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The generalized van der Corput sequence and its application to numerical integration*

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Abstract. A new class of s-dimensional uniformly distributed sequences called the generalized van der Corput sequence is defined. The sequence is constructed by using the generalized number system based on an integer matrix whose all eigenvalues reside out of the unit circle. In this talk, we show that by using the generalized van der Corput sequence we can calculate numerical integrations with the convergence speed $O(1/N)$ when integrands satisfy some regularity conditions. We also apply the sequence to a numerical integration problem and test effectiveness of the sequence.

1 Introduction

We can consider the van der Corput seqeunce to be an orbit of the origin under the adding machine transformation that is accompanied by an expanding one dimensional linear transformation [2,6–8]. Following this principle of regarding the van der Corput sequence as an orbit of the adding machine transformation, we can generalize the van der Corput sequence in various directions [1,4,6,7]. In this paper, replacing expanding one dimensional linear transformation with expanding s-dimensional linear transformation, we obtain another generalization of the van der Corput sequence. We call this sequence the generalized van der Corput sequence. We also give a theorem. According to the theorem, the required time for numerical integration of a function over s-dimensional unit cube is reduced to $O(1/\varepsilon)$ where ε denotes the accuracy, if the integrand is smooth enough. In the last of the paper, we give a numerical example.

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2 The generalized van der Corput sequence

First, we introduce the generalized number system based on an expanding integer matrix.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ are sets of all natural numbers, all integers, all real numbers, and all complex numbers, respectively. We define $\mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s$. For a real valued function f defined on $\mathbb{T}^s$, $\hat{f}(k)$ denotes the Fourier coefficients of f, that is to say

$$\hat{f}(k) = \int_{\mathbb{T}^s} f(x) e^{-2\pi\sqrt{-1}\langle x,k\rangle} dx$$

for $k \in \mathbb{Z}^s$. $M(s;\mathbb{Z})$ denotes the set of all (s,s) integer matrices and so on.

Let $A \in M(s;\mathbb{Z})$ whose all eigenvalues reside outside of the unit circle and $N = |\det A|$. Then there exist $P, Q \in GL(s;\mathbb{Z})$ and $u_1, \ldots, u_s \in \mathbb{N}$ which satisfy

$$QAP = \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_s \end{pmatrix} \tag{1}$$
$$\text{and} \quad u_i | u_{i+1} \quad \text{for } i = 1, \ldots, s-1.$$

$(u_1, \ldots, u_s)$ are elementary divisors of A. We define sets D and D' as follows:

$$\begin{aligned} D &:= Q^{-1}\left\{ {}^t(x_1, \ldots, x_s) \in \mathbb{Z}^s \,\middle|\, x_i \in \{0, \ldots, u_i - 1\}\right\} \\ D' &:= {}^tP^{-1}\left\{ {}^t(x_1, \ldots, x_s) \in \mathbb{Z}^s \,\middle|\, x_i \in \{0, \ldots, u_i - 1\}\right\}. \end{aligned} \tag{2}$$

Following relations:

$$\begin{aligned} &\mathbb{Z}^s/A\mathbb{Z}^s = \bigoplus_{i=1}^{s} \mathbb{Z}/u_i\mathbb{Z} \\ &\mathbb{Z}^s/A\mathbb{Z}^s = D, \quad \mathbb{Z}^s/{}^tA\mathbb{Z}^s = D' \\ &\sharp D = \sharp D' = \prod_{i=1}^{s} u_i = N \end{aligned} \tag{3}$$

hold immediately from above definitions. We also define

$$K := \left\{ y \,\middle|\, y = \sum_{n=1}^{\infty} A^{-n} d_{i_n}, \quad d_{i_n} \in D \right\}. \tag{4}$$

Definition 1. The quadruplet (A, D, D', K) is called the A-digit expansion. D is called the digit set of the A-digit expansion and D' its dual.

For these A, D, and N we define the generalized van der Corput sequence as follows.

Definition 2. We define $x_n \in \mathbb{R}^s$ by

$$x_n = \sum_{k=1}^{l(n)} A^{-k} d_{i_k}, \quad d_{i_k} \in D$$

$$\text{where} \quad n = \sum_{k=1}^{l(n)} N^{k-1} i_k, \quad i_k \in \{0, 1, \ldots, N-1\}.$$

The sequence $\{x_n\}_{n=0}^{\infty} \subset \mathbb{R}^s$ is called the generalized van der Corput sequence with respect to A.

When $s = 1$, this sequence becomes the van der Corput sequence [5].

We introduce two lemmas.

Lemma 1. *For any* $d' \in D' \setminus \{0\}$,

$$\sum_{d \in D} \exp\left(2\pi\sqrt{-1}\,\langle d', A^{-1}d\rangle\right) = 0.$$

Proof. There exists ${}^t(x_1, \ldots, x_s) \neq 0$ which satisfy

$$d' = {}^tP^{-1} \sum_{i=1}^{s} x_i \mathbf{e}_i$$

and $x_i \in \{0, \ldots, u_i - 1\}$ for $1 \leq i \leq s$, where $\mathbf{e}_i$ denotes the i-th unit vector in $\mathbb{Z}^s$. Then from (1) and (2),

$$\begin{aligned}
&\sum_{d \in D} \exp\left(2\pi\sqrt{-1}\,\langle d', A^{-1}d\rangle\right) \\
&= \sum_{\substack{y_i \in \{0, \ldots, u_i - 1\} \\ 1 \leq i \leq s}} \exp\left(2\pi\sqrt{-1}\,\langle {}^tP^{-1}(x_1\mathbf{e}_1 + \cdots + x_s\mathbf{e}_s), A^{-1}Q^{-1}(y_1\mathbf{e}_1 + \cdots + y_s\mathbf{e}_s)\rangle\right) \\
&= \sum_{\substack{y_i \in \{0, \ldots, u_i - 1\} \\ 1 \leq i \leq s}} \exp\left(2\pi\sqrt{-1}(x_1\mathbf{e}_1 + \cdots + x_s\mathbf{e}_s)P^{-1}A^{-1}Q^{-1}(y_1\mathbf{e}_1 + \cdots + y_s\mathbf{e}_s)\right) \\
&= \sum_{\substack{y_i \in \{0, \ldots, u_i - 1\} \\ 1 \leq i \leq s}} \exp\left(2\pi\sqrt{-1}\left(\frac{x_1 y_1}{u_1} + \cdots + \frac{x_s y_s}{u_s}\right)\right) \\
&= \prod_{i=1}^{s} \left(\sum_{\substack{y_i \in \{0, \ldots, u_i - 1\} \\ 1 \leq i \leq s}} \exp\left(2\pi\sqrt{-1}\frac{x_i y_i}{u_i}\right) \right) = 0.
\end{aligned}$$

Let $\{x_n\}_{n=0}^{\infty}$ be the s-dimensional generalized van der Corput sequence with respect to A. We define $L_i^A = {}^tA^i\mathbb{Z}^s$ for non-negative integer i. We have the following decomposition of $\mathbb{Z}^s$:

$$\mathbb{Z}^s = \bigsqcup_{i=0}^{\infty} \left(L_i^A - L_{i+1}^A\right).$$

It is easy to see the following lemma holds.

Lemma 2. *Let $k \in L_i^A - L_{i+1}^A$, $M \in \mathbb{N}$, and j be an integer which satisfies $N^j \leq M < N^{j+1}$. Then,*

$$\left|\frac{1}{M}\sum_{n=1}^{M} \exp\left(2\pi\sqrt{-1}\langle k, x_n\rangle\right)\right| = \begin{cases} 1 & \text{if } i > j, \\ N^i/M & \text{if } i \leq j. \end{cases}$$

The following theorem holds.

Theorem 1. *Following statements 1.–5. hold. Here μ_s denotes the Lebesgue measure of $\mathbb{R}^s$.*

1. *K is compact in $\mathbb{R}^s$.*
2. *$\mathbb{R}^s = \bigcup_{z\in\mathbb{Z}^s}(K + z)$.*
3. *For any $z, z' \in \mathbb{Z}^s$, if $z \neq z'$, then $\mu_s\left((K + z) \cap (K + z')\right) = 0$.*
4. *$AK = \bigcup_{i=1}^{N}(K + d_i)$.*
5. *$\mu_s(K) > 0$.*

Proof. 1.:
From the condition of eigenvalues of A, there exists a real number $\lambda > 1$ which satisfies

$$\left|A^{-1}x\right| \leq \frac{1}{\lambda}|x|$$

for any $x \in \mathbb{R}^s$. Let $(d_{i_n})_{n\in\mathbb{N}}$ be sequence of elements of D and $B = \max_{d\in D}|d|$, then

$$\left|\sum_{n=1}^{\infty} A^{-1}d_{i_n}\right| \leq \frac{B}{\lambda - 1}$$

and K is bounded. Define

$$K_m = \left\{y \in \mathbb{R}^s \,\middle|\, y = \sum_{n=1}^{m} A^{-n}d_{i_n}, \quad d_{i_n} \in D\right\},$$

and we have the following inequality:

$$d_H\left(K_m, K\right)) = d_H\left(K_m, K_m + A^{-m}K\right) \leq \left|A^{-m}K\right| \leq \lambda^{-m}\frac{B}{\lambda - 1},$$

where d_H denotes the Hausdorff distance. We define $\mathcal{K}(B/(\lambda-1))$ as follows:

$$\mathcal{K}\left(\frac{B}{\lambda-1}\right)=\left\{K\subset\mathbb{R}^s\,\middle|\,K\text{ is compact, }K\subset U\left(0,\frac{B}{\lambda-1}\right)\right\},$$

where $U(p,r)$ denote the ball in $\mathbb{R}^s$ with center p and radius r. For $\mathcal{K}(B/(\lambda-1))$ is compact with respect to d_H and $d_H(K_m,k)\to 0$ as $m\to\infty$, K is compact.
4.:
This is trivial from the following decomposition:

$$\begin{aligned}&AK\\&=\left(d_0+A\sum_{n=2}^{\infty}A^{-n}d_{i_n}\right)\bigsqcup\left(d_1+A\sum_{n=2}^{\infty}A^{-n}d_{i_n}\right)\bigsqcup\cdots\bigsqcup\left(d_N+A\sum_{n=2}^{\infty}A^{-n}d_{i_n}\right)\\&=\bigsqcup_{i=1}^{N}(K+d_i).\end{aligned}$$

5.:
We define $\mu^{(m)}$ as follows:

$$\mu^{(m)}=\sum_{y\in K_m}\frac{1}{N_m}\delta_y,\quad\text{where}\quad\delta_y(A)=\begin{cases}1&\text{if }y\in A,\\0&\text{otherwise.}\end{cases}$$

For $z \in \mathbb{Z}^s \setminus \{0\}$, let m and m' be integers and $d' \in \mathbb{Z}^s$ that satisfy $0 \leq m' < m$, $z \in L^A_{m'} \setminus L^A_{m'+1}$ and $z = {}^t A^{m'} d'$. The following equation:

$$
\begin{aligned}
\widehat{\mu^{(m)}}(z) &= \int_{\mathbb{R}^s} e^{2\pi\sqrt{-1}\langle z, x\rangle}\, d\mu^{(m)}(x) \\
&= \frac{1}{N^m} \sum_{y \in K_m} e^{2\pi\sqrt{-1}\langle z, y\rangle} \\
&= \frac{1}{N^m} \sum_{d_{i_n} \in D} \exp\left(2\pi\sqrt{-1}\left\langle z, \left(\sum_{n=1}^{m} A^{-n} d_{i_n}\right)\right\rangle\right) \\
&= \frac{1}{N^m} \prod_{n=1}^{m} \left(\sum_{y \in D} \exp\left(2\pi\sqrt{-1}\langle z, A^{-n} d\rangle\right)\right) \\
&= \frac{1}{N^m} \prod_{n=1}^{m} \sum_{y \in D} \exp\left(2\pi\sqrt{-1}\left\langle {}^t A^{m'} d', A^{-n} d\right\rangle\right) \\
&= \frac{1}{N^m} \sum_{y \in D} \exp\left(2\pi\sqrt{-1}\langle d', A^{-1} d\rangle\right) \prod_{\substack{n=1 \\ n \neq m'+1}} \exp\left(2\pi\sqrt{-1}\left\langle d', A^{-(n-m')} d\right\rangle\right) \\
&= 0
\end{aligned}
$$

holds from Lemma 1. Then,

$$
\lim_{m \to \infty} \widehat{\mu^{(m)}}(z) = \begin{cases} 1 & \text{if } z = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5}
$$

Let π be a canonical projection $\mathbb{R}^s \to \mathbb{T}^s$ and $\overline{\mu^{(m)}}$ be a measure of $\mathbb{T}^s$ defined as follows:

$$
\overline{\mu^{(m)}}(A) = \sum_{z \in \mathbb{Z}^s} \mu^{(m)}(A + z).
$$

Let μ be the s-dimensional Lebesgue measure of $\mathbb{T}^s$, then

$$
\hat{\mu}(z) = \begin{cases} 1, & \text{if } z = 0 \\ 0, & \text{otherwise.} \end{cases}
$$

From this and (5), we see that

$$
\lim_{m \to \infty} \widehat{\mu^{(m)}}(z) = \hat{\mu}(z)
$$

for any $z \in \mathbb{T}^s$, that is to say, $\overline{\mu^{(m)}}$ weakly converges to μ. For any m,

$$\begin{aligned}\overline{\mu^{(m)}}(\pi(K)) &= \sum_{z\in\mathbf{Z}^s} \mu^{(m)}(\pi(K)+z)\\ &\geq \sum_{z\in\mathbf{Z}^s} \mu^{(m)}(\pi(K_m)+z)\\ &= \mu^{(m)}(K_m) = 1.\end{aligned}$$

From this inequality and the fact that $\pi(K)$ is compact, the following inequality:

$$1 \leq \limsup_{m\to\infty} \mu^{(m)}(\pi(K)) \leq \mu(\pi(K)) \leq 1$$

holds. Then $\mu_s(K) > 0$.
2.:
From the preceding result and the "Interior Theorem" [3], $\mathring{K} \neq \emptyset$, where $\mathring{K}$ denotes the interior of K. Then,

$$\begin{aligned}\mathbb{R}^s &= \bigsqcup_{n=0}^{\infty} A^n K\\ &= \bigsqcup_{0\leq j} (K + A^j d_{i_j} + A^{j-1} d_{i_{j-1}} + \cdots + d_1)\\ &= \bigsqcup_{z\in\mathbf{Z}^s} (K+z).\end{aligned}$$

3.:
From the following inequality:

$$\begin{aligned}|\det A|\, \mu_s(K) &= \mu_s(AK)\\ &= \mu_s\left(\bigsqcup_{i=1}^{N} (K+d_i)\right)\\ &\leq \mu_s(K+d_0) + \cdots + \mu_s(K+d_N)\\ &= N\mu_s(K)\end{aligned}$$

and the preceding result,

$$\mu_s((K+d_i)\cap(K+d_j)) = 0$$

for any $i \neq j$. □

3 Application to numerical integration problems

In this section, we show that when we use the generalized van der Corput sequence to calculate numerical integrations of functions which satisfy some regularity conditions, the elapsed time for calculation is proportional to $1/\varepsilon$ where ε denotes the approximation error. Let f be a real valued function defined on $\mathbb{T}^s$. The inversion formula:

$$f(x) = \sum_{k\in\mathbb{Z}^s} \hat{f}(k)e^{2\pi\sqrt{-1}\langle x,k\rangle} \tag{6}$$

holds.

Theorem 2. *Let $\lambda_i, i = 1,\ldots,s$ be eigenvalues of A. Define*

$$\lambda = \min\{|\lambda_i| \mid i = 1,\ldots,s\},$$
$$a = \frac{\log N}{\log \lambda}.$$

If f satisfies the following regularity condition:

$$\sum_{k\in\mathbb{Z}^s} |k|^a \left|\hat{f}(k)\right| < \infty \tag{7}$$

then there exists a positive constant C which satisfies

$$\left|\int_{\mathbb{T}^s} f(x)dx - \frac{1}{M}\sum_{n=1}^{M} f(x_n)\right| < \frac{C}{M}$$

for any $M \in \mathbb{N}$.

Proof. From the inversion formula (6),

$$\begin{aligned}
&\left|\int_{\mathbb{T}^s} f(x)dx - \frac{1}{M}\sum_{n=1}^{M} f(x_n)\right| \\
&= \left|\hat{f}(0) - \frac{1}{M}\sum_{n=1}^{M}\left(\sum_{k\in\mathbb{Z}^s}\hat{f}(k)e^{2\pi\sqrt{-1}\langle k,x_n\rangle}\right)\right| \\
&= \left|\sum_{\substack{k\in\mathbb{Z}^s\\k\neq 0}}\hat{f}(k)\frac{1}{M}\sum_{n=1}^{M}e^{2\pi\sqrt{-1}\langle k,x_n\rangle}\right| \\
&\leq \sum_{\substack{k\in\mathbb{Z}^s\\k\neq 0}}\left|\hat{f}(k)\right|\left|\frac{1}{M}\sum_{n=1}^{M}e^{2\pi\sqrt{-1}\langle k,x_n\rangle}\right|.
\end{aligned} \tag{8}$$

Let $j(M)$ be an integer which satisfies $N^{j(M)} \leq M < N^{j(M)+1}$, then from Lemma 2

$$
\begin{aligned}
&\sum_{\substack{k \in \mathbf{Z}^s \\ k \neq 0}} \left|\hat{f}(k)\right| \left| \frac{1}{M} \sum_{n=1}^{M} e^{2\pi\sqrt{-1}\langle k, x_n \rangle} \right| \\
&= \sum_{v=0}^{\infty} \sum_{k \in L_v^A - L_{v+1}^A} \left|\hat{f}(k)\right| \left| \frac{1}{M} \sum_{n=1}^{M} e^{2\pi\sqrt{-1}\langle k, x_n \rangle} \right| \\
&= \sum_{j(M) \leq v} \sum_{k \in L_v^A - L_{v+1}^A} \left|\hat{f}(k)\right| \left| \frac{1}{M} \sum_{n=1}^{M} e^{2\pi\sqrt{-1}\langle k, x_n \rangle} \right| \\
&\quad + \sum_{0 \leq v \leq j(M)-1} \sum_{k \in L_v^A - L_{v+1}^A} \left|\hat{f}(k)\right| \left| \frac{1}{M} \sum_{n=1}^{M} e^{2\pi\sqrt{-1}\langle k, x_n \rangle} \right| \\
&\leq \sum_{j(M) \leq v} \sum_{k \in L_v^A - L_{v+1}^A} \left|\hat{f}(k)\right| + \sum_{0 \leq v \leq j(M)-1} \sum_{k \in L_v^A - L_{v+1}^A} \left|\hat{f}(k)\right| \frac{N^{v+1}}{M}.
\end{aligned} \tag{9}
$$

From the definition of a and the assumption that $|\lambda_i| > 1 \quad (i \in \{1, \ldots, s\})$, for any $k \in L_v^A - L_{v+1}^A$ there exists $h \in \{1, \ldots, s\}$ and

$$|k| = \left|{}^tA^v k'\right| \geq |\lambda_h|^v \geq N^{v/a}. \tag{10}$$

Then, from the regularity condition (7) and (10),

$$
\begin{aligned}
\limsup_{M \to \infty} \sum_{v=0}^{j(M)} \left|\hat{f}(k)\right| N^{v+1} &\leq \limsup_{M \to \infty} \sum_{v=0}^{j(M)} \sum_{k \in L_v^A - L_{v+1}^A} \left|\hat{f}(k)\right| |k|^a \\
&\leq \sum_{k \in \mathbf{Z}^s} \left|\hat{f}(k)\right| |k|^a < \infty.
\end{aligned} \tag{11}
$$

The following inequality:

$$\left| \int_{\mathbf{T}^s} f(x)dx - \frac{1}{M} \sum_{n=1}^{M} f(x_n) \right| \leq \sum_{k \in L_{j(M)}^A} \left|\hat{f}(k)\right| + \frac{1}{M} \sum_{k \in \mathbf{Z}^s} \left|\hat{f}(k)\right| |k|^a \tag{12}$$

holds from inequalities (8), (9), and (11). The regularity condition (7) means that the second term of the right hand side of (12) is $O(1/M)$. For the first term, we have the following estimation (13) again by virtue of (7).

$$
\begin{aligned}
\sum_{k \in L_{j(M)}^A} \left|\hat{f}(k)\right| &\leq \sum_{|k| \geq \lambda^{j(M)}} \left|\hat{f}(k)\right| \\
&\leq \sum_{|k| \geq \lambda^{j(M)}} |k|^a \left|\hat{f}(k)\right| \lambda^{-j(M)a} = O\left(\frac{1}{M}\right)
\end{aligned} \tag{13}
$$

Inequalities (12) and (13) complete the proof. □

4 Numerical example

We apply the generalized van der Corput sequence for calculating the numerical integration of the following function f which is defined on $\mathbb{T}^{10}$:

$$
\begin{aligned}
f(z) &= \frac{1}{1+\sum_{i=1}^{10} a_i(z_i - z_i^0)^2} \\
(a_i)_{i=1}^{10} &= (3.51540, 1.92331, 1.83665, 2.58459, 2.55934, \\
&\qquad 1.99071, 2.93146, 3.83957, 0.964710, 2.50068), \\
(z_i^0)_{i=1}^{10} &= (0.397903, 0.262837, 0.472738, 0.292722, 0.478440, \\
&\qquad 0.274949, 0.149833, 0.272246, 0.491894, 0.328846) \in \mathbb{T}^{10}
\end{aligned}
\tag{14}
$$

We take A to be a 10-dim companion matrix, that is,

$$
A = \begin{pmatrix}
0\,0\,0 & \dots & -11 \\
1\,0\,0 & \dots & 10 \\
0\,1\,0 & \dots & -9 \\
\dots & \dots & \dots \\
0\,0\,0 & \dots 1 & 2
\end{pmatrix}.
$$

This is an expanding matrix. We calculate the numerical integration by using 25 random number sequences and the generalized van der Corput seqeunce with respect to A. The result is displayed in Figure 1. In the figure, approximation errors resulted by using these sequences are plotted. For random number sequences, we calculate the σ of 25 sequences and plot σ, 2σ, and 3σ. Figure 1 shows that:

1. the approximation error of the generalized van der Corput sequence with respect to A converges at the speed of $O(1/M)$ where M is the number of sample points;

2. those of random number sequences at the speed of $O(1/\sqrt{M})$;

3. the generalized van der Corput sequence with respect to A achieves about 10 times worse when sample number is 10^5.

From the practical point of view, how to find a "good A" is important and this problem still remains.

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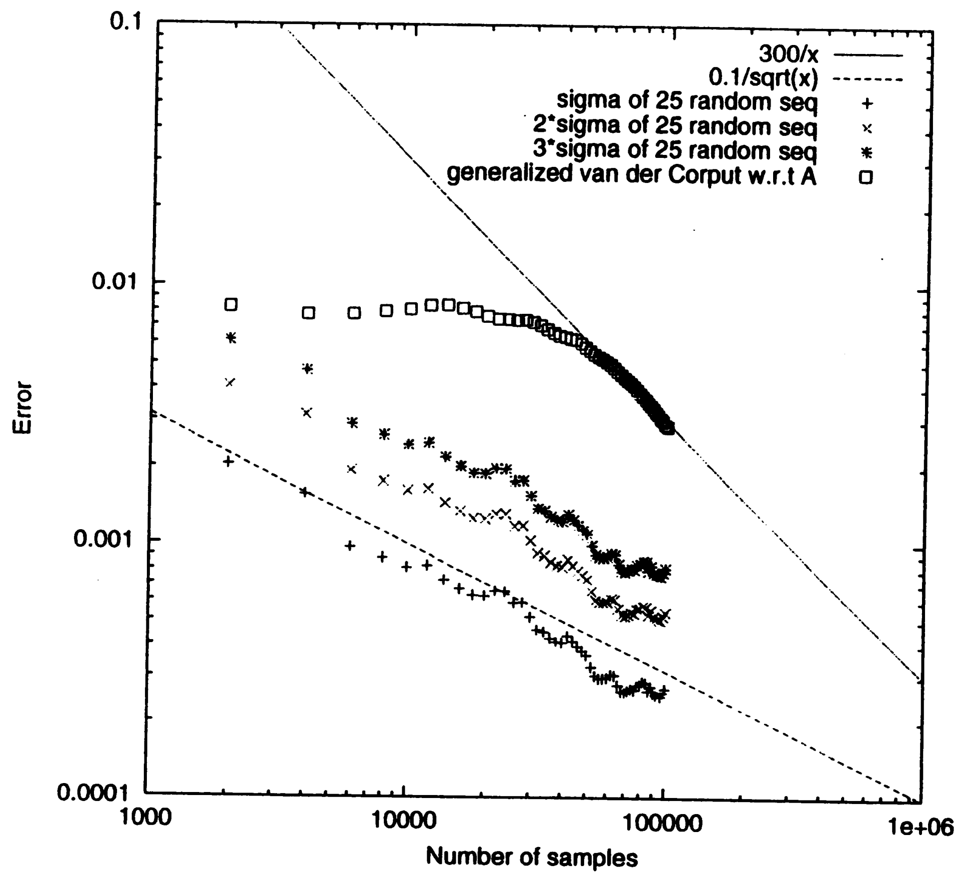

Fig. 1. Convergence performance: numerical integration of f over $\mathbb{T}^{10}$

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