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Existence of positive solutions for some nonlinear elliptic equations on unbounded domains with cylindrical ends

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1 Introduction

We consider the nonlinear elliptic boundary value problem:

$$-\Delta u + \lambda u = u^p, \quad u > 0 \quad (x \in \Omega), \quad u|_{\partial\Omega} = 0, \quad u(x) \to 0 \ (|x| \to \infty), \qquad (1)$$

where Ω is an unbounded domain in $\mathbf{R}^n$ with the boundary $\partial\Omega$ of locally piecewise C^1 class, $1 < p < (n+2)/(n-2)(n \geq 3), +\infty(n = 2)$, and λ is a parameter. We assume $\lambda \geq 0$ throughout this paper, for simplicity, although one can allow λ to be negative to some extent for domains in which Poincaré's inequality holds. In 1982, Esteban and Lions [11] discovered a certain criterion of unbounded domains Ω in which the BVP above has no solution. For example, there exist no solution for the semi-infinite cylinderical domain Ω:

$$\Omega = (0, +\infty) \times \omega,$$

where $\omega \subset \mathbf{R}^{n-1}$ is a bounded domain. Actually, they proved non-existence of non-trivial energy finite solution to (2), if there exists a constant vector $X \in \mathbf{R}^n$ such $\nu(x) \cdot X \geq 0$ and $\nu(x) \cdot X \not\equiv 0$ for $x \in \partial\Omega$, where $\nu(x)$ is the outward unit normal vector at $x \in \partial\Omega$. On the other hand, in 1983, several peoples (e.g., Esteban [10], Amick and Toland [2], Stuart [19]) proved the existence of a solution on the infinite (straight) cylindrical domain $\Omega = (-\infty, +\infty) \times \omega$. After that, in

1993 Lien, Tzeng and Wang [14] proved the existence of a solution on unbounded domains with a periodic structure and their locally deformed domains, precisely adding bounded domains, by using concentration-compactness principles. They also proved the existence of a solution on a domain

$$\Omega = \{x \in \mathbf{R}^n; |x| < R\} \cup \{x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}; x_1 \in (0, +\infty), |x'| < r\}$$

for fixed $r > 0$ and sufficiently large $R > r$. Also, del Pino and Felmer [9] proved similar results, but in slightly different situations, for more general nonlinearity by using the mountain pass approach. We also note that Bahri and Lions [3] have proved the existence of a solution on any exterior domain Ω for $\lambda > 0$. The relation between the shape of an unbounded domain Ω and the solvability of the BVP (1) is still unclear.

In this paper, we propose a class of unbounded domains, domains with semi-infinite cylindrical ends (the precise definition is given in section 2), in which the BVP will be solved. Actually we present two conjectures on the solvability of the BVP (1) and give several results to support these conjectures.

This paper is organized as follows. In section 2, we consider the elliptic boundary value problem with a general nonlinearity $f(u)$, including $f(u) = u_+^p$ as a special case. We introduce a class of unbounded domains with semi-infinite cylindrical ends and give two conjectures on the solvability on such domains. We state two results (Theorem 1, Corollary 2) on the existence of a least energy solution and a result (Theorem 4) on the existence of a higher energy solution. We also give some symmetry properties (Theorem 3) of a least energy solutions on domains with symmetries with respect to axises. In section 3, we give the proof of Theorem 1 and Theorem 3. In section 4, we give the outline of the proof of Theorem 4.

2 Main Results

We consider the nonlinear elliptic boundary value problem with a general nonlinear term $f(u)$:

$$-\Delta u + \lambda u = f(u), \quad u > 0 \quad (x \in \Omega), \quad u|_{\partial\Omega} = 0, \quad u(x) \to 0 \ (|x| \to \infty), \qquad (2)$$

where Ω is an unbounded domain in $\mathbf{R}^n$ with the boundary $\partial\Omega$ of locally piecewise C^1 class and $\lambda \geq 0$ is a parameter. Here, $f(t)$ is a C^1 function satisfying the following conditions:

(f-1) $f(t) = 0$ for $t \le 0$ and $f(t) = o(t)$ as $t \to 0$;

(f-2) there exists $p > 1$ such that $p < (n+2)/(n-2)$ for $n \ge 3$ and $p < +\infty$ for $n = 2$ and

$$\lim_{t \to +\infty} \frac{f(t)}{t^p} = 0;$$

(f-3) there exists $\theta \in (2, p+1]$ such that

$$0 < \theta F(t) \le f(t)t \quad \textit{for} \quad t > 0;$$

(f-4) the function $t \mapsto f(t)/t$ is strictly increasing on $(0, +\infty)$,

where $F(t) = \int_0^t f(s)\,ds$. To introduce the class of unbounded domains Ω to be considered in this paper, we denote by $S(\omega)$ and $A(\omega)$ the infinite cylinder and the semi-infinite cylinder, respectively, with a bounded domain $\omega \subset \mathbf{R}^{n-1}$ as its cross-section:

$$S(\omega) = \{(x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}; x_1 \in (-\infty, +\infty), x' \in \omega\},$$

$$A(\omega) = \{(x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}; x_1 \in (0, +\infty), x' \in \omega\}.$$

Especially, we use the notation $S_R = S(B'(O,R))$ and $A_R = A(B'(O,R))$ for $B'(O,R) = \{x' \in \mathbf{R}^{n-1}; |x'| < R\}$ with $R > 0$.

Definition 1 *If there exist $m \in \mathbf{N}$, a bounded domain $\omega \subset \mathbf{R}^{n-1}$ and a compact set K such*

$$\Omega \cap K^c = \cup_{j=1}^m A^{(j)}(\omega),$$

where each $A^{(j)}(\omega)$ is congruent with $A(\omega)$, then we say that Ω is a domain with m semi-infinite cylinder $A(\Omega')$ as its ends.

From S_R we construct *the $V-$ shaped cylindrical domain*, we denote by $S_R^{(V)}$, by the following procedure: cuting the domain S_R via a hyperplane, not parallel to the cross-section, and attaching again its new cross-sections so that points on one cross-section are transformed into the points of the other cross-section, which is symmetric with respect to its center. We can continue this procedure to construct a finitely times bent domain from S_R. One can also consider *the smoothly locally bent cylindrical domain* with a ball of same radius R as its cross section everywhere:

Let $y = y(s), s \in (-\infty, +\infty)$, be a smooth curve in $\mathbf{R}^n$ which is a straight line outside a compact set and let $P(s)$ be a set of unit vectors which are perpendicular to the tangent vector $y'(s)$. Then such domain Ω can be decsribed as follows:

$$\Omega = \{x = y(s) + t\nu(s); s \in (-\infty, +\infty),\ \nu(s) \in P(s),\ t \in [0, R)\},$$

We conjecture the following two statements for the solvability of (2) on a domain Ω with m semi-infinite cylinder $A(\omega)$ as its ends.

Conjecture 1 *If Ω is either a finitely times bent domain or a smoothly locally bent domain constructed from S_R by the procedure above, then there exists a least energy solution (the precise definition is given later) to (2). Actually, we conjecture the stronger statement $c(\Omega) < c(S_R)$ for such domains (the definition of $c(\Omega)$ is gevin later).*

In the proof of Theorem 1, one can see that once we know $c(\Omega) < c(S_R)$ we can show the existence of a least energy solution.

Conjecture 2 *If $m \geq 2$ and Ω is a domain with m semi-infinite cylinder $A(\omega)$ as its ends, then there exists a solution to (2).*

In general, we cannot expect the existence of least energy solution to (2) under the situation of Conjecture 2. We remark that Poincaré's ineqality holds on unbounded domains with such cylindrical ends (see, e.g., [17]). To state our first result, we denote by $S_{R,L}^{(V)}$ *the semi-infinite $V-$shaped cylindrical domain* which is constructed by cuting, perpendicularly by a hyperplane, a infinite part of one of the semi-infinite part of $S_R^{(V)}$ remaining a finite part with length L, measured from certain point on the bent region. So, $S_{R,L}^{(V)}$ tends to $S_R^{(V)}$ as $L \to \infty$. Now, we state our first result.

Theorem 1 *Suppose $\lambda \geq 0$ and* **(f-1)-(f-4)** *for $f(t)$. Let $m \geq 1$, $R > 0$ and Ω be a domain with m semi-infinite cylinder A_R as its ends which satisfies the additional condition:*

(*) *Ω contains one of S_R, $S_R^{(V)}$ and $S_{R,L}^{(V)}$ with sufficientlly large $L > 0$.*
Then, there exists a least energy solution to (2).

For the case that Ω containes S_R, Theorem 1 has been proved essentially in [9], [14]. The result for the case $\Omega = S_R^{(V)}$ seems new as far as we know and also plays an important role in the proof of Theorem 1 for the general case. As a special case, we consider the problem:

$$-\Delta u = u^p, \quad u > 0 \quad (x \in \Omega), \quad u|_{\partial\Omega} = 0, \quad u(x) \to 0 \ (|x| \to \infty), \tag{3}$$

where Ω is an unbounded domain in $\mathbf{R}^n (n \geq 2)$ and $1 < p < (n+2)/(n-2)$ for $n \geq 3$ and $1 < p < +\infty$ for $n = 2$. As a corollary of Theorem 1, we obtain the following result to (3).

Corollary 2 *Let Ω be a finitely bent domain constructed from S_R, fix its shape and consider R as a parameter. Then, there exists a sufficiently small R_0 such that for every $R \in (0, R_0)$ there exists a least energy solution to (3).*

Since one can show Corollary 2 by combining the result in Theorem 1 and the scaling argument, we omit the details of the proof of Corollary 2.

Remark 1 *One can see in the proof of Theorem 1 and Corollary 2 that the statements are true even in the case that the cross-section $B(O,R)$ is replaced by a bounded domain $\omega \in \mathbf{R}^{n-1}$ which is convex and symmetric with respect to axises $x_j, j = 2, \cdots, n$. Because we just use the existence and symmetry properties of a least energy solution on $S(\omega)$.*

Theorem 1 and Corollary 2 give partial answers to Conjecture 1. Complete answer to Conjecture 1 remains open even for finetely bent domains constructed from S_R. Moreover, existence of the least energy solution for a smoothly locally bent domain constructed from S_R is also an open problem, although we can obtain the existence result for certain smooth domains close to $V-$shaped domain, which are constructed smoothing the corner of $V-$shaped domains slightly.

Here, we briefly give the definition of *a least energy solution* to (2). The problem (2) has a variational structure and a solution u to (2) can be characterized as a non-trivial critical point of the energy functional:

$$J_\Omega(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \int_\Omega F(u)\,dx. \tag{4}$$

Under the assumptions **(f-1)**-**(f-3)**, it is well-known (see, e.g., [9], [15]) that $J_\Omega(u)$ has a mountain pass structure and, when **(f-4)** is assumed, its mountain pass value $c(\Omega)$ can be written by

$$c(\Omega) = \inf_{u \in H_0^1(\Omega), u_+ \not\equiv 0} \sup_{\tau > 0} J_\Omega(\tau u).$$

It is also known that this characterization implies that

$$c(\Omega) = \inf_{u \in M, u_+ \not\equiv 0} J(u),$$

where M is a solution manifold (called the Nehari manifold), which includes any solution to (2): namely,

$$M = \{u \in H_0^1(\Omega); \int_\Omega u(-\Delta u + \lambda u)\, dx = \int_\Omega f(u)u\, dx\}.$$

This means that if $J_\Omega(u) = c(\Omega)$ and $J'_\Omega(u) = 0$, then u has the least energy among any solutions to (2). So, we call u be a least energy solution to (2), if $J_\Omega(u) = c(\Omega)$ and $J'_\Omega(u) = 0$.

Next, we remark on symmetry property of solutions to (2) on symmetric domains. When $\Omega = S_R$, in [5] (see also [4]) they proved by the moving plane method that any solutions u to (2) has the symmetry:

$$u(x_1, x') = u(x_1, |x'|), \qquad u(x_1, x') = u(-x_1, x')$$

for any $x = (x_1, x') \in S_R$. Especially, when $n = 2$, uniqueness of solutions to (2) is also known by Dancer [7], at least for the case $f(u) = u_+^p$ and $\lambda = 0$. Now, we consider (2) on the domain:

$$\Omega = \{x = (x_1, x_2) \in \mathbf{R}^2; |x_1| \le R/2, |x_2| \le R/2\}.$$

This domain has a symmetry with respect to the x_1-axis, x_2-axis, α-axis, and β-axis;

$$\alpha = \{x; x_1 = x_2\}, \quad \beta = \{x; x_1 = -x_2\}.$$

For this domain, it is easy to see that any solution to (2) is symmetric with respect to x_1-axis and x_2-axis by using the moving plan method as in [4], [5]. For a least energy solution obtained by Theorem 1 (see also [14] for the case $f(u) = u_+^p$), we also show the symmetry with respect to α-axis and β-axis.

Theorem 3 *Let Ω be a domain above. Then any least energy solution to (2) is symmetric with respect to x_1-axis, x_2-axis, α-axis, and β-axis.*

We do not know uniqueness of solutions to (2), even if we restrict to least energy solutions.

Next, we show some result to support Conjecture 2 in certain domainw in which we cannot expect the existence of a least energy solution. There are several results in this direction (see [13], [21], [22]). In this paper, we consider the domain Ω_σ witha parameter $\sigma > 0$ satisfying the following conditions;

(D-1) $O \in \Omega_\sigma$ and Ω_σ is an unbounded domain which is symmetric with respect to the hyperplane $T_1 = \{x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}; x_1 = 0\}$,

(D-2) there exists $L > 0$ such that

$$\left(\Omega_\sigma \cap \{x = (x_1, x'); |x_1| \le \frac{L}{2}\}\right) \setminus S_\sigma = \emptyset,$$

(D-3) there exists $d(> \sigma) > 0$ such that

$$\Omega_\sigma \subset \{(x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}; |x'| < \frac{d}{2}\},$$

(D-4) $\Omega_\sigma \cap \{x \in \mathbf{R}^n; |x| < k\}$ satisfies the uniform cone condition for any sufficiently large $k > 0$.

A typical example is the unbounded dumbbell-shaped domain which is included in $S_{d/2}$ and consists of the union of two $A_{d/2}$ and the thin channell $\{(x_1, x'); |x_1| \le L/2, |x'| < \sigma/2\}$ with $\sigma < d$. For this domain, it is easy to see (e.g. [14], [22]) that there is no least energy solution, namely attains its mountain pass value $c(\Omega_\sigma)$ for small σ. Therefore, we must find a higher energy solution.

Theorem 4 *Suppose Ω_σ satisfies the conditions* **(D-1)-(D-4)**. *Then there exists a sufficientlly small $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0)$ there exists a solution $u = u_\sigma$ to (3) which is symmetric with respect to the hyperplane T_1, and satisfies the following estimates;*

$$C_1 \sigma^{-\frac{2}{p-1}} \le \|u_\sigma\|_{L^\infty(\Omega_\sigma)} \le C_2 \sigma^{-\frac{2}{p-1}},$$

$$C_3 \sigma^{\frac{n-2}{2} - \frac{2}{p-1}} \le \|\nabla u_\sigma\|_{L^2(\Omega_\sigma)} \le C_4 \sigma^{\frac{n-2}{2} - \frac{2}{p-1}},$$

where $C_j, j = 1, \cdots, 4$ are positive constants independent of σ.

The same result has been proved by by Byeon [4] (see also [7] for the case $n = 2$) for bounded dumbbell-shaped domains Ω_σ.

3 Proofs of Theorem 1, 3

In this section, first we show Thoerem 1 for the case $\Omega = S_R^{(V)}$, since this is the essential part of Theorem 1. Throughout this section, we assume that Ω is a domain with m semi-infite cylinder as its ends for some $m \in \mathbf{N}$.

First, we collect some useful known results for the proof of Theorem 1. Under the assumptions **(f-1)-(f-3)**, it is known that the energy functional $J_\Omega(u)$ defined in section 2, has a mountain pass structure and we can define its mountain value

$$c(\Omega) = \inf_{\gamma \in \Gamma}(\sup_{t \in [0,1]} J_\Omega(\gamma(t))),$$

where $\Gamma = \{\gamma \in C([0,1]; H_0^1(\Omega)); \gamma(0) = 0, \quad J_\Omega(\gamma(1)) < 0\}$. Moreover, under the additional assumption **(f-4)**, $c(\Omega)$ is characterized as by

$$c(\Omega) = \inf_{u \in H_0^1(\Omega), u_+ \not\equiv 0} \sup_{\tau > 0} J_\Omega(\tau u).$$

For large $k > 0$, we consider $\widetilde{\Omega_k} = \Omega \cap B(O,k)^c$, where $B(O,k)^c = \{x \in \mathbf{R}^n; |x| > k\}$. As before, we can define the energy functional $J_{\widetilde{\Omega_k}}(u)$ on $H_0^1(\widetilde{\Omega_k})$, and define

$$\widetilde{c_k} = c(\widetilde{\Omega_k}) = \inf_{\gamma \in \widetilde{\Gamma_k}} \sup_{t \in [0,1]} J_{\widetilde{\Omega_k}}(\gamma(t)),$$

where

$$\widetilde{\Gamma_k} = \{\gamma \in \Gamma; \gamma(t) \in H_0^1(\widetilde{\Omega_k}) \text{ for any } \ t \in [0,1]\}.$$

Since $H_0^1(\widetilde{\Omega_l}) \subset H_0^1(\widetilde{\Omega_k})$ for $k < l$, it is easy to see that $\widetilde{c_k}$ is increasing in k so that $\lim_{k \to \infty} \widetilde{c_k}$ exists. The following is a general criterion due to del Pino and Felmer [9] to assure the existence of a least energy solution to (2).

Proposition 1 *Assume $c(\Omega) < \lim_{k \to \infty} \widetilde{c_k}$. Then there exists a least energy solution u, that is $J_\Omega(u) = c(\Omega)$ and $J'_\Omega(u) = 0$.*

Actually, this proposition is true for any domain Ω for which $\Omega \cap B(O,k)$ satisfies the uniform cone property for any large $k > 0$, although this condition is not mentioned explicitely in [9]. This condition allow us to obtain the uniform constant in Sobolev's embedding theorem on $\Omega \cap B(O,k)$ for any large $k > 0$ (see Adams [1]). The following lemma is also well-known(see, e.g., [20]).

Lemma 1 *Suppose $v \in H_0^1(\Omega)$ satisfies $J_\Omega(v) = c(\Omega)$ and $J_\Omega(v) = \sup_{t>0} J_\Omega(tv)$. Then, v must be a least energy solution to (2).*

Proof of Theorem 1 for the case $\Omega = S_R^{(V)}$:

Let $\Omega = S_R^{(V)}$. First, we note that the existence of a least energy solution u_R on S_R (see, e.g., [6], [9],[15]), that is

$$c(S_R) = J_{S_R}(u_R) = \sup_{t>0} J_{S_R}(tu_R).$$

By the elliptic regularity theorem, we have $u_R \in C^2(\overline{S_R})$. It is also known that u_R satisfies

$$u_R(x_1, x') = u_R(-x_1, x') = u_R(x_1, |x'|)$$

for every $x = (x_1, x') \in S_R$. Now, we may assume that $S_R^{(V)}$ is constructed by cutting a hyperplane which passes the origin and by patching again so that a point A of one part, we say S_1, on the intersection between the plane and S_R are transformed into the symmetric point A' on the other part, say S_2, with respect to the origin. We may think $S_R = V_1 \cup V_2$ and $S_R^{(V)} = V_1' \cup V_2'$, where $V_1' = V_1$ and V_2' is congruent with V_2 and all points $x' \in V_2'$ are transformed to the point $x \in V_2$ just by rotating, coresponding to the rotation of the face S_2. Then, there is a natural way to construct a function $\tilde{u}$ on $S_R^{(V)}$ from u_R. Namely, define $\tilde{u}(x') = u_R(x')$ for $x' \in V_1'$ and $\tilde{u}(x') = u_R(x)$ for $x' \in V_2'$, where $x \in V_2$ is the point transformed by x' by the rotation above. Then, thanks to the symmetries of u_R, we see $\tilde{u}$ is continuously defined on the face $S_1 (= S_2)$ and hence $\tilde{u} \in H_0^1(S_R^{(V)})$. Furthermore, by its construction, we have $J_{S_R^{(V)}}(t\tilde{u}) = J_{S_R}(tu_R)$ for every $t > 0$ and hence

$$\begin{aligned} c(S_R^{(V)}) &= \inf_{v \in H_0^1(S_R^{(V)})} \sup_{t>0} J_{S_R^{(V)}}(tv) \\ &\leq \sup_{t>0} J_{S_R^{(V)}}(t\tilde{u}) \\ &= \sup_{t>0} J_{S_R}(tu_R) = J_{S_R}(u_R) = c(S_R). \end{aligned} \tag{5}$$

Claim 1: $c(S_R^{(V)}) < c(S_R)$ holds.

If not, the inequality in (5) should be equal and we have

$$c(S_R^{(V)}) = \sup_{t>0} J_{S_R^{(V)}}(t\tilde{u}) = J_{S_R^{(V)}}(\tilde{u}).$$

The last equality holds because $(J_{S_R^{(V)}}(t\tilde{u}) =)J_{S_R}(tu_R)$ takes its maximum at $t = 1$. Thus, it follows from Lemma 1 that $\tilde{u}$ should be a least energy solution to (2) on $S_R^{(V)}$. By the elliptic regularity theorem we have $\tilde{u} \in C^2(S_R^{(V)})$. However, considering the point P near the inner edge of $S_R^{(V)}$, it is easy to that $\tilde{u}$ has a jump in its first derivative on P because of Hopf's lemma. This is a contradiction and claim 1 is proved.

Claim 2: $c(S_R) \le \lim_{k\to\infty} c(S_R^{(V)}) \cap B(O,k)^c)$ holds.

Since $S_R^{(V)} \cap B(O,k)^c$ is a disjoint union of D_1 and D_2, we may think $D_1 \cup D_2 \subset S_R$ and therefore $c(S_R) \le c(S_R^{(V)} \cap B(O,k)^c)$. This implies Claim 2. By Claim 1 and 2, we can conclude the existence of a least energy solution on $S_R^{(V)}$ by using Proposition 1.

The claim 1 in the proof above is important, especially in the proof of Theorem 1 for the case $\Omega = S_{R,L}^{(V)}$.

The proof of Theorem 1 for the case $\Omega = S_{R,L}^{(V)}$:

For $\Omega = S_{R,L}^{(V)}$, we can prove

$$\lim_{L\to\infty} c(S_{R,L}^{(V)}) \le c(S_R^{(V)}). \tag{6}$$

Once we obtain (6), combining the estimate of Claim 1 above, we obtain for sufficientlly large $L > 0$

$$c(S_{R,L}^{(V)}) < c(S_R) \le \lim_{k\to\infty} c(S_{R,L}^{(V)}) \cap B(O,k)^c)$$

in a similar way as the proof above and conclude the existence of a least energy solution in this case. To show the estimate (6), we use a least energy solution $u \in H_0^1(S_R^{(V)})$ to (2) for $\Omega = S_R^{(V)}$. Using a cut-off function $\chi_L(x) \in C^\infty(\mathbf{R}^n)$ satisfying $0 \le \chi_L(x) \le 1$ and $|\nabla \chi_L(x)| \le M$ on $\mathbf{R}^n$ for some constant $M > 0$, $\chi_L(x) = 0$ on $S_R^{(V)} \setminus S_{R,L}^{(V)}$, and $\chi_L(x) = 1$ on $S_{R,L-1}^{(V)}$, define

$$u_L(x) = u(x)\chi_L(x) \in H_0^1(S_{R,L}^{(V)}).$$

It is easy to see $u_L \to u$ in $H_0^1(S_R^{(V)})$ as $L \to \infty$. On the other hand, there exists a unique $t(u_L) > 0$ such that

$$\sup_{t>0} J_{S_{R,L}^{(V)}}(tu_L) = J_{S_{R,L}^{(V)}}(t(u_L)u_L) = J_{S_R^{(V)}}(t(u_L)u_L),$$

since u_L can be seen as $u_L \in H_0^1(S_R^{(V)})$ by the zero-extension. It is known that $u_L \to u$ in $H_0^1(S_R^{(V)})$ implies $t(u_L) \to t(u) = 1$ as $L \to \infty$ (see, e.g., [14]). It follows that

$$\lim_{L\to\infty} \sup_{t>0} J_{S_{R,L}^{(V)}}(tu_L) = \lim_{L\to\infty} J_{S_R^{(V)}}(t(u_L)u_L) = J_{S_R^{(V)}}(u).$$

Combining $c(S_{R,L}^{(V)}) \le \sup_{t>0} J_{S_{R,L}^{(V)}}(tu_L)$, we conclude the estimate (6).

The proof of Theorem 1 for general cases:

We use the following lemma due to del Pino-Felmer [9].

Lemma 2 *Let B be a domain in $\mathbf{R}^n$ and let A is a proper subdomain of B. If there exists a least energy solution to (2) on $\Omega = A$, then we have $c(B) < c(A)$.*

Suppose Ω contains $S_R^{(V)}$ properly, then Lemma 2 implies

$$c(\Omega) < c(S_R^{(V)}). \tag{7}$$

Since we assume that Ω is a domain with m semi-infinite cylinder A_R as ends, we can show

$$c(\widetilde{\Omega_k}) \ge c(S_R) \tag{8}$$

for large k. To show this, define $\widetilde{\Omega_{k,L}} = \widetilde{\Omega_k} \cap \{x \in \mathbf{R}^n; |x| < L\}$ for large $L > 0$. Noting $c(\widetilde{\Omega_{k,L}})$ is decreasing as $L \to \infty$, we claim

$$\lim_{L\to\infty} c(\widetilde{\Omega_{k,L}}) = c(\widetilde{\Omega_k}). \tag{9}$$

For any fixed $L > 0$, it is easy to see

$$c(S_R) \le c(\widetilde{\Omega_{k,L}}),$$

because $\widetilde{\Omega_{k,L}}$ is a disjoint union of finite cylinder and can be seen as a subset of S_R. This implies $c(S_R) \le c(\widetilde{\Omega_k})$. Combining the estimate (7), (8) and $c(S_R^{(V)}) < c(S_R)$ we arrive at the conclusion. Now it remains to show the estimate (9). It suffice to show that if we assume $\lim_{L\to\infty} c(\widetilde{\Omega_{k,L}}) > c(\widetilde{\Omega_k})$, then we have a contradiction. Let

$$\delta = \lim_{L\to\infty} c(\widetilde{\Omega_{k,L}}) - c(\widetilde{\Omega_k}) > 0.$$

By the characterization of $c(\widetilde{\Omega_k})$, there exist a sequence $\{u_j\}_{j=1}^{\infty} \subset H_0^1(\widetilde{\Omega_k})$ and $t(u_j) > 0$ such that

$$J_{\widetilde{\Omega_k}}(t(u_j)u_j) = \sup_{t>0} J_{\widetilde{\Omega_k}}(tu_j) \to c(\widetilde{\Omega_k}).$$

Let $\eta_L \in C_0^\infty(\mathbf{R}^n)$ be a function satisfying $0 \leq \eta_L(x) \leq 1$ and $|\nabla \eta_L(x)| \leq M$ on $\mathbf{R}^n$ for some constant $M > 0$, $\eta_L(x) = 1$ on $\{x \in \mathbf{R}^n; |x| \leq L-1\}$, and $\eta_L(x) = 0$ on $\{x \in \mathbf{R}^n; |x| \geq L\}$. Then we have $u_{j,L} = \eta_L u_j \to u_j$ in $H_0^1(\widetilde{\Omega_k})$ and hence $t(u_{j,L}) \to t(u_j)$ as $L \to \infty$. We have

$$\sup_{t>0} J_{\widetilde{\Omega_k}}(tu_{j,L}) = J_{\widetilde{\Omega_k}}(t(u_{j,L})u_{j,L}) \to J_{\widetilde{\Omega_k}}(t(u_j)u_j)$$

as $L \to \infty$. This yields

$$\begin{aligned} c(\widetilde{\Omega_{k,L}}) &= \inf_{u \in H_0^1(\widetilde{\Omega_{k,L}})} \sup_{t>0} J_{\widetilde{\Omega_{k,L}}}(tu) \\ &\leq \sup_{t>0} J_{\widetilde{\Omega_{k,L}}}(tu_{j,L}) \leq c(\widetilde{\Omega_k}) + \frac{2\delta}{3} \\ &< \lim_{L\to\infty} c(\widetilde{\Omega_{k,L}}). \end{aligned}$$

This is a contradicition. □

Proof of Theorem 3:

The symmetry with respect to x_1 axis and x_2-axis can be proved in a similar way as in [4] (see also [5]). It suffice to show the symmetry with repect to α-axis. The symmetry with repect to β-axis can be proved in the same way. Let u be a least energy solution. Then we know

$$u(x_1, x_2) = u(x_1, -x_2) = u(-x_1, x_2)$$

for every $x = (x_1, x_2) \in \Omega$. We decompose Ω as a disjoint union of Ω_1, Ω_2 and L, where L is the intersection of Ω and $\alpha-$axis and Ω_1 and Ω_2 are domains which are symmetric with repect to L each other. Now, we consider the mapping T on Ω_2 such that $T(x) = x'$ for $x \in \Omega_2$, where $x' \in \Omega_2$ is the reflection point of x with respect to $\beta-$axis. We define v on Ω as follows:

$$v(x) = u(x) \text{ for } x \in \Omega_1 \cup L, \quad v(x) = u(T(x)) \text{ for } x \in \Omega_2.$$

Then we have $v \in H_0^1(\Omega)$ and $J_\Omega(v) = J_\Omega(u) = c(\Omega)$. Moreover, since $J_\Omega(tv) = J_\Omega(tu)$ holds for every $t > 0$, we have

$$\sup_{t>0} J_\Omega(tv) = J_\Omega(v).$$

Then, by Lemma 2 v should be a least energy solution to (2) on Ω and hence v should be symmetric with respect to x_1-axis and x_2-axis as before. It follows that u is symmetric with respect to $\alpha-$axis. □

4 Proof of Theorem 4

In this section we give an outline of the proof of Theorem 4. Although we basically follow the strategy of Byeon [4], in which the same problem was studied on a bounded domain, we should modify his argument to the problem on unbounded domains.

As in [4], by using the scaling $v^\sigma(x) = \sigma^{2/(p-1)}u(\sigma x)$ for $x \in \Omega^\sigma = \Omega/\sigma$, the problem is reduced to find a non-trivial positive solution to

$$-\Delta v = v^p, \quad v(x) > 0 \quad x \in \Omega^\sigma, \quad v|_{\partial\Omega^\sigma} = 0, \quad v(x) \to 0 \ (|x| \to \infty),$$

satisfing

$$C_1' \leq \|v\|_{L^\infty(\Omega^\sigma)} \leq C_2', \quad C_3' \leq \|\nabla v\|_{L^2(\Omega^\sigma)} \leq C_4'$$

for some positive constants $C_j', j = 1, \cdots, 4$. Take $\delta > 0$ so that $8\delta < L$, i.e. $4\delta/\sigma < L/2\sigma$. Then, define

$$H_e^\sigma = \{v \in H_0^1(\Omega^\sigma); v(x_1, x') = v(-x_1, x')\}$$

and

$$H = \{v \in H_e^\sigma; \int_{\Omega^\sigma} |v|^{p+1}\,dx = 1, \quad \int_{\Omega^\sigma} e^{\epsilon_0\sigma|x|}\chi_\sigma(x)|v(x)|^{p+1}\,dx \leq 1\},$$

where $\chi_\sigma(x) = 0$ for $x \in \{x = (x_1, x') \in \Omega^\sigma; |x_1| \leq \frac{2\delta}{\sigma}\}$ and $\chi_\sigma(x) = \sigma^{-\frac{3(p+1)}{p-1}}$ for $x \in \{x = (x_1, x') \in \Omega^\sigma; |x_1| \geq \frac{2\delta}{\sigma}\}$. Here $\epsilon_0 > 0$ is a positive number to be determined later.

Proof of Theorem 4:

(Step 1) We first solve the following minimization problem:

$$I^\sigma = \inf_{v \in H} \int_{\Omega^\sigma} |\nabla v|^2\,dx.$$

Lemma 3 *There exists a minimizer $v = v^\sigma (\geq 0)$ to attain I^σ for every $\sigma > 0$.*

Proof: The proof is done by taking a minimizing sequence and by the standard argument. We note that the compactness can be recovered by the exponential weight, even in an unbounded domain. We omit the details.

(Step 2) We claim that

$$\limsup_{\sigma\to 0} I^\sigma \le I = \inf\{\int_{S_{1/2}} |\nabla v|^2\,dx; \int_{S_{1/2}} |v|^{p+1}\,dx = 1, v \in H_0^1(S_{1/2})\}.$$

The proof is the same as in [4], but we present it for reader's convienience. It is known that I is attained by V which satisfies $V(x_1,x') = V(-x_1,x')$ and moreover V and the first derivatives of V decays exponentially (see, e.g., [4]). Taking $\xi_\sigma \in C_0^\infty(\mathbf{R}^n)$ such that $0 \le \xi_\sigma \le 1, |\nabla\xi_\sigma| \le M$ for some constant M and $\xi_\sigma(x) = 1$ on $|x| \le \delta/\sigma$ and $\xi_\sigma(x) = 0$ on $|x| \ge (\delta/\sigma) + 2$. Then, it is easy to see that for small σ we have $\xi_\sigma V \in H_e^\sigma$ and

$$\int_{\Omega^\sigma} e^{\epsilon_0\sigma|x|}\chi_\sigma(x)|\xi_\sigma(x)V(x)|^{p+1}\,dx = 0, \quad \lim_{\sigma\to 0}\int_{\Omega^\sigma} |\xi_\sigma(x)V(x)|^{p+1}\,dx = 1.$$

Then it is easy to check that

$$\lim_{\sigma\to 0}\int_{\Omega^\sigma} |\nabla(\xi_\sigma V)|^2\,dx = I,$$

and thus we have

$$\limsup_{\sigma\to 0} I^\sigma \le \lim_{\sigma\to 0}\int_{\Omega^\sigma} |\nabla(\xi_\sigma V)|^2\,dx = I.$$

(Step 3) We claim that there exists $\sigma_0 > 0$ such that for every $\epsilon > 0$ there exists a constant $C > 0$ such that

$$\int_{\Omega^\sigma\cap\{-C<x_1<C\}} v_\sigma^{p+1}\,dx \ge 1-\epsilon$$

for every $\sigma \in (0,\sigma_0)$. The proof of this part is almost the same as in [4] and is done by a concentration-compactness argument. Our modification of the minimization problem I^σ does not cause any trouble in the argument in [4]. So, we omit the details.

(Step 4) We claim the following key estimate for the minimizer v^σ.

Lemma 4 *There exist positive constants D_1, D_2 and λ_1, which are independent of σ and ϵ_0 such that*

$$e^{D_2\sigma|x_1|}v^\sigma(x) \le D_1 e^{-\frac{\sqrt{\lambda_1}\delta}{4\sigma}} \quad x \in \{|x_1| \ge \frac{5\delta}{2\sigma}\} \tag{10}$$

and

$$v^\sigma(x) \le 2e^{-\frac{\sqrt{\lambda_1}\delta}{4\sigma}} \quad \{\frac{2\delta}{\sigma} \le |x_1| \le \frac{5\delta}{2\sigma}\}. \tag{11}$$

Proof: We also take the same strategy as in [4], but we need to modify a comparison argument by using a bound state of the Laplacian on unbounded domains.

First, we claim that there exist constants $\alpha(\sigma)$ and $\beta(\sigma)$ such that

$$\Delta v^\sigma + \alpha(\sigma)(v^\sigma)^p + \beta(\sigma)\exp(\epsilon_0\sigma|x|)\chi_\sigma(v^\sigma)^p = 0. \tag{12}$$

When $\int_{\Omega^\sigma}\exp(\epsilon_0\sigma|x|)\chi_\sigma(v^\sigma)^{p+1}\,dx < 1$, it can be concluded by Lagrange's multiplier theorem as $\beta(\sigma) = 0$. When $\int_{\Omega^\sigma}\exp(\epsilon_0\sigma|x|)\chi_\sigma(v^\sigma)^{p+1}\,dx = 1$, we note that v^σ is a minimizer to the minimization problem of two constraints:

$$\inf\{\int_{\Omega^\sigma}|\nabla v|^2\,dx; \int_{\Omega_\sigma}|v|^{p+1}\,dx = 1, \quad \int_{\Omega^\sigma}\exp(\epsilon_0\sigma|x|)\chi_\sigma|v|^{p+1}\,dx = 1\}.$$

Thus Lagrange's multiplier theorem yields the desired result. We claim that $\beta(\sigma) \leq 0$. This part is the same as in [4], so we omit the proof. Then we have

$$\int_{\Omega_\sigma}|\nabla v^\sigma|^2\,dx = \alpha(\sigma) + \beta(\sigma),$$

which implies $\alpha(\sigma) \geq 0$. The uniform boundedness of $\alpha(\sigma)$ as $\sigma \to 0$ can be proved also as in [4] by using the estimate in step 3. Now, we claim that

$$\|v^\sigma\|_{L^\infty(\Omega^\sigma)} \leq M(n \geq 3), \quad \|v^\sigma\|_{L^q(\Omega^\sigma\cap\{|x_1|\leq 4\delta/\sigma\})} \leq M\ (n = 2) \tag{13}$$

for any $q > 2$, where $M > 0$ is a constant independent of $\sigma \in (0, \sigma_0)$. For the case $n \geq 3$, by Sobolev's embedding theorem and the Proposition 3.5 in [4], which is valid even for unbounded domains, we have

$$\|v^\sigma\|_{L^\infty(\Omega^\sigma)} \leq C\|v^\sigma\|_{L^{2n/(n-2)}(\Omega^\sigma)} \leq CC'\|\nabla v^\sigma\|_{L^2(\Omega^\sigma)} \leq CC'I^{1/2},$$

where C and C' are positive constants independent of σ. Here, we used the result in step 2 in the last inequality. For the case $n = 2$, we take a function $\eta \in C_0^\infty(\mathbf{R})$ such that $0 \leq \eta \leq 1, |\nabla\eta(t)| \leq 1$, $\eta(t) = 1$ for $|t| \leq 4\delta/\sigma$ and $\eta(t) = 0$ for $|t| \geq (4\delta/\sigma) + 2$. Then we can see $\eta v^\sigma \in H_0^1(S_{1/2})$, since $\Omega^\sigma \cap \{|x_1| \leq 8\delta/\sigma\} \subset S_{1/2}$ by the definition of δ. Hence, by Trudinger's inequality, we have for every $q > 2$ there exists a constant C_q such that

$$\begin{aligned} &\|\eta v^\sigma\|^2_{L^q(S_{1/2})} \leq C_q\|\eta v^\sigma\|^2_{H^1(S_{1/2})} \\ \leq\ & C + CC_q\|v^\sigma\|_{L^2(\Omega^\sigma\cap\{|x_1|\leq 4\delta/\sigma\})} + CC_q\|v^\sigma\|_{L^2(\Omega^\sigma\cap\{4\delta/\sigma\leq|x_1|\leq(4\delta/\sigma)+2\})}. \end{aligned}$$

On the other hand, by Poincaré's inequality on $S_{1/2}$, we have

$$\begin{aligned}\|\eta v^\sigma\|_{L^2(S_{1/2})} &\leq C\|\nabla(\eta v^\sigma)\|_{L^2(S_{1/2})} \leq C + C\|v^\sigma\|_{L^2(\Omega^\sigma\cap\{4\delta/\sigma\leq|x_1|\leq 4\delta/\sigma+2\})} \\ &\leq C + C|\{\Omega^\sigma \cap \{4\delta/\sigma \leq |x_1| \leq 4\delta/\sigma + 2\}|^\theta \|v^\sigma\|^{1-\theta}_{L^{p+1}(\Omega^\sigma)} \leq C,\end{aligned}$$

where $0 < \theta = \theta(p) < 1$. Combining these estimates, we obtain that for every $q > 2$ there exists a constant C which is independent of $\sigma \in (0, \sigma_0)$ such that

$$\|\eta v^\sigma\|_{L^q(S_{1/2})} \leq C$$

holds. This implies the desired result for the case $n = 2$. By the elliptic estimate in Theorem 8.25 in [12] (the same estimate also holds if we have a uniform estimate in L^q norm of the potential term with $q > n/2$, see, e.g., [18]), we have

$$|v^\sigma(x)| \leq C\Big(\frac{1}{|B(x,1)|}\int_{B(x,1)}(v^\sigma(y))^{p+1}\,dy\Big)^{\frac{1}{p+1}}$$

for every $x \in \Omega^\sigma \cap \{|x_1| \leq 4\delta/\sigma\}$, where C is a constant independent of σ. Now we note that, by the estimate in step 3, we may assume that for a fixed $\epsilon > 0$

$$\int_{\Omega^\sigma\cap\{(\delta/\sigma)-1\leq|x_1|\leq(3\delta/\sigma)+1\}}(v^\sigma)^{p+1}\,dx \leq \epsilon$$

holds for every $\sigma \in (0, \sigma_0)$. Now, let $\rho(x)$ and λ_1 be the first eigenfunction and the first eigenvalue of $-\Delta$ on $n-1$ dimensional ball $B' = \{x' \in \mathbf{R}^{n-1}; |x'| < 1\}$. We may assume $\rho(O) = 1$ and $\rho(x') > 0$. Note also that $\rho(x') > 0$ if $x = (x_1, x') \in \overline{R_{1,\sigma}}$. Then, by using the estimates above, we may also assume that

$$\alpha(\sigma)(v^\sigma(x))^{p-1} < 3\lambda_1/4 \text{ for } \sigma \in (0, \sigma_0)$$

on the region

$$R_{1,\sigma} \equiv \Omega^\sigma \cap \{\delta/\sigma \leq |x_1| \leq 3\delta/\sigma\}.$$

Consider the comparison function

$$\Phi_\sigma(x) = \Big(\exp(-\frac{\sqrt{\lambda_1}}{2}(x_1 + \frac{3\delta}{\sigma})) + \exp(\frac{\sqrt{\lambda_1}}{2}(x_1 + \frac{3\delta}{\sigma}))\Big)\rho(x').$$

Then we obtain

$$\Delta(\Phi_\sigma - v^\sigma) + \alpha(\sigma)(v^\sigma)^{p-1}(\Phi_\sigma - v^\sigma) \leq 0 \quad x \in R_{1,\sigma}, \quad \Phi_\sigma - v^\sigma \geq 0 \quad x \in \partial R_{1,\sigma}.$$

Now, we can see that the maximum principle can be applied for

$$z_\sigma(x) = \frac{\Phi_\sigma(x) - v^\sigma(x)}{\rho(x')}$$

to conclude

$$v_\sigma(x) \le \Phi_\sigma(x) \quad x \in R_{1,\sigma}.$$

This yields

$$v^\sigma(x) \le 2e^{-\frac{\sqrt{\lambda_1}\delta}{4\sigma}}$$

for $x \in \Omega^\sigma \cap \{\frac{2\delta}{\sigma} \le |x_1| \le \frac{5\delta}{2\sigma}\}$. Next, we show the estimate on

$$R_{2,\sigma} \equiv \Omega^\sigma \cap \{|x_1| \ge 5\delta/2\sigma\}.$$

Consider the domain $\Omega^{(R)} = \{x \in \mathbf{R}^n; |x| < R\} \cup \{x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}; |x'| < d/2\}$ for large $R > d/2$. Clearly, $\Omega_\sigma \subset \Omega^{(R)}$. Then it is known (see [8] for related results) that there exists the first eigenfunction ϕ^R and the first eigenvalue γ^R such that

$$-\Delta\phi^R = \gamma^R\phi^R, \quad \phi^R(x) \le \phi^R(O) \quad x \in \Omega^{(R)},$$
$$\phi^R(x) = 0 \quad x \in \partial\Omega^{(R)}, \quad \phi^R(x) \le D_1 \exp(-D_2|x_1|)$$

for some positive constants D_1 and D_2. Take ϵ so that $3\delta < \epsilon < R$. By the Harnack inequality (see, e.g., [12, Corollary 9.25]), we have

$$\phi^R(x) \le \phi^R(O) \le \sup_{B(O,\epsilon)} \phi^R(x) \le C \min_{B(O,\epsilon)} \phi^R(x), \quad x \in \Omega^{(R)}.$$

We may take ϕ^R so that $\min_{B(O,\epsilon)} \phi^R(x) = 1$. Let $\Omega^{\sigma,(R)} = \Omega^{(R)}/\sigma$ and consider

$$\Psi_\sigma(x) = 2\exp(-\frac{\sqrt{\lambda_1}\delta}{4\sigma})\phi^R(\sigma x), \quad x \in \Omega^{\sigma,(R)}.$$

Noting that $\min_{B(O,\epsilon)} \phi^R(x) = 1$ implies $\phi^R(x) \ge 1$ on $\partial R_{2,\sigma} \cap \Omega^\sigma$, we have

$$\Psi_\sigma(x) \ge v^\sigma(x), \quad x \in \partial R_{2,\sigma}.$$

On the other hand, we have

$$\int_{\Omega^\sigma \cap \{|x_1| \ge \frac{2\delta}{\sigma}\}} (v^\sigma)^{p+1}\, dx \le \int_{\Omega^\sigma \cap \{|x_1| \ge \frac{2\delta}{\sigma}\}} e^{\epsilon_0\sigma|x|}(v^\sigma)^{p+1}\, dx \le \sigma^{3(p+1)/(p-1)}.$$

By applying Theorem 8.25 in [12] again for $x \in R_{2,\sigma}$ and noting $B(x,1) \cap \Omega^\sigma \subset \Omega^\sigma \cap \{|x_1| \geq \frac{2\delta}{\sigma}\}$, we obtain

$$v^\sigma(x) \leq C\sigma^{3/(p-1)}.$$

Here, in the case $n = 2$, we use the boundedness of $\|(v^\sigma)^{p-1}\|_{L^{(p+1)/(p-1)}(\Omega^\sigma)}$ to control the uniform boundedness of the constant appeared in the generalized version of Theorem 8.25 of [12] (see [18]). Let $\tilde{\rho}$ and $\tilde{\lambda}_1$ be the first eigenfunction and the first eigenvalue of the Laplacian on $\{x' \in \mathbf{R}^{n-1}; |x'| < d\}$ and let

$$z^\sigma(x) = \frac{\Psi_\sigma(x) - v^\sigma(x)}{\tilde{\rho}(x')}.$$

Then we can see that the maximum principle can be applid to z^σ on $R_{2,\sigma}$ to obtain $z^\sigma \geq 0$ in $R_{2,\sigma}$. This yields the desired estimate on $R_{2,\sigma}$. By the estimates (10), (11), (13) and Proposition 3.5 in [4], now we have the uniform boundedness of v^σ on Ω^σ even in the case $n = 2$.

(Step 5) First, note that there exists a constant D_3 which is independent of σ that $|x| \leq D_3|x_1|$ on $\Omega^\sigma \cap \{|x_1| \geq 2\delta/\sigma\}$. Take $\epsilon_0 > 0$ so that $D_3\epsilon_0 < D_2(p+1)$, where D_2 is the constant appeared in the estimate of step 4. Then, dividing Ω^σ into two parts and using estimates in step 4, we can easily see

$$\int_{\Omega^\sigma} e^{\epsilon_0\sigma|x|}\chi_\sigma(x)(v^\sigma(x))^{p+1}\,dx \leq C\sigma^{-(n+\frac{3(p+1)}{p-1})}e^{-\frac{(p+1)\sqrt{\tilde{\lambda}_1}\delta}{4\sigma}} \to 0$$

as $\sigma \to 0$. Thus there exists a constant $\sigma_0 > 0$ such that

$$\int_{\Omega^\sigma} e^{\epsilon_0\sigma|x|}\chi_\sigma(x)(v^\sigma(x))^{p+1}\,dx < 1$$

holds for $\sigma \in (0, \sigma_0)$. Therefore, difining $u^\sigma(x) = (I^\sigma)^{1/(p-1)}v^\sigma(x)$, we obtain

$$-\Delta u^\sigma = (u^\sigma)^p.$$

Note that we can see from the the estimate in Step 4 $\liminf_{\sigma\to 0} I^\sigma \geq I$. Then the uniform lower bounds for u^σ follow from estimates in Step 2, Step 3 and Step 4. □

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