数理解析研究所講究録 1232 巻 2001 年 109-124

Brill-Noether problem for sheaves on K3 surfaces

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1 Introduction

This paper consists of a work with Toshiya Kawai [K-Y, sect. 5] and some remarks on my paper [Y1]. In [K-Y, sect. 5], we tried to understand the meaning of string partition function on elliptically and K3 fibered Calabi-Yau 3-folds in terms of $D0$-$D2$ branes. We conjectured that string partition function is constructed by lifting procedure from a jacobi form of weight 0

$$\Phi_0(\tau, z, \nu) = \frac{\Psi_{10,m}(\tau, z)}{\chi_{10,1}(\tau, \nu)} \tag{1.1}$$

where $\Psi_{10,m}(\tau, z)$ is a jacobi form of weight 10 and index m and $\chi_{10,1}(\tau, \nu)$ is the cusp jacobi form of weight 10 and index 1 [K], [K-Y, sect. 4]. $\Psi_{10,m}(\tau, z)$ depends on the choice of Calabi-Yau 3-fold. In [K-Y, sect. 5], we understand independent term $1/\chi_{10,1}(\tau, \nu)$ as a contribution of $D0$-$D2$ branes on a fixed K3 surface. We interpret $D0$-$D2$ branes as pairs (L, s) of dimension 1 sheaves L and sections $s \in H^0(L)$. Then $1/\chi_{10,1}(\tau, \nu)$ is regarded as Euler characteristics of moduli spaces of these pairs (more precisely, moduli spaces of coherent systems) on a fixed K3 fiber (Theorem 3.24).

As far as I know, moduli spaces of stable pairs, or coherent systems are used as a tool for investigating moduli spaces of vector bundles. For example, they are used to show Verlinde formula by Thaddeus [T], to compute Donaldson invariant by O'Grady [O], Le-Potier [Le], He [He],... and to compute Hodge numbers of moduli spaces by Göttsche-Huybrechts [G-H]. ¿From this point of view, our result is interesting. That is, our result make us to expect that moduli spaces of coherent systems have good structure.

For our computation of Euler characteristics, we need to control $\dim H^0(L)$. Hence we need to analyse Brill-Noether locus (BN locus) of moduli spaces of sheaves. In general this is a difficult problem, but in our case BN locus behaves very well. Hence we can compute Euler characteristics of moduli spaces of coherent systems. For more details, see our paper [K-Y].

*The second part of this paper was done during my stay at Max-Planck Institut für Mathematik in

In the second part, we consider the contraction of BN locus and the ample cone of moduli spaces. We also give some examples of birational maps.

Finally we remark that Markman [Mr] also studied (-2)-reflections and Brill-Noether locus of moduli spaces as an example of his generalized elementary transformation of symplectic manifold.

2 Preliminaries

Hodge polynomials: For a smooth complex projective variety V, we define the Hodge polynomial by

$$\chi_{t,\bar{t}}(V) := \sum_{p,q=0}^{\dim(V)} (-1)^{p+q} h^{p,q}(V) t^p \bar{t}^q \,, \tag{2.1}$$

where $h^{p,q}(V) = \dim H^q(V, \Omega_V^p)$. We also introduce

$$\chi_t(V) := \chi_{t,1}(V) \,, \tag{2.2}$$

which is essentially the Hirzebruch χ_y genus of V. Note that the Euler characteristic of V is given by $\chi(V) = \chi_1(V)$.

Mukai lattice: Let X be a K3 surface. The Mukai lattice of X is the total integer cohomology group

$$H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \,, \tag{2.3}$$

endowed with the symmetric bilinear form

$$\langle v, v' \rangle = \int_X (c_1 \wedge c_1' - r \wedge a' \varrho - r' \wedge a \varrho) \,, \tag{2.4}$$

for any $v = (r, c_1, a) \in H^*(X,\mathbb{Z})$ and $v' = (r', c_1', a') \in H^*(X,\mathbb{Z})$. Here the notation $v = (r, c_1, a)$ means $v = r \oplus c_1 \oplus a\varrho$ with $r \in H^0(X,\mathbb{Z})$, $c_1 \in H^2(X,\mathbb{Z})$, $a \in \mathbb{Z}$ and $\varrho \in H^4(X,\mathbb{Z})$ is the fundamental cohomology class of X so that $\int_X \varrho = 1$. We have $H^*(X,\mathbb{Z}) \cong \mathsf{E}_8(-1)^{\oplus 2} \oplus H^{\oplus 4}$ where E_8 is the positive definite even unimodular lattice of rank 8.

The Grothendieck group $K(X)$ is defined to be the quotient of the free abelian group generated by all the coherent sheaves (up to isomorphisms) on X by the subgroup generated by the elements $F - E - G$ for each short exact sequence

$$0 \to E \to F \to G \to 0 \tag{2.5}$$

of coherent sheaves on X. In what follows, we shall use the same notation E for both a coherent sheaf on X and its image in $K(X)$.

Let $v : K(X) \to \oplus_i H^{2i}(X,\mathbb{Q})$ be the module homomorphism defined by Mukai vectors, namely $E \mapsto v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$. Explicitly we have

$$v(E) = \left(\mathrm{rk}(E),\, c_1(E),\, \mathrm{rk}(E)\,\varrho + \frac{1}{2} c_1(E)^2 - c_2(E)\right) . \tag{2.6}$$

Thus actually we have $v(K(X)) \subset H^{2*}(X, \mathbb{Z})$ since $H^2(X, \mathbb{Z})$ is even. The image $v(K(X))$ is $\mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\varrho$. This definition is such that

$$\chi(E, F) := \sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(E, F) = -\langle v(E), v(F) \rangle, \qquad (2.7)$$

by the Hirzebruch-Riemann-Roch formula.
Isometry of Mukai lattice: The Mukai lattice has several distinguished isometries.

(i) Let N be a line bundle on X. Since $\langle x\,\mathrm{ch}(N), y\,\mathrm{ch}(N) \rangle = \langle x, y \rangle$, the homomorphism

$$\begin{array}{rccc} T_N: & H^*(X,\mathbb{Z}) & \to & H^*(X,\mathbb{Z}) \\ & x & \mapsto & x\,\mathrm{ch}(N) \end{array}$$

is an isometry.

(ii) $\mathrm{O}(H^2(X, \mathbb{Z}))$ acts on $H^*(X, \mathbb{Z})$.

(iii) Let $v_1 \in H^*(X, \mathbb{Z})$ be a Mukai vector of $\langle v_1^2 \rangle = -2$. Then the (-2)-reflection

$$\begin{array}{rccc} R_{v_1}: & H^*(X,\mathbb{Z}) & \to & H^*(X,\mathbb{Z}) \\ & x & \mapsto & x + \langle v_1, x \rangle v_1 \end{array}$$

is an isometry.

(iv)

$$\begin{array}{rccc} D: & H^*(X,\mathbb{Z}) & \to & H^*(X,\mathbb{Z}) \\ & x = (r, c_1, a) & \mapsto & x^\vee = (r, -c_1, a) \end{array}$$

is an isometry.

It is known that $\mathrm{O}(H^*(X, \mathbb{Z}))/\pm 1$ is generated by these transformations and $\mathrm{O}(H^*(X, \mathbb{Z}))$ acts transitively on the set of primitive Mukai vectors v of the same $\langle v^2 \rangle$. Hence it is important to study (-2)-reflections.
Muduli spaces of stable sheaves: Let $M_H(v)$ be the moduli space of stable sheaves E of $v(E) = v$. If v is primitive, then for a suitable polarization, $M_H(v)$ becomes a smooth projective manifold.

We need the following theorem [Y3, Thm. 5.1].

Theorem 2.8. *Let v be a primitive Mukai vector such that $\mathrm{rk}\, v > 0$, or $\mathrm{rk}\, v = 0$ and $c_1(v)$ is ample. Then $M_H(v)$ is deformation equivalent to $X^{[\langle v^2 \rangle/2+1]}$. If $X \to \mathbb{P}^1$ is an elliptic K3 and f is a fiber, then the same result holds for $M_H(0, f, a)$. In particular, $\chi_{t,\bar{t}}(M_H(v)) = \chi_{t,\bar{t}}(X^{[\langle v^2 \rangle/2+1]})$.*

3 Coherent systems

Let C_h be an effective divisor of $(C_h^2)/2 = h - 1$.

Definition 3.1. *We set* $v = (r, C_h, a)$. *Let*

$$\operatorname{Syst}^n(v) := \{(E, U) | E \in M_H(v), U \subset H^0(X, E), \dim U = n\} \tag{3.2}$$

be the moduli space of coherent systems and $p_v : \operatorname{Syst}^n(v) \to M_H(v)$ *the natural projection.*

In order to consider fibers of p_v, we introduce a stratification.

Definition 3.3. *For* $i \geq \max\{0, \langle v, v_1 \rangle\}$, *we set*

$$\begin{aligned} M_H(v)_i &:= \{E \in M_H(v) | \dim H^0(X, E) = -\langle v, v_1 \rangle + i\}, \\ \operatorname{Syst}^n(v)_i &:= p_v^{-1}(M_H(v)_i). \end{aligned} \tag{3.4}$$

We consider the following two conditions on C_h:

($\star$1) There is an ample line bundle H such that

$$(C_h, H) = \min\{(L, H) | L \in \operatorname{Pic}(X), (L, H) > 0\}. \tag{3.5}$$

($\star$2) Every member of $|C_h|$ is irreducible and reduced.

Obviously, condition ($\star$1) implies condition ($\star$2).

Assume that $n \leq r$. We set $w = (r - n, C_h, a - n)$ and $m = n - (r + a)$. Then we have a morphism

$$\begin{array}{rcl} q_v : \operatorname{Syst}^n(v) & \to & M_H(w) \\ (f : U \otimes \mathcal{O}_X \to E) & \mapsto & \operatorname{coker} f \end{array} \tag{3.6}$$

and we get the following diagram:

$$\begin{array}{ccccc} & & \operatorname{Syst}^n(v)_i & & \\ & {}^{p_v}\swarrow & & \searrow{}^{q_w} & \\ M_H(v)_i & & & & M_H(w)_{i+n}, \end{array} \tag{3.7}$$

where p_v is an étale locally trivial $Gr(-\langle v, v_1 \rangle + i, n)$-bundle and q_w is an étale locally trivial $Gr(i + n, n)$-bundle.

Lemma 3.8. *[K-Y] Under the condition ($\star$1),* $\operatorname{Syst}^n(v)$ *is a smooth scheme of dimension* $\langle v^2 \rangle + 2 - n(n + \langle v_1, v \rangle)$, *where* $v_1 = (1, 0, 1)$.

Outline of the proof. Let $\Lambda = (E, U)$ be a point of $\operatorname{Syst}^n(v)$. Then the Zariski tangent space of $\operatorname{Syst}^n(v)$ at Λ is given by $\mathbb{E}\mathrm{xt}^1(U \otimes \mathcal{O}_X \to E, E)/\operatorname{Hom}(U, U)$, and the obstruction of infinitesimal liftings belong to the kernel of the composition of homomorphisms

$$\tau : \mathbb{E}\mathrm{xt}^2(U \otimes \mathcal{O}_X \to E, E) \to \operatorname{Ext}^2(E, E) \xrightarrow{tr} H^2(X, \mathcal{O}_X), \tag{3.9}$$

where $\mathbb{E}\mathrm{xt}^*(U\otimes\mathcal{O}_X \to E,*)$ is the hypercohomology associated to the complex $U\otimes\mathcal{O}_X \to E$. By using the universal extension

$$0 \longrightarrow \mathcal{O}_X \otimes \mathrm{Ext}^1(E,\mathcal{O}_X)^\vee \longrightarrow G \longrightarrow E \longrightarrow 0, \tag{3.10}$$

we can show that $\mathbb{E}\mathrm{xt}^2(U \otimes \mathcal{O}_X \to E, E) \cong \mathbb{C}$, which implies that $\mathrm{Syst}^n(v)$ is smooth at Λ. □

By using Lemma 3.8, we see that

Corollary 3.11. *[Y1, Cor. 5.3] Assume that* $i > \max\{0, \langle v, v_1\rangle\}$. *Under the condition* ($\star$*1),*

(i) BN locus $\overline{M_H(v)_i}$ *has a stratification* $\overline{M_H(v)_i} = \cup_{j\geq i} M_H(v)_j$,

(ii) $\overline{M_H(v)_i}$ *has the expected dimension* $\langle v^2\rangle + 2 - i(i - \langle v_1, v\rangle)$.

(iii) $\overline{M_H(v)_i}$ *is singular along* $\cup_{j>i} M_H(v)_j$,

(iv) $q_v : \mathrm{Syst}^i(v + iv_1) \to \overline{M_H(v)_i}$ *is a desingularization of* $\overline{M_H(v)_i}$.

Remark 3.12. We can define scheme structure on $\overline{M_H(v)_i}$ by using fitting ideal [ACGH]. Then we see that $\overline{M_H(v)_i}$ is Cohen-Macaulay, reduced and normal (see [ACGH]).

Remark 3.13. We have another desingularization:

$$\begin{array}{ccc} \mathrm{Syst}^i(v+iv_1) & \leftarrow \cdots\cdots \rightarrow & \mathrm{Syst}^{i-\langle v_1,v\rangle}(v) \\ & \searrow \quad \swarrow & \\ & \overline{M_H(v)_i} & \end{array} \tag{3.14}$$

The following proposition which plays important roles is due to Markman [Mr, Thm. 39].

Proposition 3.15. *[K-Y] Assume that* C_h *satisfies condition (*$\star$*1). For* $n \geq r$, *we have an isomorphism*

$$\delta : \mathrm{Syst}^n(r, C_h, a) \to \mathrm{Syst}^n(n-r, C_h, n-a). \tag{3.16}$$

If $n = 1$ *and* $r = 0$, *then the same assertion holds under the condition (*$\star$*2).*

Outline of the proof. For a coherent system $f : U \otimes \mathcal{O}_X \to E$, by our assumptions, we see that

(i) f is surjective in codimension 1 (and hence $\dim\operatorname{coker} f = 0$) and $\ker f$ is a (slope) stable sheaf, or

(ii) f is injective and $\operatorname{coker} f$ is a (slope) stable sheaf

according as (i) $n > r$ or (ii) $n = r$. For the second case, f is also generically surjective. Hence we get an exact sequence

$$\begin{aligned} 0 \to \mathcal{H}om_{\mathcal{O}_X}(E,\mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(U \otimes \mathcal{O}_X, \mathcal{O}_X) \to \\ \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X) \to \mathcal{E}xt^1_{\mathcal{O}_X}(E,\mathcal{O}_X) \to 0 \end{aligned} \tag{3.17}$$

We set $D(E) := \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X)$. Then $U^\vee \otimes \mathcal{O}_X \to D(E)$ is an element of $\mathrm{Syst}^n(n-r, C_h, n-a)$. Hence we get a map $\delta : \mathrm{Syst}^n(r, C_h, a) \to \mathrm{Syst}^n(n-r, C_h, n-a)$. It is not difficult to see that δ is an isomorphism. □

Corollary 3.18. *[K-Y] By the above isomorphism, we get the following diagram:*

$$\begin{tikzcd} & \mathrm{Syst}^n(v)_i \cong \mathrm{Syst}^n(w)_{r+a-n+i} \arrow[dl, "p_v"'] \arrow[dr, "p_w"] & \\ M_H(v)_i & & M_H(w)_{r+a-n+i} \end{tikzcd} \tag{3.19}$$

where $v = (r, C_h, a)$ and $w = (n-r, C_h, n-a)$.

Proof. Let $U \otimes \mathcal{O}_X \to E$ be an element of $\mathrm{Syst}^n(v)$. Since $\mathcal{E}xt^i_{\mathcal{O}_X}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X) = 0$ for $i \neq 1$, we get

$$\mathbb{E}\mathrm{xt}^{i+1}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X) \cong H^i(X, \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X)). \tag{3.20}$$

Since $\mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X)$ is a stable sheaf of positive degree, Serre duality and (3.20) imply that

$$\mathbb{E}\mathrm{xt}^3(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X) = H^2(X, \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X)) = 0. \tag{3.21}$$

By using the canonical exact sequence

$$\begin{aligned} 0 = \mathrm{Ext}^1(U \otimes \mathcal{O}_X, \mathcal{O}_X) \to \mathbb{E}\mathrm{xt}^2(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X) \to \\ \mathrm{Ext}^2(E, \mathcal{O}_X) \to \mathrm{Ext}^2(U \otimes \mathcal{O}_X, \mathcal{O}_X) \to 0, \end{aligned} \tag{3.22}$$

we see that

$$\begin{aligned} \dim H^1(X, \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X)) &= \dim \mathbb{E}\mathrm{xt}^2(U \otimes \mathcal{O}_X \to E, \mathcal{O}_X) \\ &= \dim \mathrm{Ext}^2(E, \mathcal{O}_X) - n \\ &= \dim H^0(X, E) - n = r + a + i - n. \end{aligned} \tag{3.23}$$

□

By using the diagram (3.7), Corollary 3.18 and Theorem 2.8, we can show our main assertion of the talk at RIMS.

Theorem 3.24. *[K-Y] Assume that C_h satisfies $(\star 1)$ for all $h \geq 0$. Then, for $0 < |q| < |y| < 1$,*

$$\begin{aligned} &\sum_{h=0}^{\infty} \sum_{d=0}^{\infty} \chi_{t,\tilde{t}}(\mathrm{Syst}^1(0, C_h, d+1-h))(t\tilde{t})^{1-h} q^{h-1} y^{d+1-h} \\ &= \frac{-1}{q(y)_\infty (q/y)_\infty ((t\tilde{t}y)^{-1})_\infty (t\tilde{t}yq)_\infty (t\tilde{t}^{-1}q)_\infty (q)_\infty^{18} (t^{-1}\tilde{t}q)_\infty}, \end{aligned} \tag{3.25}$$

where

$$(\xi)_\infty = \prod_{n=0}^{\infty} (1 - \xi q^n). \tag{3.26}$$

In particular, by setting $t = \tilde{t} = 1$, we obtain

$$\sum_{h=0}^{\infty} \sum_{d=0}^{\infty} \chi(\mathrm{Syst}^1(0, C_h, d+1-h)) q^{h-1} y^{d+1-h} = \frac{1}{\chi_{10,1}(\tau, \nu)}. \tag{3.27}$$

Moreover, if C_h is ample and satisfies $(\star 2)$, then $\chi_{t,\tilde{t}}(\mathrm{Syst}^1(0, C_h, d+1-h))$ is meaningful and can be obtained from (3.25) as if C_h satisfied $(\star 1)$.

As we explained in Introduction, (3.27) gives the meaning of $1/\chi_{10,1}(\tau,\nu)$ which appears in the string partition function of elliptically and K3 fibered Calabi-Yau 3-fold. For the last claim, we use the following lemma and deformation argument.

Lemma 3.28. *Under the condition (⋆2),* $\mathrm{Syst}^1(0, C_h, a)$ *is smooth of dimension* $2h + a - 1$.

Proof. By Proposition 3.15, $\mathrm{Syst}^1(0, C_h, a)$ is isomorphic to $\mathrm{Syst}^1(1, C_h, 1-a)$. Hence we shall prove that $\mathrm{Syst}^1(1, C_h, 1-a)$ is smooth. Let $f : \mathcal{O}_X \to I_Z(C)$ be an element of $\mathrm{Syst}^1(1, C_h, 1-a)$. Then condition ($\star$2) implies that f is injective and $L := \operatorname{coker} f$ is a torsion free sheaf on C. In order to prove the smoothness of $\mathrm{Syst}^1(1, C_h, 1-a)$ at $f : \mathcal{O}_X \to I_Z(C)$, it is sufficient to prove that $\mathrm{Hom}(I_Z(C), L) \cong \mathbb{C}$. Since $I_Z(C)_{|C}/(torsion) \cong L$ and L is simple, we get our claim. □

4 Contraction of Brill-Noether loci

4.1 Line bundles on $M_H(v)$

Theorem 4.1. *[Y2, Thm. 0.1] Let v be a primitive Mukai vector of* $\operatorname{rk} v > 0$ *or* $c_1(v)$ *is ample. Let $B_{M_H(v)}$ be Beauville's bilinear form on $H^2(M_H(v), \mathbb{Z})$. Then*

$$\theta_v : (v^\perp, \langle\ \ ,\ \ \rangle) \to (H^2(M_H(v), \mathbb{Z}), B_{M_H(v)})$$

is an isometry which preserves Hodge structures for $\langle v^2 \rangle \geq 2$, where $\theta_v : v^\perp \to H^2(M_H(v), \mathbb{Z})$ is the canonical homomorphism defined by

$$\theta_v(x) := \frac{1}{\rho}\left[p_{M_H(v)*}((\operatorname{ch}\mathcal{E})\sqrt{\operatorname{td}_X} x^\vee)\right]_1,$$

and $\mathcal{E}$ is a quasi-universal family of similitude ρ on $M_H(v) \times X$, that is, $\mathcal{E}_{|\{E\}\times X} \cong E^{\oplus\rho}$ for all $E \in M_H(v)$ ([Mu3]).

By this theorem, we can identify $\mathrm{Pic}(M_H(v))$ with $(\mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\varrho) \cap v^\perp = v(K(X)) \cap v^\perp$. If $x \in v^\perp$ belongs to $\mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\varrho$, then we can construct $\theta_v(x)$ as a determinant line bundle:

There are at least two method to construct determinant line bundles. One method is to use a standard family on a quot-scheme. The other is to use local universal family. Here we explain the second method. Let $\{U_i\}$ be an analytic open covering of $M_H(v)$ such that there is a universal family $\mathcal{E}_v^i$ on each $U_i \times X$. We may assume that $(\mathcal{E}_v^i)_{|U_i \cap U_j} \cong (\mathcal{E}_v^j)_{|U_i \cap U_j}$. Since $\mathcal{E}_v^i$ is a family of simple sheaves, $\mathrm{Hom}_{p_{U_i \cap U_j}}((\mathcal{E}_v^i)_{|U_i \cap U_j}, (\mathcal{E}_v^j)_{|U_i \cap U_j}) \cong \mathcal{O}_{U_i \cap U_j}$. So the isomorphism $\varphi_{i,j} : (\mathcal{E}_v^i)_{|U_i \cap U_j} \cong (\mathcal{E}_v^j)_{|U_i \cap U_j}$ is determined up to the choice of $t \in \mathcal{O}_{U_i \cap U_j}^\times$. For $\alpha \in K(X)$, we consider line bundles $\det p_{U_i !}(\mathcal{E}_v^i \otimes \alpha^\vee)$ on U_i. We consider an automorphism $t : \mathcal{E}_v^i \to \mathcal{E}_v^i$, $t \in \mathcal{O}_{U_i}^\times$. Then it acts on $\det p_{U_i !}(\mathcal{E}_v^i \otimes \alpha^\vee)$ multiplication by $t^{\langle v(\alpha), v \rangle}$. Therefore if $\langle v(\alpha), v \rangle = 0$, then we can patch up $\{\det p_{U_i !}(\mathcal{E}_v^i \otimes \alpha^\vee)\}_i$ to get a line bundle $\mathcal{L}_v(\alpha)$ on $M_H(v)$. Then we can show that $c_1(\mathcal{L}_v(\alpha)) = \theta_v(v(\alpha))$.

Definition 4.2. *$M_H(v)^{\mu,loc}$ is the open subscheme of $M_H(v)$ consisting of μ-stable vector bundles and $N_L(v)$ the Uhlenbeck compactification of $M_H(v)^{\mu,loc}$.*

We quote the following fundamental result of J. Li.

Theorem 4.3. *[Li] The linear system $|\theta_v(n(0, rL, (c_1, L)))|$, $n \gg 0$ is base point free. If $r > 1$, then the image is $N_L(v)$, if $r = 1$, then the image is the symmetric product of X.*

If $r = 0$, then we have the following.

Lemma 4.4. *We set $v := (0, L, a)$. Let $j : M_H(v) \to \mathbb{P}^n$ be the map sending $E \in M_H(v)$ to $\mathrm{Supp}(E) \in |L|$, that is, j is the Jacobian fibration, where $n = \dim M_H(v)/2$. Then $\theta_v(\varrho) = j^*(\mathcal{O}_{\mathbb{P}^n}(1))$.*

Proof. Let $q : Q(v) \to M_H(v)$ be a standard covering of $M_H(v)$, where $Q(v)$ is an open subscheme of a quot scheme. It is sufficient to prove that $q^*\theta_v(\varrho) = q^*j^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Let $\mathcal{Q}$ be the universal quotient sheaf on $Q(v) \times X$. Let

$$0 \to V_1 \to V_0 \to \mathcal{Q} \to 0 \tag{4.5}$$

be a locally free resolution of $\mathcal{Q}$. Let $\mathcal{D}$ be an effective divisor on $Q(v) \times X$ defined by $\det V_1 \to \det V_0$. By the construction of $\mathcal{D}$, $\mathcal{D}_{|\{x\} \times X} = \mathrm{Supp}(\mathcal{Q}_x) \in |L|$. Hence we get a morphism $Q(v) \to \mathbb{P}(H^0(X, L)^\vee)$ which factors through $Q(v) \xrightarrow{q} M_H(v) \xrightarrow{j} \mathbb{P}(H^0(X, L)^\vee)$. Hence $q^*\theta_v(\varrho) = q^*j^*(\mathcal{O}_{\mathbb{P}^n}(1))$. □

4.2 Birational correspondence

Let $v_1, v \in H^*(X, \mathbb{Z})$ be Mukai vectors such that

$$\begin{cases} v_1 = (r_1, L_1, a_1), \\ v = (r, L, a), \\ \langle v_1^2 \rangle = -2, \end{cases} \tag{4.6}$$

where $r_1, r > 0$ and $a_1, a \in \mathbb{Z}$.

We assume that there is an ample divisor H such that

($\star$3)

$$r_1(L, H) - r(L_1, H) = \min\{(D, H) | D \in \mathrm{Pic}(X), (D, H) > 0\}.$$

Throughout this section, we choose this ample divisor as a polarization of X.

Remark 4.7. L.H.S. is called twisted degree of v with respect to v_1. If $v_1 = v(\mathcal{O}_X)$, then twisted degree is nothing but the usual degree of v.

Example 4.8. $\mathcal{O}_X$ satisfies that $\langle v(\mathcal{O}_X)^2 \rangle = -2$.

Let E_1 be an element of $M_H(v_1)$. Then E_1 is locally free and satisfies that

$$\begin{cases} \mathrm{Hom}(E_1, E_1) = \mathbb{C}, \\ \mathrm{Ext}^1(E_1, E_1) = 0, \\ \mathrm{Ext}^2(E_1, E_1) = \mathbb{C}. \end{cases} \tag{4.9}$$

Definition 4.10. *(1) Let*

$$\mathrm{Syst}^n(v_1, v) := \{(E, U) | E \in M_H(v), U \subset \mathrm{Hom}(E_1, E), \dim U = n\} \tag{4.11}$$

be the moduli space of (twisted) coherent systems and $p_v : \mathrm{Syst}^n(v_1, v) \to M_H(v)$ *the natural projection.*

(2) For $i \geq \max\{0, \langle v, v_1\rangle\}$*, we set*

$$\begin{aligned} M_H(v)_i &:= \{E \in M_H(v) | \dim \mathrm{Hom}(E_1, E) = -\langle v, v_1\rangle + i\}, \\ \mathrm{Syst}^n(v_1, v)_i &:= p_v^{-1}(M_H(v)_i). \end{aligned} \tag{4.12}$$

Then we can easily generalize Lemma 3.8, Corollary 3.11, Proposition 3.15, and Corollary 3.18 to our situation. For example, Proposition 3.15 is generalized as follows: For $nr_1 \geq r$, we have an isomorphism

$$\delta : \mathrm{Syst}^n(v_1, v) \to \mathrm{Syst}^n(v_1^\vee, nv_1^\vee - v^\vee) \tag{4.13}$$

by sending $U \otimes E_1 \to E$ to $U^\vee \otimes E_1^\vee \to \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes E_1 \to E, \mathcal{O}_X)$.

Assume that $n := -\langle v_1, v\rangle > 0$. We consider a correspondence defined by $\mathrm{Syst}^n(v_1, v)$:

$$\begin{tikzcd} & \mathrm{Syst}^n(v_1, v) \arrow[dl, "\pi_v"'] \arrow[dr, "\pi_w"] & \\ M_H(v) & & M_H(w) \end{tikzcd} \tag{4.14}$$

where

(i) $\pi_v = p_v$,

(ii)

$$w = \begin{cases} R_{v_1}(v), & r \geq nr_1 \\ - - D \circ R_{v_1}(v), & r < nr_1, \end{cases} \tag{4.15}$$

(iii)

$$\pi_w((E, U)) = \begin{cases} \mathrm{coker}(U \otimes E_1 \to E), & r \geq nr_1 \\ \mathcal{E}xt^1_{\mathcal{O}_X}(U \otimes E_1 \to E, \mathcal{O}_X) = p_w \circ \delta((E, U)), & r < nr_1. \end{cases} \tag{4.16}$$

Then we proved the following result in [Y1].

Theorem 4.17. *[Y1, Thm. 2.5] We assume that* $r \geq nr_1$*. Then,*

(1) $M_H(v)_0$ *and* $M_H(w)_n$ *are open dense subschemes of* $M_H(v)$ *and* $M_H(w)$ *respectively.*

(2) $\pi_{v|\mathrm{Syst}^n(v_1, v)_0}$ *and* $\pi_{w|\mathrm{Syst}^n(v_1, v)_0}$ *are isomorphisms. In particular* $M_H(v)$ *and* $M_H(w)$ *are birationally equivalent.*

(3) We assume that $M_H(v)_i \neq \emptyset$. We set $u_i := v + iv_1$. Then there are morphisms

$$\begin{aligned} \varpi_v &: M_H(v)_i \to M_H(u_i)_0, \\ \varpi_w &: M_H(w)_{i+n} \to M_H(u_i)_0, \end{aligned} \tag{4.18}$$

and the restriction of the diagram (4.14) to $\mathrm{Syst}^n(v_1, v)_i$ is displayed as follows:

$$\begin{tikzcd} & \mathrm{Syst}^n(v_1,v)_i \arrow[dl, "\pi_v"'] \arrow[dr, "\pi_w"] & \\ M_H(v)_i \arrow[dr, "\varpi_v"'] & & M_H(w)_{n+i} \arrow[dl, "\varpi_w"] \\ & M_H(u_i)_0 & \end{tikzcd} \tag{4.19}$$

where

(3-1) $\varpi_v(E)$, $E \in M_H(v)_i$ is defined by the universal extension

$$0 \to E_1 \otimes \mathrm{Ext}^1(E, E_1)^{\vee} \to \varpi_v(E) \to E \to 0. \tag{4.20}$$

$\varpi_w(F), F \in M_H(w)_{i+n}$ is also defined by the universal extension.

(3-2) ϖ_v is an étale locally trivial $Gr(2i+n, i)$-bundle.

(3-3) ϖ_w is an étale locally trivial $Gr(2i+n, n+i)$-bundle, which is the dual of ϖ_v.

(3-4) $\mathrm{Syst}^n(v_1, v)_i$ is the incidence correspondence of these two bundles.

By similar method as in [Y1], we can show the following result due to Markman [Mr].

Theorem 4.21. *We assume that $n := -\langle v_1, v\rangle > 0$ and $r < nr_1$. We set $w = -D \circ R_{v_1}(v)$.*

(1) $M_H(v)_0$ and $M_H(w)_0$ are open dense subschemes of $M_H(v)$ and $M_H(w)$ respectively.

(2) $\pi_{v|\mathrm{Syst}^n(v_1,v)_0}$ and $\pi_{w|\mathrm{Syst}^n(v_1,v)_0}$ are isomorphisms. In particular $M_H(v)$ and $M_H(w)$ are birationally equivalent.

(3) We assume that $M_H(v)_i \neq \emptyset$. We set $u_i := v + iv_1$. Then there are morphisms

$$\begin{aligned} \varpi_v &: M_H(v)_i \to M_H(u_i)_0, \\ \varpi_w &: M_H(w)_i \to M_H(u_i)_0, \end{aligned} \tag{4.22}$$

and the restriction of the diagram (4.14) to $\mathrm{Syst}^n(v_1, v)_i$ is displayed as follows:

$$\begin{tikzcd} & \mathrm{Syst}^n(v_1,v)_i \arrow[dl, "\pi_v"'] \arrow[dr, "\pi_w"] & \\ M_H(v)_i \arrow[dr, "\varpi_v"'] & & M_H(w)_i \arrow[dl, "\varpi_w"] \\ & M_H(u_i)_0 & \end{tikzcd} \tag{4.23}$$

where

(3-1) ϖ_v *is the same as in Theorem 4.17 and* $\varpi_v(F)$, $F \in M_H(w)_i$ *is defined by*

$$\varpi_w(F) := \mathcal{E}xt^1_{\mathcal{O}_X}(\operatorname{Hom}(E_1^\vee, F) \otimes E_1^\vee \to F, \mathcal{O}_X). \tag{4.24}$$

(3-2) ϖ_v *is an étale locally trivial* $Gr(2i+n,i)$*-bundle.*

(3-3) ϖ_w *is an étale locally trivial* $Gr(2i+n,n+i)$*-bundle, which is the dual of* ϖ_v.

(3-4) $\operatorname{Syst}^n(v_1,v)_i$ *is the incidence correspondence of these two bundles.*

Remark 4.25. $\varpi_w : M_H(w)_i \cong \operatorname{Syst}^{n+i}(v_1^\vee, w)_i \overset{\delta}{\cong} \operatorname{Syst}^{n+i}(v_1, u_i)_0 \overset{p_{u_i}}{\to} M_H(u_i)_0$.

We shall show that the exceptional locus (BN locus) of the birational transformation can be contracted:

$$\begin{array}{ccccc} & & \operatorname{Syst}^n(v_1,v) & & \\ & {}^{\pi_v}\swarrow & & \searrow^{\pi_w} & \\ M_H(v) & & \leftarrow \cdots\cdots \rightarrow & & M_H(w) \\ & \searrow & & \swarrow & \\ & & \cup_{i \geq 0} M_H(u_i)_0 & & \end{array} \tag{4.26}$$

Example 4.27. We assume that X is a K3 surface of $\operatorname{Pic}(X) = \mathbb{Z}H$ and $(H^2) = 2r > 0$. We set $v = (r, H, 0)$ and $w = (0, H, -r)$. Then $w = R_{v(\mathcal{O}_X)}(v)$ and $R_{v(\mathcal{O}_X)}$ induces an elementary transformation $M_H(v) \cdots \to M_H(w)$.

We set

$$\begin{aligned} B_v &:= \{E \in M_H(v) | h^0(X,E) = r+1\}, \\ B_w &:= \{F \in M_H(w) | h^0(X,F) = 1\}. \end{aligned}$$

Then there is an exceptional vector bundle G of $v(G) = (r+1, H, 1)$ such that $B_v \cong \mathbb{P}(H^0(X,G)^\vee)$ and $B_w \cong \mathbb{P}(H^0(X,G))$. The exceptional set of the elementary transformation $M_H(v) \cdots \to M_H(w)$ are $r+1$-dimensional projective spaces B_v and B_w. Let $j : M_H(w) \to \mathbb{P}^{r+1}$ be Jacobian fibration sending $F \in M_H(w)$ to the support $C \in |H|$. Then B_w is the 0-section of this fibration. By Lemma 4.4, $j^*\mathcal{O}_{\mathbb{P}^{r+1}}(1) = \theta_w(\varrho)$. We note that

$$\begin{aligned} R_{\mathcal{O}_X}((-1,0,0)) &= (0,0,1) \\ R_{\mathcal{O}_X}((0,H,2)) &= (-2,H,0). \end{aligned}$$

Hence $\theta_v((1,0,0))$ is nef on $M_H(v) \setminus B_v$. We shall prove that

$$\theta_v(x)_{|B_v} = -\langle v(\mathcal{O}_X), x\rangle c_1(\mathcal{O}_{\mathbb{P}^{r+1}}(1)). \tag{4.28}$$

Proof of (4.28). Let $\mathcal{F}$ be a family of sheaves on $B_v \times X$ which is defined by the exact sequence

$$0 \to \mathcal{O}_{B_v}(-1) \boxtimes \mathcal{O}_X \to \mathcal{O}_{B_v} \boxtimes G \to \mathcal{F} \to 0.$$

Then we see that

$$\begin{aligned} \mathcal{L}_v(\alpha)_{|B_v} &= \det p_{B_v !}(\mathcal{F} \otimes \alpha^\vee) \\ &= \det p_{B_v !}(-\mathcal{O}_{B_v}(-1)) \otimes \alpha^\vee) \\ &= \mathcal{O}_{B_v}(-1)^{\otimes \langle v(\mathcal{O}_X), \alpha\rangle} \\ &= \mathcal{O}_{\mathbb{P}^{r+1}}(1)^{\otimes(-\langle v(\mathcal{O}_X), \alpha\rangle)}. \end{aligned}$$

Hence $\theta_v(x)$ is nef on B_v if and only if $\langle v(\mathcal{O}_X), x\rangle \leq 0$. Therefore $\theta_v(a(0,H,2)+b(1,0,0))$, $2a \leq b \leq 0$ is a nef divisor on $M_H(v)$. Under the same conditions, we get that $\langle (a(0,H,2)+b(1,0,0))^2\rangle \geq 0$ and the equality holds if $a=0$. By $\theta_v((-1,0,0)+2(0,H,2))$, we can contract the exceptional set B_v.

$$
\begin{array}{ccccc}
 & & \mathrm{Syst}^n(v_1,v) & & \\
 & {}^{\pi_v}\swarrow & & \searrow^{\pi_w} & \\
M_H(v) & & \leftarrow\cdots\cdots\rightarrow & & M_H(w) \\
 & \searrow & & \swarrow & \\
 & & M_H(v)_0 \cup \{G\} & &
\end{array}
\tag{4.29}
$$

We can also compute the ample cone.

$$
\begin{aligned}
A(M_H(v)) &= \{x(0,H,2)+y(-1,0,0) | 2x > y > 0\} \\
A(M_H(w)) &= \{x(0,0,1)+y(-2,H,0) | x/2 > y > 0\}.
\end{aligned}
\tag{4.30}
$$

In particular, $M_H(v)$ is not isomorphic to $M_H(w)$.

Proof of (4.30). By Theorem 4.1, $\operatorname{rk}\operatorname{Pic}(M_H(v)) = 2$. In particular $\operatorname{Pic}(M_H(v)) \otimes \mathbb{Q}$ is generated by $(0,H,2)$ and $(-1,0,0)$. We note that $\langle (v+\varrho)^2\rangle = 0$. Hence if $r > 1$, then $M_H(v) \setminus M_H(v)^{\mu,loc}$ is not empty. By Theorem 4.3, $\theta_v((0,H,2))$ is not ample. If $r=1$, then $M_H(v) = \mathrm{Hilb}^2_X$, and hence $\theta_v((0,H,2))$ is not ample either. Therefore we get that

$$A(M_H(v)) = \{x(0,H,2)+y(-1,0,0) | 2x > y > 0\}.$$

Hence there is no morphism $M_H(v) \to \mathbb{P}^{r+1}$. In the same way, we get the description of $A(M_H(w))$.

Construction of the contraction map: For a Mukai vector v, we set

$$
\begin{aligned}
\lambda_v &:= -\langle \varrho, v\rangle H + \langle H, v\rangle \varrho, \\
\mu_v &:= -\langle \varrho, R_{v_1}(v)\rangle R_{v_1}(H) + \langle H, R_{v_1}(v)\rangle R_{v_1}(\varrho).
\end{aligned}
\tag{4.31}
$$

Then $\mu_v = R_{v_1}(\lambda_{R_{v_1}(v)}) = R_{v_1} \circ D(\lambda_{-D\circ R_{v_1}(v)})$. Since $\theta_v \circ (R_{v_1} \circ D) = \theta_{-D\circ R_{v_1}(v)}$, we get

$$\theta_v(\mu_v) = \theta_{R_{v_1}(v)}(\lambda_{R_{v_1}(v)}) = \theta_{-D\circ R_{v_1}(v)}(\lambda_{-D\circ R_{v_1}(v)}).$$

By Theorem 4.3, $\theta_v(\lambda_v)$ is nef and big and it gives a contraction $M_H(v) \to N_H(v)$. Also $\theta_w(\lambda_w)$ gives a contraction $M_H(w) \to N_H(w)$, or $M_H(w) \to \mathbb{P}^m$, $(2m = \dim M_H(w))$. We see that $\langle \lambda_v + \mu_v, v_1\rangle = 0$. So we can expect that $\theta_v(\lambda_v+\mu_v)$ and $\theta_w(\lambda_w+\mu_w)$ give contractions

$$
\begin{aligned}
q_1 &: M_H(v) \to M' = \cup_{i\geq 0} M_H(u_i)_0 \\
q_2 &: M_H(w) \to M' = \cup_{i\geq 0} M_H(u_i)_0
\end{aligned}
\tag{4.32}
$$

such that $q_2^{-1} \circ q_1 : M_H(v) \cdots \to M_H(w)$ is generalized elementary transformation.

We claim that

(*) the restriction of $\theta_v(\lambda_v+\mu_v)$ to $M_H(v)_i$ is the pull-back of an ample line bundle on $M_H(u_i)_0$.

Proof of (*): We note that R_{v_1} or $-D \circ R_{v_1}$ induces an isomorphism $M_H(u_i)_0 \to M_H(w_i)_0$, where $w_i = R_{v_1}(u_i)$ or $w_i = -D \circ R_{v_1}(u_i)$ according as $\operatorname{rk} R_{v_1}(u_i) \geq 0$ or $\operatorname{rk} R_{v_1}(u_i) < 0$. By Theorem 4.3, $\theta_{u_i}(\lambda_{u_i})$ is nef on $M_H(u_i)$. Also $\theta_{w_i}(\lambda_{w_i})$ is nef on $M_H(w_i)$, and hence $\theta_{u_i}(\mu_{u_i}) = \theta_{w_i}(\lambda_{w_i})$ is nef on $M_H(u_i)_0$. It is known from the construction of $M_H(u_i)$ that $(\mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\varrho) \cap u_i^\perp$ contains ample divisors. It is easy to see that $(a,b,c) \in u_i^\perp$ satisfies $a < 0$ if it is ample and $M_H(u_i) \neq M_H(u_i)^{\mu,loc}$. By a simple calculation, we see that $\operatorname{rk} \mu_{u_i} = \operatorname{rk} v_1(\operatorname{rk} v L_1 - \operatorname{rk} v_1 L, H) < 0$. Hence $\lambda_{u_i} + \epsilon\mu_{u_i}$, $0 < \epsilon \ll 1$ is ample on $M_H(u_i)$. The same is true for $\lambda_{w_i} + \epsilon\mu_{w_i}$, $0 < \epsilon \ll 1$. Therefore $\theta_{u_i}(\lambda_{u_i} + \mu_{u_i})$ is ample on $M_H(u_i)_0$. Since $\lambda_v + \mu_v = \lambda_{u_i} + \mu_{u_i}$, our claim follows from the following:

Lemma 4.33. *$\theta_v(\lambda_v + \mu_v)_{|M_H(v)_i}$ comes from $\theta_{u_i}(\lambda_{u_i} + \mu_{u_i})$.*

Proof. Let $\{\mathcal{U}^j\}$ be an analytic open covering of $M_H(u_i)_0$ such that there is a universal family $\mathcal{E}^j_{u_i}$ on $\mathcal{U}^j \times X$. We set $V^j := \operatorname{Hom}_{p_{\mathcal{U}^j}}(E_1, \mathcal{E}^j_{u_i})$. V^j is a locally free sheaf. Let $g : Gr(V^j, i) \to \mathcal{U}^j$ be the Grassmann bundle of i-dimensional subspaces. Let W^j be the universal subbundle of V^j. Then we have an exact sequence

$$0 \to W^j \boxtimes E_1 \to g^*\mathcal{E}^j_{u_i} \to \mathcal{E}^j_v \to 0, \tag{4.34}$$

where ${\mathcal{E}_v}^j$ is a family of stable sheaves which belongs to $M_H(v)_i$. $\mathcal{E}_v$ gives an open immersion $Gr(V,i) \to M_H(v)_i$. Then

$$\begin{aligned} \mathcal{L}_v(\alpha)_{|Gr(V^j,i)} &= \det p_{Gr(V^j,i)!}({\mathcal{E}_v}^j \otimes \alpha^\vee) \\ &= \det p_{Gr(V^j,i)!}(\mathcal{E}^j_{u_i} \otimes \alpha^\vee) \otimes \det p_{Gr(V^j,i)!}(W^j \boxtimes E_1 \otimes \alpha^\vee)^\vee \\ &= \det p_{Gr(V^j,i)!}(\mathcal{E}^j_{u_i} \otimes \alpha^\vee) \otimes \det(W^j)^{\otimes \langle v_1, x\rangle}. \end{aligned} \tag{4.35}$$

If $x = \lambda_v + \mu_v$, then we get a canonical identification

$$\mathcal{L}_v(\alpha)_{|Gr(V^j,i)} = g^* \det p_{\mathcal{U}^j!}(\mathcal{E}^j_{u_i} \otimes \alpha^\vee). \tag{4.36}$$

Therefore we get

$$\mathcal{L}_v(\alpha)_{|M_H(v)_i} = \varpi^*_{u_i}\mathcal{L}_{u_i}(\alpha). \tag{4.37}$$

□

Remark 4.38. Although we used local universal family to prove the lemma, we can prove the lemma by using quasi-universal family or the canonical family on a quot-scheme.

By (*), $\theta_v(\lambda_v + \mu_v)$ is nef and big. Since $K_{M_H(v)}$ is trivial, base point free theorem implies that $\theta_v(\lambda_v + \mu_v)$ is base point free. By this map, all fibers of Grassmann bundle $M_H(v)_i \to M_H(u_i)_0$ are contracted.

Proposition 4.39. *If $M_H(v)_1 \neq \emptyset$, then $\mathbb{R}_+(\lambda_v + \mu_v)$ is a boundary of the ample cone. In particular, if $\operatorname{Pic}(X) = \mathbb{Z}H$, $M_H(v)_1 \neq \emptyset$ and $M_H(v) \neq M_H(v)^{\mu,loc}$, then the nef cone is spanned by $\lambda_v + \mu_v$ and λ_v.*

Some examples of birational maps:

Example 4.40. We shall give an example of $M_H(v)$ whose elementary transformation is isomorphic to $M_H(v)$. Assume that $\operatorname{Pic}(X) = \mathbb{Z}H$ and $(H^2) = 10$. We set $v = (1, H, 4)$. Then $M_H(v) \cong \operatorname{Hilb}_X^2$. We set $v_1 := (3, 2H, 7)$. Then $\langle v_1^2 \rangle = -2$. Hence there is an exceptional bundle E_1 of $v(E_1) = v_1$. We set $B_v := \{E \in M_H(v) | h^0(X, E_1^\vee \otimes E) = 1\}$. Then B_v is isomorphic to $\mathbb{P}^2$ and R_{v_1} induces an elementary transformation $M_H(v) \cdots \to M_H(4, 3H, 11)$ along B_v.

On the other hand, $-R_{\mathcal{O}_X}$ induces an isomorphism $M_H(v) \cong M_H(4, H, 1)$ and since $M_H(4, H, 1) \cong M_H(4, 3H, 11)$, we get $\operatorname{elm}_{B_v}(M_H(v)) \cong M_H(v)$.

The following proposition shows that the divisorial contraction $M_H(2, c_1, a) \to N_H(2, c_1, a)$ is different from the Hilbert-Chow morphism.

Lemma 4.41. *We assume that* $\operatorname{rk} v = 2$ *and* $D := M_H(v) \setminus M_H(v)^{\mu, loc}$ *is not empty. Then* $\mathcal{O}_{M_H(v)}(D)$ *is defined by* $\theta_v((2, c_1, 2 + (c_1^2)/2 - \langle v(\mathcal{O}_X), v \rangle))$. *In particular, D is primitive. Since the exceptional divisor of Hilbert-Chow morphism is divisible by 2, the two divisorial contractions are different.*

Proof. Let $\mathcal{F}$ be a universal family of stable sheaves on X parametrized by an open subscheme Q of a suitable quot scheme. Let $0 \to V_1 \to V_0 \to \mathcal{F} \to 0$ be a locally free resolution of $\mathcal{F}$. Then we get an exact sequence

$$0 \to \mathcal{F}^\vee \to V_0^\vee \to V_1^\vee \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{Q \times X}) \to 0.$$

We note that $\mathcal{F}$ is reflexive and $\mathcal{F}^\vee \cong \mathcal{F} \otimes \det \mathcal{F}^\vee$. We denote the pull-back of D to Q by D'. Since the multiplicity of $p_{Q*}(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{Q \times X}))$ at the generic point of D' is 1, $\det p_{Q*}(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{Q \times X})) \cong \mathcal{O}_Q(D')$. By using relative duality, we see that

$$\begin{aligned}
p_{Q!}(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{Q \times X})) &= p_{Q!}(V_1^\vee) - p_{Q!}(V_0^\vee) + p_{Q!}(\mathcal{F}^\vee) \\
&= p_{Q!}(V_0, \mathcal{O}_{Q \times X}) - p_{Q!}(\mathcal{F}, \mathcal{O}_{Q \times X}) - p_{Q!}(V_0^\vee) + p_{Q!}(\mathcal{F}^\vee) \\
&= -p_{Q!}(\mathcal{F}, \mathcal{O}_{Q \times X}) + p_{Q!}(\mathcal{F}^\vee) \\
&= p_{Q!}(\mathcal{F}) + p_{Q!}(\mathcal{F}^\vee) \\
&= p_{Q!}(\mathcal{F}) + p_{Q!}(\mathcal{F}(-c_1)) + \langle v, v(\mathcal{O}_X) \rangle \alpha
\end{aligned}$$

where $\alpha = c_1(\mathcal{F}) - c_1$. Hence $\mathcal{O}_Q(D') \cong \det p_{Q!}(\mathcal{F} \otimes (\mathcal{O}_X + \mathcal{O}_X(c_1) + \langle v, v(\mathcal{O}_X) \rangle \mathbb{C}_P)^\vee)$. Since $v(\mathcal{O}_X + \mathcal{O}_X(c_1) + \langle v, \mathcal{O}_X \rangle \mathbb{C}_P) = (2, c_1, 2 + (c_1^2)/2 - \langle v, v(\mathcal{O}_X) \rangle)$, we get our lemma. □

The following exapme shows that the reflection changes holomorphic structures in general.

Example 4.42. Assume that $\operatorname{Pic}(X) = \mathbb{Z}H$. $R_{v(\mathcal{O}_X)}$ induces a birational map

$$M_H(r, H, -a) \leftarrow \cdots \to M_H(a, H, -r), \ r > a > 0.$$

Since $M_H(r, H, -a + 1), M_H(a, H, -r + 1) \neq \emptyset$, $D_r := M_H(r, H, -a) \setminus M_H(r, H, -a)^{\mu, loc}$ (resp. $D_a := M_H(a, H, -r) \setminus M_H(a, H, -r)^{\mu, loc}$) is a non-empty subset of codimension $r - 1$ (resp. $a - 1$). Hence if $(r, a) \neq (2, 1)$, then in the same way as in Example 4.27, we see that $M_H(r, H, -a) \not\cong M_H(a, H, -r)$. If $(r, a) = (2, 1)$, then by Lemma 4.41, we see that $M_H(2, H, -1) \not\cong M_H(1, H, -2)$.

We give an example of moduli spaces such that $M_H(v) \not\cong M_H(v^\vee)$.

Example 4.43. We assume that $\operatorname{Pic}(X) = \mathbb{Z}H$ and $(H^2) = 4$. $R_{v(\mathcal{O}_X)}$ induces a birational map

$$M_H(1, H, 0) \cdots \to M_H(0, H, -1),$$

which is an elementary transform along $\mathbb{P}^2$-bundle over X. In the same way as in Example 4.27, we see that $M_H(1, H, 0) \not\cong M_H(0, H, -1)$. By the action of $T_{\mathcal{O}(H)}$, we have isomorphisms $M_H(0, H, -1) \cong M_H(0, H, 1) \cong M_H(0, H, 3)$. By $R_{v(\mathcal{O}_X)}$, we get an isomorphism $M_H(0, H, -3) \cong M_H(3, H, 0)$.

On the other hand, we set $v_0 := (2, -H, 1)$. Then by using reflection $R_{v(\mathcal{O}_X)}$, we see that $(M_H(v_0), \widehat{H}) = (X, H)$, where $\widehat{H} := \theta_{v_0}((0, H, -2))$. Since there is a universal sheaf on $M_H(v_0) \times X$, we can consider Fourier-Mukai transform. Then we get an isomorphism $M_H(1, H, 0) \cong M_H(3, -H, 0)$ ([Y3, Thm. 3.11]). Hence $M_H(3, H, 0) \not\cong M_H(3, -H, 0)$. Moreover we see that the birational map $D : M_H(3, H, 0) \cdots \to M_H(3, -H, 0)$ is an elementary transformation along the set of non-locally free sheaves.

Finally we give a remark on Riemann-Roch number $\chi(M_H(v), \theta_v(x))$, $x \in v(K(X))$.

Proposition 4.44.

$$\chi(M_H(v), \theta_v(x)) = \frac{(\langle x^2 \rangle + 4)(\langle x^2 \rangle + 6) \ldots (\langle x^2 \rangle + 2n + 2)}{2^n n!},$$

where $n = \langle v^2 \rangle / 2 + 1$.

Outline of the proof. By Fujiki's result [F], $\chi(M_L(v), \theta_v(x))$ is written as a polynomial of $\langle x^2 \rangle$. If $\operatorname{rk} v = 1$, then by a direct computation for $x = (0, L, (L, v))$ where L is ample, the claim easily follows. For general cases, we use the proof of Theorem 4.1. □

This formula enable us to compute the dimension of linear systems. For example, assume that $\operatorname{rk} v > 0$. Then $\theta_v(\lambda_v)$ is nef and big. By using Kawamata-Viehweg vanishing theorem, we get

$$\begin{aligned} \dim H^0(M_H(v), \theta_v(l\lambda_v)) &= \frac{(\langle (l\lambda_v)^2 \rangle + 4)(\langle (l\lambda_v)^2 \rangle + 6) \ldots (\langle (l\lambda_v)^2 \rangle + 2n + 2)}{2^n n!} \\ &= \frac{(r^2 l^2 (H^2) + 4)(r^2 l^2 (H^2) + 6) \ldots (r^2 l^2 (H^2) + 2n + 2)}{2^n n!}, \end{aligned} \tag{4.45}$$

where $n = \langle v^2 \rangle / 2 + 1$ and $r = \operatorname{rk} v$.

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