is that after noticing the special role that the classes λ_g and $\lambda_g\lambda_{g-1}$ play, one is tempted to consider $\lambda_g\lambda_{g-1}\lambda_{g-2}$ as well. This picks out a point class $[X_g]$ in M_g and the fiber $X^{\mathrm{rt}}_{g,n}$ over it in $M^{\mathrm{rt}}_{g,n}$. To obtain a natural map on the level of Chow groups it seems best to take X_g to be the generic curve of genus g. The second reason is that in the study of $R^*(M^{\mathrm{rt}}_{g,n})$ as a module over $R^*(M_g)$ one needs to understand the fiber first.

Thus we were led to considering the tautological ring $R^*(X_g^n)$ of the cartesian product X_g^n (this should be easier than yet essentially the same as $R^*(X^{\mathrm{rt}}_{g,n})$). It is clear what $R^*(X_g^n)$ should be: the $\mathbb{Q}$-subalgebra of $A^*(X_g^n)\otimes\mathbb{Q}$ generated by the divisor classes K_i (the canonical class on the ith factor) and D_{ij} (the class of the diagonal $\{x_i = x_j\}$). The question is what the relations between these classes

Our result (yet to be written up) is that we have an explicit description of the image $RH^*(X_g^n)$ of $R^*(X_g^n)$ in cohomology and that this image is a Gorenstein algebra (with socle in algebraic degree n). On the other hand, already for $n = 2$ we don't understand $R^*(X_g^n)$ at the moment. The crucial question is whether

$$K_1K_2 - (2g-2)K_1D_{12}$$

is zero (torsion) in $A^2(X_g \times X_g)$. This holds for $g \leq 3$ but we don't know the answer for general curves of genus $g \geq 4$.

The evaluation of ψ-monomials against $\lambda_g\lambda_{g-1}\lambda_{g-2}$ is governed by the string and dilaton equations and this determines the intersection products in $R^*(X_{g,n}^{\mathrm{rt}})$.

Finally, formula (2) determines the evaluation on $\overline{M}_{g,1}$ of a single λ class times a power of ψ_1. Similarly, formula (5) determines the $\lambda_g\lambda_{g-1}$-evaluation on $\overline{M}_{g,1}$ of such a product. Somewhat surprisingly, a formula for the λ_g-evaluation of such a product is as yet missing.

3. An example

We present here in some detail the proof of the third result in §1:

$$\int_{[\overline{M}_g(d)]^{\mathrm{vir}}} c_{\mathrm{top}}(R^1\pi_*\mu^*N) = \frac{|(2g-1)B_{2g}|}{(2g)!}\, d^{2g-3}.$$

First we check dimensions. By the well-known formula for the virtual dimension of $\overline{M}_{g,n}(X,\beta)$:

$$3g-3+n+(\dim X)(1-g)+\beta\cdot c_1(T_X),$$

the virtual dimension of $\overline{M}_g(d) = \overline{M}_{g,0}(\mathbf{P}^1, d)$ equals $3g-3+1-g+2d = 2g-2+2d$. Since $H^1(\mathcal{O}(-d))$ has for $d \geq 1$ dimension $g-1+d$ on a genus g curve, $R^1\pi_*\mu^*N$ is a bundle of rank $2g-2+2d$, which checks.

Our conventions for the $\mathbb{C}^*$-action on $\mathbf{P}^1$ are:

$$V = \mathbb{C}\oplus\mathbb{C}, \qquad \mathbf{P}^1 = \mathbf{P}(V), \qquad \xi\cdot(v_1,v_2) = (v_1,\xi v_2).$$

The fixed points on $\mathbf{P}(V)$ are $p_1 = [1,0]$ and $p_2 = [0,1]$. An equivariant lifting of the $\mathbb{C}^*$-action to a line bundle L over $\mathbf{P}(V)$ is uniquely determined by the weights $[l_1,l_2]$ on the fibers $L_1 = L|_{p_1}$ and $L_2 = L|_{p_2}$ at the fixed points. With this notation, the canonical lifting of the $\mathbb{C}^*$-action to the tangent bundle $T_{\mathbf{P}(V)}$ has weights $[1,-1]$; in general, $l_1 - l_2 = c_1(L)$. The possible linearizations on $\mathcal{O}_{\mathbf{P}}(-1)$ are $[\alpha,\alpha+1]$ with $\alpha\in\mathbb{Z}$. Note that we can choose two different linearizations for the two summands $\mathcal{O}_{\mathbf{P}}(-1)$ of N.

As explained in [K2], the $\mathbb{C}^*$-fixed loci in $\overline{M}_g(d)$ are conveniently indexed by graphs Γ. The vertices of these graphs lie over p_1 or p_2 and correspond to the connected components of the inverse image of p_i. The vertices are labeled with the arithmetic genera of the corresponding connected components, which sum over the graph to $g - h^1(\Gamma)$. The edges lie over $\mathbf{P}^1$ and are labeled with positive degrees that sum to d. Each edge e corresponds to a $\mathbf{P}^1$ mapping to the target $\mathbf{P}^1$ with degree d_e, with two points of total ramification over p_1 and p_2.

For such a graph, define a flag f as an incident edge-vertex pair (e, v). To every edge correspond two flags. Let $x_f = C_{e(f)} \cap C_{v(f)}$ denote the incidence point of a flag f. One has the exact sequence [GrP]:

$$0 \to \mathcal{O}_C \to \bigoplus_v \mathcal{O}_{C_v} \oplus \bigoplus_e \mathcal{O}_{C_e} \to \bigoplus_f \mathcal{O}_{x_f} \to 0\,.$$

This is the partial normalization sequence that arises from resolving the nodes of C corresponding to flags. The third map is

$$\big((a_v)_v, (a_e)_e\big) \mapsto \big(a_{v(f)}(x_f) - a_{e(f)}(x_f)\big)_f\,.$$

Twisting by $\mu^*\mathcal{O}(-1)$ and taking cohomology yields:

$$0 \to \bigoplus_v \mathcal{O}_{p_{i(v)}}(-1) \to \bigoplus_f \mathcal{O}_{p_{i(v(f))}}(-1) \to$$

$$H^1(C, \mu^*\mathcal{O}(-1)) \to \bigoplus_v H^1(C_v, \mu^*\mathcal{O}(-1)) \oplus \bigoplus_e H^1(C_e, \mu^*\mathcal{O}(-1)) \to 0$$

where $i(v) \in \{1, 2\}$ is defined by $\mu(C_v) = p_{i(v)}$.

We apply the localization formula of Graber and Pandharipande [GrP] to compute the integral. The integrand is the top Chern class of the bundle with fibers $H^1(C, \mu^*(\mathcal{O}(-1)) \oplus H^1(C, \mu^*(\mathcal{O}(-1))$. ¿From the cohomology sequence, a vertex with valence m contributes $m - 1$ terms $\mathcal{O}_{p_{i(v)}}(-1)$ to $H^1(C, \mu^*(\mathcal{O}(-1))$. These terms are pure weight. If the linearization $[\alpha, \alpha+1]$ is chosen for $\mathcal{O}_{\mathbf{P}}(-1)$, then $\mathcal{O}_{p_1}(-1)$ has weight α and $\mathcal{O}_{p_2}(-1)$ has weight $\alpha + 1$.

Choose the linearization of N given by $[0, 1]$ on one summand and $[-1, 0]$ on the other summand. Then *any* graph that has a vertex of valence at least 2 doesn't contribute to the integral: $H^1(C, \mu^*N)$ contains terms $\mathcal{O}_{p_1}(-1)$ or $\mathcal{O}_{p_2}(-1)$ of pure weight zero, so the integrand vanishes.

Contributing graphs have therefore one edge, labeled with degree d, and two vertices, labeled with genera g_1 and g_2 with sum g. For such a graph, the cohomology sequence yields an isomorphism

$$H^1(C, \mu^*\mathcal{O}(-1)) \cong (\mathbb{E}_{g_1}^\vee \otimes \mathcal{O}_{p_1}(-1)) \oplus (\mathbb{E}_{g_2}^\vee \otimes \mathcal{O}_{p_2}(-1)) \oplus H^1(\mathbf{P}^1, \mathcal{O}(-d)).$$

With linearization $[\alpha, \alpha+1]$, the top Chern class of $H^1(C, \mu^*\mathcal{O}(-1))$ contributes

$$c_{\alpha^{-1}}(\mathbb{E}_{g_1}^\vee)\alpha^{g_1} \cdot c_{(\alpha+1)^{-1}}(\mathbb{E}_{g_2}^\vee)(\alpha+1)^{g_2} \cdot \prod_{\substack{a,b<0:\\ a+b=-d}} \frac{a(-\alpha)+b(-\alpha-1)}{d}$$

(compare [GrP,p.505] and [K2,p.358]). So with the chosen linearization of N, the contribution of $c_{\mathrm{top}}(R^1\pi_*\mu^*N)$ becomes

$$(-1)^{g_1}\lambda_{g_1}^{(1)} \cdot c(\mathbb{E}_{g_2}^\vee) \cdot \frac{d!}{d^d} \cdot (-1)^{g_1} c(\mathbb{E}_{g_1}) \cdot (-1)^{g_2}\lambda_{g_2}^{(2)} \cdot (-1)^{d-1}\frac{d!}{d^d}.$$

This has to be multiplied with $1/e(N^{\mathrm{vir}})$, the inverse Euler class of the virtual normal bundle to the fixed point locus corresponding to the graph. In case g_1 and g_2 are both positive, we can use the formula from [GrP,p.505]:

$$\frac{1}{e(N^{\mathrm{vir}})} = \frac{1}{\frac{1}{d}-\psi^{(1)}} \cdot 1 \cdot \frac{1}{-\frac{1}{d}-\psi^{(2)}} \cdot (-1) \cdot c(\mathbb{E}_{g_1}^\vee) \cdot c(\mathbb{E}_{g_2})(-1)^{g_2-1} \cdot \frac{(-1)^d d^{2d}}{(d!)^2 \cdot 1}.$$

Multiplication gives

$$\int_{\overline{M}_{g_1,1}} \frac{\lambda_{g_1}}{\frac{1}{d}-\psi_1} \cdot \int_{\overline{M}_{g_2,1}} \frac{\lambda_{g_2}}{\frac{1}{d}+\psi_1}$$

since $c(\mathbb{E})c(\mathbb{E}^\vee) = 1$. The contribution of the graph is then

$$\frac{1}{d} \cdot d^{2g_1-1} b_{g_1} \cdot d^{2g_2-1} b_{g_2} = d^{2g-3} b_{g_1} b_{g_2} \tag{7}$$

(the extra $\frac{1}{d}$ comes from automorphisms of the map [GrP,p.501]).

In case g_1 resp. g_2 is zero, replace $(\frac{1}{d}-\psi^{(1)})^{-1}$ by $\frac{1}{d}$ resp. $(-\frac{1}{d}-\psi^{(2)})^{-1}$ by $-\frac{1}{d}$ (see [K2,p.356]). The result is that formula (7) continues to hold when g_1 or g_2 is zero.

Therefore

$$\int_{[\overline{M}_g(d)]^{\mathrm{vir}}} c_{\mathrm{top}}(R^1\pi_*\mu^*N) = \sum_{\substack{g_1,g_2\geq 0:\\ g_1+g_2=g}} d^{2g-3} b_{g_1} b_{g_2} = \frac{|(2g-1)B_{2g}|}{(2g)!} d^{2g-3},$$

where the last equality follows from Bernoulli identities [FP1,§4.2].

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