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Stable Pattern of FitzHugh-Nagumo Equation for Higher Dimensions and its Limiting Problem

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概要

FitzHugh Nagumo equation has been studied extensively in the field of mathematical biology. It has the mechanism of "lateral inhibition" which seems to play a big role in the pattern formation of plancton distribution. We consider FitzHugh Nagumo equation in high dimension and show the existence of stable nonconstant stationary solutions which have fine structures in a mesoscopic scale. We construct the spatially periodic stationary stable solution. Moreover we consider the singular limit energy.

1 Introduction and Results

In this paper, we consider the activator-inhibitor system called FitzHugh Nagumo equation in higher dimension. We are particularly interested in the problem with a small positive parameter and the related singular limit problem. FitzHugh Nagumo equation was first introduced as the simplified equation of Hodgkin-Huxley system which is a model of conduction and excitation of nerve impulses in physiology. High dimensional problem appears in neural net models for short term memory or in nerve cells of heart muscle. Afterward Extended FitzHugh Nagumo equation was proposed as a mathematical model of biological pattern formation. It has been suggested that lateral inhibition may contribute to pattern formation in plancton distribution [6]. It is a system of reaction-diffusion equations with two independent variables, u and v. Here u denotes an activator and v acts as its inhibitor. Such an activator-inhibitor sysytem has also been studied in the field of chemical reaction.

One of our main results concerns the Neumann problem:

$$(1.1)\qquad \begin{cases} u_t = D_1\Delta u + f(u) - \kappa v & x\in\Omega, t>0,\\ \tau v_t = D_2\Delta v + u - \gamma v & x\in\Omega, t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, t\geq 0,\end{cases}$$

where $\Omega\subset\mathbb{R}^n$ is a smooth domain, $\nu(x)$ is the outer normal at $x\in\partial\Omega$; the functions u, v are real-valued and denotes an activator and an inhibitor, respectively; κ, τ, D_2 are positive constants; $\gamma, D_1=\varepsilon^2$ are positive small parameters; $\Delta=\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is Laplacian; $f(u)$ is a cubic nonlinearity like $f(u)=u(1-u)(u-a)$, $(0<a<1/2)$. More generally we assume that

(F1) $f\in C^1(\mathbb{R})$.

(F2) There exist constants $0 < a_0 < a_1 < a_2 < 1$ such that

$$f(u)\begin{cases} >0 & \text{for } u<0, a_1<u<1,\\ <0 & \text{for } 0<u<a_1, u>1,\end{cases} \quad f'(u)\begin{cases} <0 & \text{for } u<a_0, u>a_2,\\ >0 & \text{for } a_0<u<a_2.\end{cases}$$

(F3) $\liminf_{|u|\to\infty} \frac{|f(u)|}{|u|} > 0$.

(F4) $\int_0^1 f(v)\,dv > 0$.

Note that (F1) and (F2) imply that $f(0) = f(a_1) = f(1) = 0$ and $f'(a_0) = f'(a_2) = 0$. There exists a unique number $b > 0$ such that the equation $f(s) = b$ has three different solutions $s_0 < s_1 < s_2$ which satisfy $\int_{s_0}^{s_2}(f(v) - b)\,dv = 0$. We assume that

(H)
$$\frac{\kappa}{\gamma} > \frac{b}{s_2}.$$

In case γ small, the problem (1.1) has no constant stationary solution except for $(u, v) = (0, 0)$. We call this case monostable. In case γ large, there are three constant stationary solutions, two of which are stable and one unstable. We call this case bistable. Assumption (H) does not completely exclude bistable parameter region. Note that $(u, v) = (0, 0)$ is always a stable stationary solution. The positive stable constant solution in bistable case is called an excited state. The space independent system of (1.1) exhibits excitability. Our system of equations has both excitability and lateral inhibition.

Our existence and stability result of Neumann Boundary Problem (1.1) is the following:

Theorem 1.1 *Let Ω be a bounded domain and assume (H). Then for D_1 sufficiently small, the problem (1.1) has a nonconstant stationary solution $(u_\varepsilon, v_\varepsilon)$ such that the total variation of u_ε goes to ∞ as $D_1 \to 0$. If, in addition, $\tau\kappa < \gamma^2$, then there exists a stable nonconstant stationary solution.*

In case Ω is a bounded open interval of one dimensional space, it is known that there are stable stationary solutions which have multi internal layers for D_1 small. Moreover solutions with finite number of layers remain stable as $D_1 \to 0$ (see [8]). Unlike one dimensional case, in higher dimensions, any stable stationary internal layered solutions to a class of reaction-diffusion systems cannot have a smooth limiting interfacial configurations when the thickness of the interface tends to zero [9]. Our solutions in high dimension oscillate in a mesoscopic scale and the interfacial energy of u_ε tends to infinity. Hence the interface of our solutions does not converge to any smooth $(n-1)$ dimensional surface. Moreover the pattern of u_ε does not exhibit point condensation. We express the richness of spatial patterns of u_ε in terms of Young measure (see Theorem 2.1). We also estimate the amount of interface.

A periodically undulating cylindrical domain Ω is the set of the form

$$\bigcup_{s\in\mathbb{R}} \{s\} \times \Omega_s$$

where $\Omega_s \subset \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ is a family of bounded domains which is periodic in $s \in \mathbb{R}$, that is, there exists a number $T > 0$ such that $\Omega_{s+T} = \Omega_s$ for all $s \in \mathbb{R}$.

Our result in a periodically undulating cylindrical domain is the following:

Theorem 1.2 *Let Ω be a periodically undulating cylindrical domain and assume (H). Then for D_1 sufficiently small, the problem (1.1) has a nonconstant periodic stationary solution.*

In particular, when Ω is a cylindrical domain, we construct the spacially periodic stationary solutions. In a cylindrical domain, our construction might give a standing wave which is not a trivial extension of one dimensional solutions, which is suggested by the comparison of energy of patterns in the limiting case $D_1 \to 0$. Another main result of the present paper concerns the singular limit problem which may characterize internal layers of our solutions (see section 4).

In one dimensional space, it is known that there exists a travelling wave with a front (or interface). Moreover when $f(u)$ is cubic, there exists a nonconstant stationary solution which is decaying at infinity or spatially periodic [2], [5]. We have the following:

Corollary 1.1 *Assume (H). Then for D_1 sufficiently small, the problem*

$$(1.2) \qquad \begin{cases} u_t = D_1 \Delta u + f(u) - \kappa v, & x \in \mathbb{R}, t > 0, \\ \tau v_t = D_2 \Delta v + u - \gamma v, & x \in \mathbb{R}, t > 0. \end{cases}$$

has an infinite number of nonconstant periodic stationary solutions modulo translation equivalence.

We remark that in [2], Ermentrout, Hastings and Troy considered the case

$$(1.3) \qquad f(u) = u(1-u)(u-a), \quad 0 < a < 1/2$$

with

$$(1.4) \qquad \frac{\kappa}{\gamma} > \frac{(a-1)^2}{4} = \max_{u>0}(1-u)(u-a).$$

It is easy to see that (1.4) implies assumption (H), which is explicitly written

$$\frac{\kappa}{\gamma} > \frac{(a+1)(1-2a)(2-a)}{9(a+1+\sqrt{3(a^2-a+1)})}.$$

We also remark that any stable stationary solutions of the single equation

$$u_t = d\Delta u + f(u), \quad d > 0,\ u \in L^\infty(\mathbb{R}^n)$$

are translation and rotation invariant, hence constant. See [10].

This paper is organized as follows: In section 2 and 3, we show the existence of nonconstant stationary solution in bounded domains and periodically undulating cylindrical domains, respectively. Our approach is the following: We define some functional (which we call energy) on $H^1(\Omega)$, and find a critical point with Morse index 0 which is typically a local minimizer. Our functional has a nonlocal term, which plays a big role in existence of nonconstant local minimizer. In section 4, we define the limiting problem for the constrained class of functions with periodic structures. We estimate the energy for both one dimensional lamellar pattern and two dimensional square structure. These estimates suggests that minimizers in two space dimension can have actually two dimensional periodic structures in some parameter region. In section 5, we give a stability result to the solution obtained in section 2 and 3. We consider the spectrum of linearized problem and study carefully the relation between the original linearized operator and the operator corresponding to the second derivative of energy functional. Note that our equation is not a gradient system. Theorems 1.1 and 1.2 follow from a series of propositions in sections 2–5.

Remark 1-1. Hereafter, for the sake of notational simplicity, we will use the same letters C to denote some positive constants whose values may vary from line to line. This notational convention does not apply to such letters $C_1, C_2, \ldots$.

Remark 1-2. We will express any sequence $a_k \to 0$, $k \to \infty$ as the same notation $a_k = o(1)$, $k \to \infty$.

2 Neumann Problem on Bounded Domain

In sections 2–4, for simplicity, we assume that $D_2 = \kappa = 1$. In this section, we are concerned with the existence of stationary solutions for Neumann problem on bounded domain. Denote by $(\cdot,\cdot)$ L^2-inner product. By linear transformathion of u, v (and still using the same notation u, v), we reformulate the problem into

$$\left\{ \begin{array}{rcl} u_t &=& \varepsilon^2 \Delta u - W'(u) - v, \\ \tau v_t &=& \Delta v + u - m_0 - \gamma v, \end{array} \right. \tag{2.1}$$

where m_0 is a constant and $W(u)$ is a double-well potential with equal depth which satisfies the following.

(W1) $W \in C^2(\mathbb{R})$.

(W2) $W(u) = 0$ if $u = \pm 1$, and $W(u) > 0$, otherwise.

(W3) $W''(\pm 1) > 0$.

(W4) $\liminf_{|u|\to\infty} \frac{W(u)}{|u|^2} > 0$.

We realize that $m_0 \in (-1, 1)$ by the assumption (H). Introduce an inverse operator K of $-\Delta + \gamma$ with Neumann boundary condition:

$$K = (-\Delta + \gamma)^{-1}. \tag{2.2}$$

The stationary solution (u, v) of (2.1) solves the system of equations

$$\begin{array}{rcl} v &=& \varepsilon^2 \Delta u - W'(u) \\ v &=& K(u - m_0) \end{array} \tag{2.3}$$

and hence

$$K(u - m_0) = \varepsilon^2 \Delta u - W'(u). \tag{2.4}$$

Suppose u is a solution of (2.4) with Neumann boundary condition, then we get the stationary solution u and $v = Ku$ of equation (1.1). Problem (2.4) corresponds to the Euler-Lagrange equation of the functional

$$I(u) = I_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx + \frac{1}{2}(K(u - m_0), u - m_0), \tag{2.5}$$

on the space $H^1(\Omega)$. We note that in case $n \geq 5$, I may take the value $+\infty$ for some $u \in H^1(\Omega)$. (It does not matter for the existence of global minimizers.)

Lemma 2.1 *Let $u_c \equiv c$ be constant. Then $\min_{c \in \mathbb{R}} I(u_c) > 0$.*

Proof. Noting that $Ku_c = \frac{1}{\gamma} c$,

$$I(u_c) = \int_\Omega W(c)\, dx + \frac{1}{2}(\frac{1}{\gamma}(c - m_0), c - m_0).$$

Lemma 2.1 follows from the fact $m_0 \neq \pm 1$. □

Since K is a positive operator and there exist positive constants c, C' such that $W(u) \geq -C' + \frac{c}{2} u^2$ for all $u \in \mathbb{R}$, we have

$$I(u) \geq \frac{\varepsilon^2}{2} \|\nabla u\|_{L^2}^2 + \frac{c}{2} \|u\|_{L^2}^2 - C', \tag{2.6}$$

uniformly in $u \in H^1(\Omega)$. Hence I is coercive on $H^1(\Omega)$. Moreover I is weakly lower semi-continuous on $H^1(\Omega)$. Indeed, If $u_m \rightharpoonup u$ weakly in $H^1(\Omega)$, by Rellich's theorem $u_m \to u$ strongly in $L^2(\Omega)$ and hence

$$(Ku, u) = \lim_{m \to \infty} (Ku_m, u_m).$$

Since the function $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$G(u, p) = \frac{\varepsilon^2}{2} |p|^2 + W(u)$$

is convex in p, and bounded from below, the first term of I is weakly lower semi-continuous with respect to H^1-norm. (See for example Struwe [13, Theorem 1.6].) Thus we see

$$I(u) \leq \liminf_{m \to \infty} I(u_m)$$

as we desired. By the standard variational method, I is bounded from below on $H^1(\Omega)$ and there exists a global minimizer $u_\varepsilon \in H^1(\Omega)$ of I solving (2.4). Generally, the global minimizer u_ε may be constant. However we have the following:

Proposition 2.1 *The minimum of I_ε goes to 0 as $\varepsilon \to 0$ and hence u_ε is not constant.*

Proof. For a positive integer $m \in \mathbb{N}$, we consider the parameter such that $D_1 = \varepsilon^2 := \frac{1}{m^6}$. Since $\min_{H^1(\Omega)} I$ is non-decreasing with respect to ε, it suffices to show that the minimum of I converges to 0 as $m \to \infty$. To see this, let $\theta = \frac{m_0+1}{2} \in (0, 1)$. Let $u_m \in C^\infty \cap L^\infty(\mathbb{R}^n)$

be the function which depends only on x_1 and satisfies $u_m(-x_1, \cdots, x_n) = u_m(x_1, \cdots, u_n)$, $u_m(x_1 + \frac{2}{m}, \cdots, x_n) = u_m(x_1, \cdots, u_n)$ for all $x \in \mathbb{R}^n$, and

$$u_m(x) = \begin{cases} -1, & 0 \le x_1 \le a_m, \\ \chi(m^3(x_1 - a_m)), & a_m \le x_1 \le b_m, \\ 1, & b_m \le x_1 \le \frac{1}{m} \end{cases}$$

where $\chi \in C^\infty(\mathbb{R}, [-1, 1])$ is a function with

$$\chi(s) = \begin{cases} -1, & s \le 0 \\ 1, & s \ge 1 \end{cases}$$

$a_m = (1-\theta)(\frac{1}{m} - \frac{1}{m^3})$, $b_m = (1-\theta)\frac{1}{m} + \theta\frac{1}{m^3}$. Since the wave length of u_m is $\frac{2}{m}$ and Ω is bounded, we estimate H^1-seminorm of u_m as follows:

$$\begin{aligned} \int_\Omega |\nabla u_m|^2\, dx &\le Cm \cdot (m^3)^2 \int_{-\infty}^{\infty} \chi'(m^3 s)^2\, ds \\ &= Cm^4 \int_{-\infty}^{\infty} \chi'(t)^2\, dt. \end{aligned}$$

Now let us consider the second and the third tirm of $I(u_m)$. The sequence (u_m) is bounded in $L^2(\Omega)$ but does not have the convergent subsequence since u_m oscillates rapidly in x_1 as $m \to \infty$. We show that

$$u_m \rightharpoonup m_0, \quad W(u_m) \rightharpoonup 0,$$

weakly in $L^2(\Omega)$. Although this can be proved directly, we use the notion of Young measure for understanding the behavior of sequence (u_m). Young measure $\mu = (\mu_x)_{x\in\Omega}$ is a family of probability measures on $\mathbb{R}$.

Lemma 2.2 *The sequence $(u_m)_{m=1}^\infty \subset L^2(\Omega)$ generates Young measure $\mu = (\mu_x)_{x\in\Omega}$ with $\mu_x = (1-\theta)\delta_{-1} + \theta\delta_1$ a.e. $x \in \Omega$, where δ_s denotes Dirac measure centered at $s \in \mathbb{R}$.*

We note that for large m, the functions u_m can be approximated by periodic (with periodicity $\frac{2}{m}$) step function in L^2-topology.

Proof. Since (u_m) is a bounded sequence in $L^2(\Omega)$, by the fundamental existence theorem (See [12], Theorem 6.2), there exists a parametrized measure (called Young measure) associated to sequence (u_m) (if necessary passing to the subsequence, not relabeled). Let $Q \subset \Omega$ be n-dimensional cube. For simplicity, put

$$Q_\delta = (0,\delta)^n = (0,\delta) \times Q'_\delta, \quad Q'_\delta = (0,\delta)^{n-1}.$$

Finally let φ_η be the characteristic function of $(-1-\eta, -1+\eta)$, which is defined by

$$\varphi_\eta(s) = \begin{cases} 1, & \text{if } s \in (-1-\eta, -1+\eta), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have for $\eta \in (0,1)$ and $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$,

$$\begin{aligned} \int_{Q_\delta} \varphi_\eta \circ u_m\, dx &= \int_{Q'_\delta} dx' \int_0^\delta \varphi_\eta \circ u_m\, dx_1 \\ &= \delta|Q'_\delta|(ma_m + o(1)) \\ &= \delta|Q'_\delta|((1-\theta)(1 - \frac{1}{m^2}) + o(1)) \\ &= (1-\theta)|Q_\delta| + o(1), \end{aligned}$$

where $o(1) \to \infty$ as $m \to \infty$. Since

$$\lim_{m\to\infty} \int_{Q_\delta} \varphi_\eta \circ u_m \, dx = \int_{Q_\delta} \int_{-\infty}^{\infty} \varphi_\eta(\xi) \, d\mu_x(\xi) dx.$$

by the property of Young measure, it follows that

$$\frac{1}{|Q_\delta|} \int_{Q_\delta} \mu_x(-1-\eta, -1+\eta) \, dx = (1 - \theta).$$

Letting $\delta \to 0$, we obtain

$$\mu_x(-1-\eta, -1+\eta) = 1 - \theta, \quad \text{a.e. in } x \in \Omega.$$

Since $0 < \eta \ll 1$ is arbitrary, we have

$$\mu_x(\{-1\}) = 1 - \theta, \quad \text{a.e. in } x \in \Omega.$$

Similarly we have $\mu_x(\{1\}) = \theta$ a.e. The proof is complete. □

Completion of Proof of Proposition 2.1. By Lemma 2.2, we have $u_m \rightharpoonup m_0$ weakly in $L^2(\Omega)$, and

$$\begin{aligned} W(u_m) &\rightharpoonup \int_{-\infty}^{\infty} W(\xi) \, d\mu_x(\xi) \\ &= \theta W(1) + (1-\theta) W(-1) \\ &= 0 \end{aligned}$$

weakly in $L^2(\Omega)$. Since K is compact on $L^2(\Omega)$, $Ku_m \to \frac{m_0}{\gamma}$ strongly in $L^2(\Omega)$. Hence for large m (by $D_1 = \frac{1}{m^6}$),

$$\begin{aligned} I(u_m) &= \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u_m|^2 + W(u_m) \right) dx + \frac{1}{2}(K(u_m - m_0), u_m - m_0) \\ &\to 0 \end{aligned}$$

as $m \to \infty$. We have proved Proposition 2.1. □

Our next goal is the behavior of our solutions as ε tends to zero. We get some information of global minimizers from the associated Young measure.

Theorem 2.1 *As $\varepsilon \to 0$, any sequence of global minimizers (u_ε) generates Young measure $\mu = (\mu_x)_{x\in\Omega}$ with*

$$\mu_x = \frac{1-m_0}{2}\delta_{-1} + \frac{1+m_0}{2}\delta_1 \quad a.e.\ x.$$

Proof. From $I_\varepsilon(u_\varepsilon) \to 0$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega W(u_\varepsilon) \, dx = 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} (K(u_\varepsilon - m_0), u_\varepsilon - m_0) = 0.$$

In particular u_ε is bounded in $L^2(\Omega)$ and generates some Young measure μ (if necessary, passing to the subsequence). Hence we have

$$\int_\Omega \int_{-\infty}^{\infty} W(\lambda) \, d\mu_x(\lambda) dx = 0. \tag{2.7}$$

$$(K(u - m_0), u - m_0) = 0, \tag{2.8}$$

where

$$u := \int_{-\infty}^{\infty} \lambda \, d\mu_x(\lambda)$$

is the first moment of μ. Since $W \geq 0$, (2.7) implies that

$$\int_{-\infty}^{\infty} W(\lambda) \, d\mu_x(\lambda) = 0 \quad \text{a.e. } x. \tag{2.9}$$

Letting $U_\eta = \{\lambda \in \mathbb{R} \,;\, (\lambda^2 - 1)^2 > \eta\}$ for $\eta > 0$, there holds $\inf_{U_\eta} W > 0$. By (2.9), $\mu_x(U_\eta) = 0$ for any $\eta > 0$. Hence there exists a measurable function θ such that $0 \leq \theta \leq 1$ and

$$\mu_x = (1 - \theta(x))\delta_{-1} + \theta(x)\delta_1.$$

On the other hand, (2.8) implies that $u(x) \equiv m_0$. Hence

$$m_0 = \int_{-\infty}^{\infty} \lambda \, d\mu_x(\lambda) = 2\theta(x) - 1.$$

Therefore $\theta(x) = \frac{m_0+1}{2}$. The proof is complete. □

Remark 2-1. From this result we see that

(1) u_ε are internal layered solutions.

(2) The interface of solutions u_ε does not converge to any smooth $(n-1)$ dimensional surface.

(3) The interfacial energy (proportional to the area of interface) tends to ∞.

3 Existence of Periodic Solutions

In this section, we prove the existence of periodic stationary solutions of (1.1) when

$$\Omega = \bigcup_{s \in \mathbb{R}} \{s\} \times \Omega_s$$

is a periodically undulating cylindrical domain. First we define the function space with periodic structures. Denote by $\mathbf{e}_i$, $i = 1, 2, \ldots, n$ be a i-th unit vector and

$$\Omega' = \{0 < x_1 < T\} \cap \Omega,$$

where $T > 0$ is a number such that

$$\Omega_{s+T} = \Omega s, \quad \text{for all } s \in \mathbb{R}.$$

We consider the energy functional I_ε defined below on the space

$$H^1_{\text{per}}(\Omega) = \{u \in H^1_{\text{loc}}(\Omega) \,;\, u(x + T\mathbf{e}_1) = u(x), \text{ a.e. } x \in \Omega\}.$$

The functional I_ε is defined by

$$I_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_{\Omega'} |\nabla u|^2\,dx + \int_{\Omega'} W(u)\,dx + \frac{1}{2}\int_{\Omega'} (u - m_0)\psi\,dx, \tag{3.1}$$

where W is the double-well potential and m_0 is the constant as defined in seciton 2, and ψ is the solution of

$$\begin{cases} (-\Delta + \gamma)\psi = u - m_0 & \text{in } \Omega', \\ \psi \in H^1_{\mathrm{per}}(\Omega), \\ \text{Neumann boundary condition on } \partial\Omega. \end{cases} \tag{3.2}$$

Let d_ε be the infimum of I_ε:

$$d_\varepsilon := \inf_{H^1_{\mathrm{per}}(\Omega)} I_\varepsilon.$$

By applying the direct method of the calculus of variations, d_ε is attained at a point $u^\varepsilon \in H^1_{\mathrm{per}}(\Omega)$. As in section 2, the global minimizer is not constant in case $D_1 = \varepsilon^2$ is sufficiently small. Moreover, letting ψ^ε be the solution of (3.2) corresponding to u^ε, we obtain the stationary solution of (1.1) after the change of variables. Hence we have

Proposition 3.1 *For sufficiently small ε, the problem (1.1) has a nonconstant periodic stationary solutions $(u_\varepsilon, v_\varepsilon)$ with $u_\varepsilon \in H^1_{per}(\Omega)$.*

In particular, when $\Omega = \mathbb{R}$, we can choose T arbitrarily. Using algebraically independent numbers $T > 0$, we obtain infinitely many solutions which do not become identically equal by translation.

Next we consider the lattice periodicity in higher dimendions. First we define the function space with periodic structures. For each $L \in GL(n, \mathbb{R})$, let

$$\Omega_L = \{\sum_{i=1}^{n} \zeta^i(L\mathbf{e}_i)\,;\, -\frac{1}{2} \le \zeta^i \le \frac{1}{2}, \text{ for } i = 1, 2, \ldots, n\}$$

be a unit lattice corresponding to $L \in GL(n, \mathbb{R})$. Defin the function space

$$H^1_L(\mathbb{R}^n) = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)\,;\, u(x + L\mathbf{e}_i) = u(x), \text{ a.e.}, 1 \le i \le n\}.$$

and the functional E_ε

$$E_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_{\Omega_L} |\nabla u|^2\,dx + \int_{\Omega_L} W(u)\,dx + \frac{1}{2}\int_{\Omega_L} (u - m_0)\psi\,dx, \tag{3.3}$$

where W and m_0 is the same as in the definition of I_ε and ψ is the solution of

$$\begin{cases} (-\Delta + \gamma)\psi = u - m_0 & \text{in } \Omega_L, \\ \text{periodic boundary condition on } \partial\Omega_L. \end{cases} \tag{3.4}$$

Let u^ε minimize E_ε on $H^1_L(\mathbb{R}^n)$. Moreover, letting v^ε be the solution of (3.4) corresponding to u^ε, we obtain the spatially periodic stationary solution of

$$\begin{cases} u_t &= \varepsilon^2\Delta u - W'(u) - v & \text{in } \mathbb{R}^n, \\ \tau v_t &= \Delta v + u - m_0 - \gamma v & \text{in } \mathbb{R}^n. \end{cases}$$

Remark 3-1. The solutions for $n \ge 2$ may be a trivial extension of the solution in one dimensional space. However the results of the next section imply that various periodic solutions can appear in higher dimension when parameters change.

4 Limiting Problem

In this section, we consider the limiting problem, namely the problem obtained by letting $\varepsilon \to 0$ in (2.5) or (3.3). The energy functional consists of three competing terms. The double-well energy $W(u)$ prefers the segregated states $\{u \approx 1\}$ and $\{u \approx -1\}$ to intermediate states. The term $\varepsilon^2|\nabla u|^2$ is the interfacial energy which penalizes interfaces between such two states and thus prefers large domains of a single state. The third term of I_ε is non local and represents long range energy which does not become small if a single state cover all the domain. In order to make the third term small, the ratio of domains of two states approaches $(m_0+1)/2$ and oscillates rapidly around $(m_0+1)/2$. The effect of all three is to determine the optimal domain size compromising these opposite tendencies. We are interested in the characterization of the periodic structure of the global minimizer of the energy. We consider the geometrical pattern of such two states in case phase separation is very strong. The singular limit of the energy consists of two competeing terms: one is interfacial energy and the other non local long range energy.

Denote by $\mathcal{M}$ the space of the pairs of a function u and a matrix $L \in GL(n,\mathbb{R})$, $\|L\|_\infty \le 1$ which satisfy $|u(x)| = 1$, $u(x + L\mathbf{e}_i) = u(x)$, $1 \le i \le n$, a.e. $x \in \mathbb{R}^n$ and $u|_{\Omega_L} \in BV(\Omega_L)$.

Remark 4-1. This lattice periodicity includes the hexagonal structure in two space dimension.

Define the functional I^*_α on $\mathcal{M}$ as follows: Let $\Gamma(u,L)$ be the average total variation per unit volume:

$$\Gamma(u,L) := \lim_{R\to\infty} \frac{1}{|\Omega_{RL}|}\int_{\Omega_{RL}} |Du|,$$

and $J(u,L)$ be the long-distance energy dencity:

$$J(u,L) := \frac{1}{|\Omega_L|}\int_{\Omega_L} (u-m_0)\psi\, dx,$$

where ψ is the solution of (3.4). Let $I^*_\alpha(u)$ be the sum of $\frac{\alpha}{2}\Gamma(u,L)$ and $\frac{1}{2}J(u,L)$:

$$I^* = I^*_\alpha = \frac{\alpha}{2}\Gamma + \frac{1}{2}J,$$

where α is a small positive parameter.

Remark 4-2. This limiting problem has two interpretations as follows:
(i) One is that when $|\Omega| = 1$ and $\alpha = \alpha_0\varepsilon$ where $\alpha_0 = \sqrt{2}\int_{-1}^{1}\sqrt{W(u)}\,du$ is the interfacial energy per unit area, I^*_α approximates (2.5).
(ii) The other is that using the rescaling function $u_x(y) := \hat{u}(x + \varepsilon^{\frac{1}{3}}y)$, for $\hat{u} \in H^1(\Omega)$ and some $x \in \Omega$, the quantity $\varepsilon^{-\frac{2}{3}} I_\varepsilon(\hat{u})/|\Omega|$ in (2.5) or (3.3) is approximated by $I^*_{\alpha_0}(u_x)$.
In any cases, we do not consider low frequncy periodicity but confine ourselves to mesoscopic scale. See also the observation in the last part of this section.

Fix $L \in GL(n,\mathbb{R})$. Then it is easy to show that the infimum of I^*_α on

$$\mathcal{M}_L = \{u : \mathbb{R}^n \to \{+1,-1\}\,;\, u(x+L\mathbf{e}_i) = u(x),\ \text{a.e. } 1 \le i \le n,\ u|_{\Omega_L} \in BV(\Omega_L)\}$$

is positive and attained at a point $u_1 \in \mathcal{M}_L$. Indeed note that the embedding $BV \hookrightarrow L^1$ is compact and the function $u \mapsto \int_\Omega |Du|$ is lower semi-continuous with respect to L^1 convergence. In fact, I^*_α has a positive minimum not only on $\mathcal{M}_L$ but also on $\mathcal{M}$. (See Lemma 4.1 below.)

Note that for $|\det L|$ is very small, the functional $J(u,L)$ is approximated by

$$J(u,L) \approx \frac{1}{\gamma|\Omega_L|^2}\left(\int_{\Omega_L}(u-m_0)\,dx\right)^2 + \frac{1}{|\Omega_L|}\int_{\Omega_L} u\psi_0\, dx$$

where ψ_0 is defined by

$$
\begin{cases} -\Delta\psi_0 = u - \frac{1}{|\Omega_L|}\int_{\Omega_L} u\,dx, & \text{in } \Omega_L, \\ \int_{\Omega_L} \psi_0 = 0, \\ \text{Periodic B.C.} \end{cases} \tag{4.1}
$$

Here for convenience, we define for $(u, L) \in \mathcal{M}$,

$$
\tilde{I}^*(u) = \tilde{I}^*_\alpha(u) := \frac{\alpha}{2}\Gamma(u, L) + \frac{1}{2\gamma|\Omega_L|^2}\left(\int_{\Omega_L}(u - m_0)\,dx\right)^2 + \frac{1}{2}J_0(u, L),
$$

where

$$
J_0(u, L) := \frac{1}{|\Omega_L|}\int_{\Omega_L} u\psi_0\,dx.
$$

and ψ_0 is the solution of (4.1). Then

Theorem 4.1 *I^* has a positive minimum on $\mathcal{M}$ and*

$$
\min_{\mathcal{M}} \tilde{I}^*_\alpha \geq \min_{\mathcal{M}} I^*_\alpha > 0.
$$

Proof. It suffices to show that $\inf_{\mathcal{M}} I^*_1 > 0$. Assume by contradiction that $\inf_{\mathcal{M}} \tilde{I}^*_1 = 0$. Let (u_k, L_k) be a minimizing sequence. Since $\|L_k\|_\infty$ is bounded, there exists a convergent subsequence (not relabelled). Let $L_0 := \lim_{k\to\infty} L_k$. We claim that $\det L_0 \neq 0$. In fact, $\lim_{k\to\infty} J(u_k, L_k) = 0$ implies that

$$
\left|\frac{1}{|\Omega_{L_k}|}\int_{\Omega_{L_k}} u_k\,dx - m_0\right| \to 0.
$$

On the other hand, by $\lim_{k\to\infty}\Gamma(u, L) = 0$, we have

$$
\frac{1}{|\Omega_{L_k}|}\int_{\Omega_{L_k}} |Du_k| \to 0.
$$

By the isoperimetric inequality, $\lim_{k\to\infty}|\Omega_{L_k}| = 0$ deduces

$$
\frac{1}{|\Omega_{L_k}|}\int_{\Omega_{L_k}} |Du_k| \to \infty.
$$

which is the desired contradiction. Thus $L_0 \in GL(n, \mathbb{R})$.

Since u_k is bounded in BV_{loc}, we may assume that $u_k \to u_0$ strongly in L^1_{loc}, and hence pointwise almost everywhere. Hence we get $(u_0, L_0) \in \mathcal{M}$. Moreover since Γ is lower semicontinuous and J is continuous with respect to L^1 convergence, we obtain

$$
\min_{\mathcal{M}} I^*_\alpha = I^*_\alpha(u_0) > 0.
$$

The proof is complete. □

Now we proceed to the comparison of minimal energy of one dimensional pattern and two dimensional one. We will show that in some parameter region, the minimal energy is not attained at the configuration of one dimension.

Definition. For each n, α, m_0, let $d^*_n = d^*_n(\alpha, m_0) = \min_{\mathcal{M}} I^*$. □

We can regard n_1 dimensional pattern as a subset of n_2 dimensional pattern by the natural inclusion for $n_1 < n_2$. Our goal is to show the following:

Theorem 4.2 *For m_0 sufficiently close to -1, there holds the strict inequality $d_1^* > d_2^*$. That is, two dimensional global minimizer does not have the lamellar structure.*

This theorem follows from Propositions 4.1 and 4.2 below. The proofs of Proposition 4.1 and 4.2 suggests that for $m_0 \approx -1$, there exists a two dimensional configuration of droplets whose energy is less than one dimensional pattern.

Proposition 4.1 *Let $L = diag[l, 1, \ldots, 1], l > 0$ be an diagonal matrix and $(u, L) \in \mathcal{M}$. Assume that u depends only on x_1 (that is, u has one dimensional structure). Then*

$$I_\alpha^*(u) \geq \min_{0 \leq \phi \leq 1/2} \left\{ \frac{2}{\gamma}(\phi - \theta)^2 + \frac{1}{4}\alpha^{\frac{2}{3}} \left(\frac{c_\gamma}{2}\right)^{\frac{1}{3}} (3\phi)^{\frac{2}{3}} \right\},$$

where $\theta := (m_0 + 1)/2$ and $c_\gamma = \frac{(2\pi)^2}{(2\pi)^2+\gamma}$.

Proof. In order to prove Proposition 4.1, it is sufficient to consider the case $n = 1$. Let $u : \mathbb{R} \to \{\pm 1\}$ be one dimensional periodic function with $u(x + l) = u(x)$, $0 < l \leq 1$ a.e. $x \in \mathbb{R}$. Put $\frac{1}{l}\int_0^l u\,dx := 2\phi - 1$. Without loss of generality, we may assume that $\phi \in [0, 1/2]$. Since the second eigenvalue of $-\frac{d^2}{dx^2}$ acting on $H_L^1(0, l)$ is $(\frac{2\pi}{l})^2 \geq (2\pi)^2$, we have

$$J(u, l) \geq \frac{4(\phi - \theta)^2}{\gamma} + \frac{(2\pi)^2}{(2\pi)^2 + \gamma} J_0(u, l).$$

Thus we estimate

$$\begin{aligned} I_\alpha^*(u) &\geq \frac{2}{\gamma}(\phi - \theta)^2 + \frac{\alpha}{2}\Gamma(u, l) + \frac{c_\gamma}{2} J_0(u, l) \\ &= \frac{2}{\gamma}(\phi - \theta)^2 + \frac{\alpha}{2l}\Gamma(u_l, 1) + \frac{c_\gamma l^2}{2} J_0(u_l, 1) \\ &\geq \frac{2}{\gamma}(\phi - \theta)^2 + \frac{3}{2 \cdot 2^{\frac{2}{3}}}\alpha^{\frac{2}{3}}(c_\gamma)^{\frac{1}{3}}\Gamma(u_l, 1)^{\frac{2}{3}} J_0(u_l, 1)^{\frac{1}{3}}, \end{aligned}$$

where $u_l(x) := u(lx)$. Let $0 < x_1 < x_2 < \cdots < x_N < 1$ be the discontinuity points of u_l in the interval $(0, 1)$. Then $\Gamma(u_l, 1) \geq N$. Letting ψ be the solution of

$$\begin{cases} -\psi'' = u_l - (2\phi - 1), \\ \psi(0) = \psi(1), \psi'(0) = \psi'(1), \\ \int_0^1 \psi(x)\,dx = 0, \end{cases}$$

we have

$$J_0(u_l, 1) = \int_0^1 \psi(u_l - (2\phi - 1))\,dx = \int_0^1 (\psi'(x))^2\,dx.$$

To establish the lower bound for $J_0(u_l, 1)$, we show the following:

Lemma 4.1 *Let $u : [0, 1] \to \mathbb{R}$ be a piecewise linear function such that $u'(x)$ is constant on each interval (x_i, x_{i+1}) and takes either ϕ_1 or ϕ_2, where $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ and $\phi_1, \phi_2 \in \mathbb{R}$. Then*

$$\int_0^1 u^2\,dx \geq \frac{1}{12}\min\{\phi_1^2, \phi_2^2\}(N + 1)^{-2}.$$

Proof. Assume that $u'(x) = \phi_1$ for $x_i < x < x_{i+1}$. Then

$$\int_{x_i}^{x_{i+1}} u^2\,dx = \frac{(u(x_i)+u(x_{i+1}))^2}{4}(x_{i+1}-x_i) + \frac{\phi_1^2}{12}(x_{i+1}-x_i)^3.$$

Summing over i, we get

$$\begin{aligned}\int_0^1 u^2\,dx &\geq \frac{1}{12}\min\{\phi_1^2,\phi_2^2\}\sum_{i=0}^{N}(x_{i+1}-x_i)^3\\ &\geq \frac{1}{12(N+1)^2}\min\{\phi_1^2,\phi_2^2\}.\end{aligned}$$

Here in the second inequality we used the Jensen's inequality. □

Applying Lemma 4.1 to ψ', we have

$$J_0(u_l, 1) \geq \frac{\phi^2}{3}(N+1)^{-2},$$

hence

$$I_\alpha^*(u) \geq \frac{2}{\gamma}(\phi-\theta)^2 + \frac{1}{2}\left(\frac{N}{N+1}\right)^{\frac{2}{3}}\alpha^{\frac{2}{3}}(c_\gamma)^{\frac{1}{3}}\left(\frac{3\phi}{2}\right)^{\frac{2}{3}},$$

which proves Proposition 4.1. □

Next we consider the two dimensional pattern.

Proposition 4.2 *There exists a positive constant C_0 such that for any $\phi \in [0,1/2]$, there exist $L \in GL(2,\mathbb{R})$ and a function $\tilde{u}_\phi \in H_L^1(\mathbb{R}^2)$ such that $(u,L) \in \mathcal{M}$ and*

$$\tilde{I}_\alpha^*(\tilde{u}_\phi) \leq \frac{2}{\gamma}(\phi-\theta)^2 + 3\alpha^{\frac{2}{3}}(C_0)^{\frac{1}{3}}\phi|\log\phi|^{\frac{1}{3}}$$

for $\frac{\alpha}{C_0} \leq \phi^{\frac{3}{2}}|\log\phi|$. Here θ is the constant defined in Proposition 4.1.

Proof. For $\phi \in [0,1/2]$, let $u_\phi : \mathbb{R}^2 \to \{\pm 1\}$ be the function such that

$$u_\phi(x_1,x_2) := \begin{cases} 1, & (x_1,x_2) \in (N+\frac{1-\sqrt{\phi}}{2}, N+\frac{1+\sqrt{\phi}}{2})^2 \text{ for some integer } N,\\ -1, & \text{otherwise}\end{cases}$$

It is easily seen that $(u_\phi, id) \in \mathcal{M}$ (id is unit matrix), $\Gamma(u_\phi, id) = 4\sqrt{\phi}$. There holds

$$\int_Q u_\phi\,dx = 2\phi - 1, \quad Q = (0,1)^2.$$

If $\phi = 0$, then $u_0 = const.$ and $\tilde{I}_\alpha^*(u_0) = \frac{4}{\gamma}(\phi-\theta)^2$. Hence we assume $\phi > 0$ and give an upper bound for $J_0(u_\phi, id)$. The function u_ϕ has the following Fourier expansion:

$$\begin{aligned}u_\phi(x_1,x_2) = (2\phi-1) + \sum \frac{2}{nm\pi^2}&\sin(n\pi(1+\sqrt{\phi}))\sin(m\pi(1+\sqrt{\phi}))\\ &\times \cos 2n\pi x_1 \cdot \cos 2m\pi x_2.\end{aligned}$$

Here the summation means summing over

$$\Lambda := \{(n,m) \in \mathbb{Z}^2\,;\, n,m = 0,1,\ldots,\ (n,m) \neq (0,0)\}.$$

Hence we calculate

$$J_0(u_\phi, id) = \sum \frac{1}{4\pi^2(n^2+m^2)} \frac{4\sin^2 n\pi(1+\sqrt{\phi})\sin^2 m\pi(1+\sqrt{\phi})}{n^2m^2\pi^4}$$
$$= \sum \frac{1}{4\pi^2(n^2+m^2)} \frac{4\sin^2(n\pi\sqrt{\phi})\sin^2(m\pi\sqrt{\phi})}{n^2m^2\pi^4}.$$

Let

$$\Lambda_1 = \{(n,m) \in \Lambda;\, n^2 \le \frac{1}{\phi\pi^2}, m^2 \le \frac{1}{\phi\pi^2}\},$$
$$\Lambda_2 = \{(n,m) \in \Lambda;\, n^2 > \frac{1}{\phi\pi^2}, m^2 \le \frac{1}{\phi\pi^2}\},$$
$$\Lambda_3 = \{(n,m) \in \Lambda;\, n^2 \le \frac{1}{\phi\pi^2}, m^2 > \frac{1}{\phi\pi^2}\},$$
$$\Lambda_4 = \{(n,m) \in \Lambda;\, n^2 > \frac{1}{\phi\pi^2}, m^2 > \frac{1}{\phi\pi^2}\}.$$

Then we estimate

$$\sum_{\Lambda_1} \frac{\sin^2(n\pi\sqrt{\phi})\sin^2(m\pi\sqrt{\phi})}{(n^2+m^2)n^2m^2} \le \sum_{\Lambda_1} \frac{(n\pi\sqrt{\phi})^2(m\pi\sqrt{\phi})^2}{(n^2+m^2)n^2m^2}$$
$$\le \sum_{\Lambda_1} \frac{\pi^4\phi^2}{n^2+m^2} \le C\phi^2|\log\phi|,$$

and

$$\sum_{\Lambda_2} \frac{\sin^2(n\pi\sqrt{\phi})\sin^2(m\pi\sqrt{\phi})}{(n^2+m^2)n^2m^2} \le \sum_{\Lambda_2} \frac{(m\pi\sqrt{\phi})^2}{(n^2+m^2)n^2m^2}$$
$$\le \sum_{\Lambda_2} \frac{\pi^2\phi}{2n^3m} \le C\phi^2|\log\phi|.$$

Since there hold the same estimates for summing over Λ_3, Λ_4, there exists a positive constant C_0 such that

$$J_0(u_\phi, id) \le C_0\phi^2|\log\phi|.$$

Finally putting $\tilde{u}_\phi(x) = u_\phi(\frac{x}{R})$, $R = (\frac{\alpha}{C_0|\log\phi|})^{\frac{1}{3}}\phi^{-\frac{1}{2}}$, we have

$$\tilde{I}^*_\alpha(\tilde{u}_\phi) \le \frac{2}{\gamma}(\phi-\theta)^2 + 3\alpha^{\frac{2}{3}}(C_0)^{\frac{1}{3}}\phi|\log\phi|^{\frac{1}{3}}.$$

The proof is complete. □

Finally we consider the limit case $\|L\|_\infty$ is small. Namely, neglecting low frequency mode, we get scaling law of the energy. A universal constant appearing in the limiting problem characterize the principal part of the asymptotic expansion of minimal energy as $\varepsilon \to 0$ and hence the geometrical pattern of the global minimizer of (2.5).

Lemma 4.2 *There exists a positive constant $d > 0$ depending only on n such that*

$$\inf_{\mathcal{A}_{m_0}} (\alpha\Gamma + J_0) = 2\alpha^{\frac{2}{3}} d,$$

where $\mathcal{A}_{m_0} = \{u \in \mathcal{M}_L\,;\, L \in GL(n, \mathbb{R}),\, \frac{1}{|\Omega_L|}\int_{\Omega_L} u\,dx = m_0\}$.

Proof. We note that if $(u, L) \in \mathcal{A}_{m_0}$, then we have $(u_R, RL) \in \mathcal{A}_{m_0}$, where $u_R(x) = u(\frac{x}{R})$, $R > 0$ and there holds

$$\alpha\Gamma(u_R, RL) + J_0(u_R, RL) = \frac{\alpha}{R}\Gamma(u, L) + R^2 J_0(u, L).$$

In particular, for $R = \alpha^{\frac{1}{3}}$,

$$\alpha\Gamma(u_R, RL) + J_0(u_R, RL) = \alpha^{\frac{2}{3}} \tilde{I}_1^*(u, L),$$

hence we obtain

$$\inf_{\mathcal{A}_{m_0}} (\alpha\Gamma + J_0) = \alpha^{\frac{2}{3}} \inf_{\mathcal{A}_{m_0}} (\Gamma + J_0).$$

It remains to show that $\inf_{\mathcal{A}_{m_0}}(\Gamma + J_0) > 0$. Assume by contradiction that $\inf_{\mathcal{A}_{m_0}}(\Gamma + J_0) = 0$. Let (u_k, L_k) be a minimizing sequence. Then

$$0 < \frac{3}{2^{2/3}}\Gamma(u_k, L_k)^{\frac{2}{3}} J_0(u_k, L_k)^{\frac{1}{3}} \le \Gamma(u_k, L_k) + J_0(u_k, L_k) \to 0$$

We claim that there exists a sequence (v_k, L_0) such that

$$\Gamma(v_k, L_0)^{\frac{2}{3}} J_0(v_k, L_0)^{\frac{1}{3}} \to 0 \tag{4.2}$$

Indeed, noting that the quantity $\Gamma^{\frac{2}{3}} J_0^{\frac{1}{3}}$ is invariant under the scaling transformation $u \mapsto u_R(x) := u(\frac{x}{R})$, we may assume that L_k is convergent. Moreover as in the proof of Lemma 4.1, we have $L_0 := \lim_{k\to\infty} L_k \in GL(n, \mathbb{R})$.

Let $0 = \lambda_0 \le \lambda_1 \le \ldots$ be the eigenvalues of $-\Delta$ on $L^2(\Omega_{L_0})$ with periodic boundary condition. Now we apply the following lemma:

Lemma 4.3 *([1], Lemma 2.3) Let $u \in BV(\Omega_{L_0})$, $|u| \le 1$. Then there exists a constant $c_1 > 0$ such that for all positive integers N,*

$$\frac{1}{N}\int_{\Omega_{L_0}} |Du| + \sum_{i=1}^{\infty} \min\left\{1, \frac{N^2}{\lambda_i}\right\} |\hat{u}_i|^2 \ge c_1$$

where $\hat{u}_i$ denotes the i^{th} Fourier coefficient of u.

The proof of the above lemma is very similar to that of [1]. We omit the details. Let N_k be a positive integer such that $\frac{2}{c_1}\Gamma(v_k, L_0) < N_k \le \frac{2}{c_1}\Gamma(v_k, L_0) + 1$. Then we have

$$\frac{1}{N_k}\int_{\Omega_{L_0}} |Dv_k| \le \frac{1}{N_k}\Gamma(v_k, L_0) \le \frac{c_1}{2}$$

$$\sum_{i=1}^{\infty} \min\left\{1, \frac{N_k^2}{\lambda_i}\right\} |\hat{v}_{k,i}|^2 \le N_k^2 J_0(v_k, L_0) \to 0$$

as $k \to \infty$. In the second inequality, we used (4.2). This is a desired contradiction. The proof is complete. $\square$

Observation. Our ansatz of the global minimizer of (3.3) is as follows: global minimizers have the mesoscopic scale fine structure such that the average wave length is of order $\varepsilon^{\frac{1}{3}}$ and the minimial energy is of order $\varepsilon^{\frac{2}{3}}$. More presicely,

$$d_\varepsilon = \varepsilon^{\frac{2}{3}}(\alpha_0^{\frac{2}{3}} d + o(1))|\Omega|, \quad \text{as } \varepsilon \to 0,$$

where $\alpha_0 = \sqrt{2}\int_{-1}^{1}\sqrt{W(u)}\,du$ is the interfacial energy per unit area.

Remark 4-3. The above estimate may also holds in case Neumann Boundary Problem (2.5).

5 Stability of Local Minimizers

Throughout this section, Ω is a bounded domain in $\mathbb{R}^n$ or Ω_L in section 3. Consider the linearized equation about the stationary solution (u, v):

$$\text{(P)} \qquad \begin{cases} D_1\Delta\phi + f'(u)\phi - \kappa\psi &= \lambda\phi, \\ D_2\Delta\psi + \phi - \gamma\psi &= \tau\lambda\psi. \end{cases}$$

In this section, we denote by u, v the solution of the original equation (1.1) or (1.2), by $\bar{I}$ the associated functional

$$\bar{I}(u) := \frac{D_1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx + \frac{\kappa}{2}(Ku, u),$$

where $F(u) := \int_0^u f(v)\,dv$ is a primitive of f and $K = (-D_2\Delta + \gamma)^{-1}$ with Neumann or periodic boundary condition. Note that by the linear transformation of u, the set of minimizers of I maps one to one and onto those of $\bar{I}$ (with the trivial change of parameters). The goal of this section is to show the following stability result:

Proposition 5.1 *Assume that u is a local minimizer of $\bar{I}$ and $\tau\kappa < \gamma^2$. Then the spectrum of (P) lies in the stable region.*

In order to consider the spectrum of the linear operator, we denote by the notation $\langle\cdot,\cdot\rangle$ the complex inner product on the complex extended Hilbert space L^2, that is,

$$\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}\,dx.$$

Let T denote the linear operator on $(L^2(\Omega))^2$ defined by

$$T(\Phi) = (D_1\Delta\phi + f'(u)\phi - \kappa\psi, \tfrac{1}{\tau}(D_2\Delta\psi + \phi - \gamma\psi))$$

for $\Phi = (\phi, \psi)$. Denote by $\Sigma_p, \Sigma_c, \Sigma_r$ the point spectrum, the continuous spectrum, and the residual spectrum of T respectively. Let $\Sigma := \Sigma_p \cup \Sigma_c \cup \Sigma_r$ be the spectrum of (P). Let

$$N = \{\zeta \in \mathbb{C};\, \mathrm{Re}\,(\zeta) > \max\{-\tfrac{\gamma}{\tau}, f'(0)\}\}$$

Suppose $\lambda \in N$ is an eigenvalue of (P) with eigenfunction (ϕ, ψ). Since $\psi = (-D_2\Delta + (\gamma + \tau\lambda))^{-1}\phi$, we have $\mathcal{L}_\lambda\phi = \lambda\phi$, where

$$\mathcal{L}_\lambda = D_1\Delta + f'(u) - \kappa(-D_2\Delta + (\gamma + \tau\lambda))^{-1}.$$

Lemma 5.1 *(1) If* $\lambda \in \Sigma \cap N$*, then there holds either*
(a) λ *is an eigenvalue of* $\mathcal{L}_\lambda$ *or*
(b) $\overline{\lambda}$ *is an eigenvalue of* $\mathcal{L}_{\overline{\lambda}}$.
(2) Assume $\tau\kappa < \gamma^2$*. Then* $\Sigma \cap \{\zeta \in \mathbb{C};\, Re(\zeta) > \max\{-\frac{\gamma-\sqrt{\tau\kappa}}{\tau}, f'(0)\}\}$ *consists of real numbers.*

Proof. (1) First, if $\lambda \in \Sigma_p \cup \Sigma_c$, then we have sequences $\phi_k, \psi_k, f_k, g_k \in L^2(\Omega)$ such that

$$\begin{aligned} D_1\Delta\phi_k + f'(u)\phi_k - \kappa\psi_k &= \lambda\phi_k + f_k \\ D_2\Delta\psi_k + \phi_k - \gamma\psi_k &= \tau\lambda\psi_k + g_k \\ \|\phi_k\|_{L^2} + \|\psi_k\|_{L^2} &= 1 \\ \|f_k\|_{L^2}, \|g_k\|_{L^2} \to 0, &\quad k \to \infty \end{aligned}$$

Since $\psi_k = (-\Delta + \gamma + \tau\lambda)^{-1}(\phi_k - g_k)$, it follows that $\|\phi_k\|_{L^2}$ is bounded away from 0, that is,

$$\liminf_{k\to\infty} \|\phi_k\|_{L^2} > 0$$

Indeed,

$$\begin{aligned} 1 &\le \|\phi_k\|_{L^2} + C\|\phi_k - g_k\|_{L^2} \\ &\le C\|\phi_k\|_{L^2} + o(1), \end{aligned}$$

where $o(1) \to 0$ as $k \to 0$. On the other hand, we have

$$\mathcal{L}_\lambda\phi_k - \lambda\phi_k = f_k - \kappa(-D_2\Delta + \gamma + \tau\lambda)^{-1}g_k \to 0 \tag{5.1}$$

as $k \to \infty$. Since

$$D_1\Delta\phi_k + f'(0)\phi_k$$

is bounded in $L^2(\Omega)$, we may assume that $\phi_k \rightharpoonup \phi \in H^1(\Omega)$, weakly in $H^1(\Omega)$, $\phi_k \to \phi$ strongly in $L^2_{\mathrm{loc}}(\Omega)$, and

$$g'(u)\phi_k \to z_0 := g'(u)\phi$$

where $g(u) := f(u) - f'(0)u$. From (5.1), it follows that $\mathcal{B}_\lambda\phi_k - \lambda\phi_k \to -z_0$, where

$$\mathcal{B}_\lambda := D_1\Delta + f'(0) - \kappa(-D_2\Delta + \gamma + \tau\lambda)^{-1}.$$

Since $\lambda \notin \sigma(\mathcal{B}_\lambda)$ by $\mathrm{Re}\,\lambda > f'(0)$, we see that $\phi_k \to -(\mathcal{B}_\lambda - \lambda)^{-1}z_0$. Therefore

$$\mathcal{B}_\lambda\phi - \lambda\phi = -z_0 = -g'(u)\phi$$

Hence λ is an eigenvalue of $\mathcal{L}_\lambda$ with an eigenfunction ϕ.
Next if $\lambda \in \Sigma_r$, then we have

$$\begin{gathered} (D_1\Delta\phi + f'(u)\phi - \kappa\psi - \lambda\phi, \tilde{\phi}) + (D_2\Delta\psi + \phi - \gamma\psi - \tau\lambda\psi, \tilde{\psi}) = 0 \\ \|\tilde{\phi}\|_{L^2} + \|\tilde{\psi}\|_{L^2} = 1 \end{gathered}$$

$$(\phi, D_1\Delta\tilde{\phi} + f'(u)\tilde{\phi} - \overline{\lambda}\tilde{\phi} + \tilde{\psi}) + (\psi, D_2\Delta\tilde{\psi} - \kappa\tilde{\phi} - \gamma\tilde{\psi} - \tau\overline{\lambda}\tilde{\psi}) = 0$$

This implies that $\tilde{\phi}, \tilde{\psi} \in H^2(\Omega)$ and $\tilde{\phi}, \tilde{\psi}$ solves the system of linear equations:

$$\begin{aligned} D_1\Delta\tilde{\phi} + f'(u)\tilde{\phi} - \overline{\lambda}\tilde{\phi} + \tilde{\psi} &= 0, \\ D_2\Delta\tilde{\psi} - \kappa\tilde{\phi} - \gamma\tilde{\psi} - \tau\overline{\lambda}\tilde{\psi} &= 0. \end{aligned}$$

Using the inverse operator, we summarize the equation into $\mathcal{L}_{\overline{\lambda}}\tilde{\phi} = \overline{\lambda}\tilde{\phi}$. Since $\|\tilde{\phi}\|_{L^2} \neq 0$, this means that $\overline{\lambda}$ is an eigenvalue of $\mathcal{L}_{\overline{\lambda}}$ with an eigenfunction $\tilde{\phi}$.

(2) It suffices to show that if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{L}_\lambda$ with $\operatorname{Re}\lambda > \max\{-\frac{\gamma-\sqrt{\tau\kappa}}{\tau}, f'(0)\}$, then λ is real. Let $\lambda = x + iy$ be an eigenvalue of $\mathcal{L}_\lambda$ such that $x > \max\{-\frac{\gamma-\sqrt{\tau\kappa}}{\tau}, f'(0)\}$, $y \in \mathbb{R}$, and let ϕ be an eigenfunction corresponding to λ satisfying $\|\phi\|_{L^2} = 1$. Then there holds

$$\begin{aligned}
|y| &= |\operatorname{Im}\langle\mathcal{L}_\lambda\phi, \phi\rangle| \\
&= \kappa\left|\operatorname{Im}\int_0^\infty \frac{1}{D_2\xi + \gamma + \tau\lambda}\, d(E(\xi)\phi, \phi)\right| \\
&= \kappa\int_0^\infty \frac{\tau|y|}{(D_2\xi + \gamma + \tau x)^2 + (\tau y)^2}\, d(E(\xi)\phi, \phi) \\
&\leq \kappa\int_0^\infty \frac{\tau|y|}{(\gamma + \tau x)^2}\, d(E(\xi)\phi, \phi) \\
&\leq \frac{\tau\kappa}{(\gamma + \tau x)^2}|y|
\end{aligned}$$

As $(\gamma + \tau x)^2 > \tau\kappa$, we have $y = 0$. The proof is complete. □

Now we consider $\mathcal{L}_\lambda$ for $\lambda \in I := (-\frac{\gamma}{\tau}, \infty)$. In this case, we see that $\mathcal{L}_\lambda$ is a self-adjoint operator. Moreover, for any $\lambda \in I$, there holds

$$\langle\mathcal{L}_\lambda\phi, \phi\rangle \leq \sup_{x\in\Omega} f'(u(x)) := d$$

for all $\phi \in H^1(\Omega)$ with $\|\phi\|_{L^2} = 1$. It follows that the spectrum of $\mathcal{L}_\lambda$ lies in

$$\{\zeta \in \mathbb{R};\, \zeta \leq d\}$$

with a uniform constant d. By applying the perturbation theory of linear operators, we can easily deduce that $\sigma(\mathcal{L}_\lambda) \cap N$ consists of eigenvalues with finite muliplicities. (See [4, Theorem 5.35].) Denote by $h(\lambda)$ the maximum value of $\sigma(\mathcal{L}_\lambda)$ which is characterized as

$$h(\lambda) = \max_{\|\phi\|_{L^2}=1} \langle\mathcal{L}_\lambda\phi, \phi\rangle \leq d. \tag{5.2}$$

Lemma 5.2 *(1) If $\lambda \in I$ is an eigenvalue of $\mathcal{L}_\lambda$, then $\lambda \leq h(\lambda)$ and λ is an eigenvalue of (P).*
(2) If $\lambda \in I$, $\lambda = h(\lambda)$, then λ is an eigenvalue of (P).

Proof. The first statement of (1) is an easy consequence of the definition of $h(\lambda)$. The second statement of (1) is trivial. (2) follows from the variational characterization of eigenvalues. We omit the details. □

Let $E(\xi)$ be spectral resolution associated to $-\Delta$ with Neumann boundary condition. Then there holds

$$\langle(-D_2\Delta + \gamma + \tau\lambda)^{-1}\phi, \phi\rangle = \int_0^\infty \frac{1}{D_2\xi + \gamma + \tau\lambda}\, d(E(\xi)\phi, \phi). \tag{5.3}$$

Lemma 5.3 *(1) $h(0) \leq 0$.*
(2) The function $h : I \to \mathbb{R}$ is non-decreasing and locally Lipshitz continuous. Furthermore the local Lipshitz constant of h at $\lambda \in I$ is less than or equal to $\frac{\kappa\tau}{(\gamma+\tau\lambda)^2}$.

Proof. (1) Since u is a local minimizer of I,

$$\begin{aligned}\langle \mathcal{L}_0\phi, \phi\rangle &= -D_1\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega f'(u)\phi^2\,dx - \kappa\langle(-D_2\Delta+\gamma)^{-1}\phi, \phi\rangle \\ &= -I''(u)(\phi,\phi) \le 0\end{aligned}$$

for any $\phi \in H^1(\Omega)$. Hence we have $h(0) \le 0$.
(2) For $\lambda_1 < \lambda_2$, there holds

$$\begin{aligned}\langle \mathcal{L}_{\lambda_1}\phi, \phi\rangle &= -D_1\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega f'(u)\phi^2 - \kappa\int_0^\infty \frac{1}{D_2\xi+\gamma+\tau\lambda_1}\,d(E(\xi)\phi,\phi) \\ &\le -D_1\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega f'(u)\phi^2 - \kappa\int_0^\infty \frac{1}{D_2\xi+\gamma+\tau\lambda_2}\,d(E(\xi)\phi,\phi) \\ &\le h(\lambda_2)\end{aligned}$$

for any ϕ with $\|\phi\|_{L^2} = 1$. Hence we have $h(\lambda_1) \le h(\lambda_2)$. Now if ϕ_2 is an eigenfunction with $\|\phi_2\|_{L^2} = 1$ corresponding to $h(\lambda_2)$, we have

$$\begin{aligned}h(\lambda_2) &= -D_1\int_\Omega |\nabla\phi_2|^2\,dx + \int_\Omega f'(u)\phi_2^2 - \kappa\int_0^\infty \frac{1}{D_2\xi+\gamma+\tau\lambda_2}\,d(E(\xi)\phi_2,\phi_2) \\ h(\lambda_1) &\ge -D_1\int_\Omega |\nabla\phi_2|^2\,dx + \int_\Omega f'(u)\phi_2^2 - \kappa\int_0^\infty \frac{1}{D_2\xi+\gamma+\tau\lambda_1}\,d(E(\xi)\phi_2,\phi_2)\end{aligned}$$

Hence for some $\lambda \in (\lambda_1, \lambda_2)$,

$$\begin{aligned}h(\lambda_2) - h(\lambda_1) &\le \kappa\int_0^\infty \left(\frac{1}{D_2\xi+\gamma+\tau\lambda_1} - \frac{1}{D_2\xi+\gamma+\tau\lambda_2}\right) d(E(\xi)\phi_2,\phi_2) \\ &= \kappa\int_0^\infty \frac{\tau(\lambda_2-\lambda_1)}{(D_2\xi+\gamma+\tau\lambda)^2}\,d(E(\xi)\phi_2,\phi_2) \\ &\le \frac{\kappa\tau}{(\gamma+\tau\lambda)^2}(\lambda_2-\lambda_1)\end{aligned}$$

Therefore, h is locally Lipshitz continuous. □

Lemma 5.4 *Assume $\tau\kappa < \gamma^2$. Then there holds $h(\lambda) < \lambda$ for all $\lambda > 0$.*

Proof. From Lemma 5.3, we have for $\lambda > 0$,

$$h(\lambda) \le h(0) + \frac{\kappa\tau}{\gamma^2}\lambda \le \frac{\kappa\tau}{\gamma^2}\lambda < \lambda$$

as desired. □

Lemma 5.5 *There holds $\Sigma \subset \{0\} \cup \{\zeta\,;\,\mathrm{Re}\,\zeta \le \alpha\}$ for some $\alpha < 0$. If, in addition, $h(0) < 0$, then $\Sigma \subset \{\zeta\,;\,\mathrm{Re}\,\zeta \le \alpha\}$.*

Proof. Let $\beta = \max\{-\frac{\gamma-\sqrt{\tau\kappa}}{\tau}, f'(0)\} < 0$. If $\lambda \in \Sigma \cap \{\mathrm{Re}\,\zeta > \beta\}$, then from Lemma 5.1, λ is real and hence an eigenvalue of $\mathcal{L}_\lambda$. By Lemma 5.2 (1) and 5.4, we have $\lambda \le 0$. Moreover, from Lemma 5.2 (1), λ is an eigenvalue of (P) with finite multiplicity. Since $T - \zeta$ is semi-Fredholm operator for $\mathrm{Re}\,(\zeta) > \beta$, The number $\lambda \in (\beta, 0]$ is not in the essential spectrum of T and hence is isolated in Σ. Therefore noting that$\{\zeta\,;\,\mathrm{Re}\,\zeta > \beta\}\backslash(\beta,0] \subset \rho(T)$, there exist finite numbers $0 \ge \lambda_1 \ge \ldots \lambda_m > \beta$ which are eigenvalues of T with finite algebraic multiplicity and $\Sigma \subset \{\zeta\,;\,\mathrm{Re}\,\zeta \le \beta\} \cup \{\lambda_1, \ldots, \lambda_m\}$. If, in addition, $h(0) < 0$, then we have $\lambda_1 < 0$. The proof is

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