数理解析研究所講究録 1244 巻 2002 年 24-31

A MATHEMATICAL APPROACH TO INTERMITTENCY (1),(2)

Michiko Yuri
Department of Business Administration, Sapporo University

平成 13 年 12 月 16 日

1 Piecewise C^0-invertible Systems

Let $(T, X, Q=\{X_i\}_{i\in I})$ be a piecewise C^0-invertible system i.e., X is a compact metric space with metric d, $T: X\to X$ is a noninvertible map which is not necessarily continuous, and $Q=\{X_i\}_{i\in I}$ is a countable disjoint partition $Q=\{X_i\}_{i\in I}$ of X such that $\bigcup_{i\in I} intX_i$ is dense in X and satisfy the following properties.

(01) For each $i\in I$ with $intX_i\neq\emptyset, T|_{intX_i}: intX_i\to T(intX_i)$ is a homeomorphism and $(T|_{intX_i})^{-1}$ extends to a homeomorphism v_i on $cl(T(intX_i))$.

(02) $T(\bigcup_{intX_i=\emptyset} X_i)\subset\bigcup_{intX_i=\emptyset} X_i$.

(03) $\{X_i\}_{i\in I}$ generates $\mathcal{F}$, the sigma algebra of Borel subsets of X.

Let $\underline{i}=(i_1\ldots i_n)\in I^n$ satisfy $int(X_{i_1}\cap T^{-1}X_{i_2}\cap\ldots T^{-(n-1)}X_{i_n})\neq\emptyset$. Then we define $X_{\underline{i}}:=X_{i_1}\cap T^{-1}X_{i_2}\cap\ldots T^{-(n-1)}X_{i_n}$ which is called a cylinder of rank n and write $|\underline{i}|=n$. By (01), $T^n|_{intX_{i_1\ldots i_n}}: intX_{i_1\ldots i_n}\to T^n(int(X_{i_1\ldots i_n}))$ is a homeomorphism and $(T^n|_{intX_{i_1\ldots i_n}})^{-1}$ extends to a homeomorphism $v_{i_1}\circ v_{i_2}\circ\ldots\circ v_{i_n}=v_{i_1\ldots i_n}: cl(T^n(intX_{\underline{i}}))\to cl(intX_{\underline{i}})$.

We impose on (T, X, Q) the next condition which gives a nice countable states symbolic dynamics similar to sofic shifts (cf. [11]):

(Finite Range Structure) $\mathcal{U}=\{int(T^nX_{i_1\ldots i_n}):\forall X_{i_1\ldots i_n},\forall n>0\}$ consists of finitely many open subsets $U_1\ldots U_N$ of X.

In particular, we say that (T, X, Q) satisfies Bernoulli property if $cl(T(intX_i))=X(\forall i\in I)$ so that $\mathcal{U}=\{intX\}$ and that (T, X, Q) satisfies Markov property if $int(cl(intX_i)\cap cl(intTX_j))\neq\emptyset$ implies $cl(intTX_j)\supset cl(intX_i)$. (T, X, Q) satisfying Bernoulli (Markov) property is called a piecewise C^0-invertible Bernoulli (Markov) system respectively. We say that $X_i\in Q$ is a *full cylinder* if $cl(T(intX_i))=X$. We assume further the next condition :

(Transitivity) $intX = \cup_{k=1}^{N} U_k$ and $\forall l \in \{1, 2, \ldots N\}, \exists 0 < s_l < \infty$ such that for each $k \in \{1, 2, \ldots N\}$, U_k contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank s_l such that $T^{s_l}(intX^{(k,l)}(s_l)) = U_l$.

2 Topological pressure for potentials of weak bounded variation

Definition We say that ϕ is a potential of *weak bounded variation*(WBV) if there exists a sequence of positive numbers $\{C_n\}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n = 0$ and $\forall n \geq 1, \forall X_{i_1 \ldots i_n} \in \bigvee_{j=0}^{n-1} T^{-j}Q$,

$$\frac{\sup_{x\in X_{i_1\ldots i_n}} \exp(\sum_{j=0}^{n-1}\phi(T^j x))}{\inf_{x\in X_{i_1\ldots i_n}} \exp(\sum_{j=0}^{n-1}\phi(T^j x))} \leq C_n.$$

(C.f.[11,13,15-19])

We define a partition function for each $n > 0$ and for each $U_k \in \mathcal{U}$ as follows :

$$Z_n(U_k, \phi) := \sum_{\underline{i}:|\underline{i}|=n, int(TX_{i_n})=U_k\supset intX_{i_1}} \ \sum_{v_{\underline{i}}x=x\in cl(intX_{\underline{i}})} \exp[\sum_{h=0}^{n-1} \phi T^h(x)].$$

We further define :

$$\overline{Z}_n(U_k, \phi) = \sum_{\underline{i}:|\underline{i}|=n, int(TX_{i_n})=U_k\supset intX_{i_1}} \sup_{x\in X_{\underline{i}}} \exp[\sum_{h=0}^{n-1} \phi T^h(x)]$$

and

$$\underline{Z}_n(U_k, \phi) = \sum_{\underline{i}:|\underline{i}|=n, int(TX_{i_n})=U_k\supset intX_{i_1}} \inf_{x\in X_{\underline{i}}} \exp[\sum_{h=0}^{n-1} \phi T^h(x)].$$

Lemma 2.1 *([17]) Let (T, X, Q) be a piecewise C^0-invertible Markov system with finite range structure satisfying the transitivity. Let ϕ be a potential of WBV. For each $U_k \in \mathcal{U}$, $\lim_{n\to\infty} \frac{1}{n}\log\overline{Z}_n(U_k,\phi)$, $\lim_{n\to\infty}\frac{1}{n}\log\underline{Z}_n(U_k,\phi)$ exist and the limits does not depend on k. Furthermore,*

$$P_{top}(T,\phi) := \lim_{n\to\infty}\frac{1}{n}\log Z_n(X,\phi)$$

$$= \lim_{n\to\infty}\frac{1}{n}\log Z_n(U_k,\phi) = \lim_{n\to\infty}\frac{1}{n}\log\overline{Z}_n(U_k,\phi) = \lim_{n\to\infty}\frac{1}{n}\log\underline{Z}_n(U_k,\phi),$$

where

$$\log Z_n(X, \phi) := \sum_{\underline{i}:|\underline{i}|=n, int(TX_{i_n})\supset intX_{i_1}} \ \sum_{v_{\underline{i}}x=x\in cl(intX_{\underline{i}})} \exp[\sum_{h=0}^{n-1} \phi T^h(x)].$$

We define

$$\mathcal{W}_0(T) := \{\phi : X \to \mathbf{R} | \phi \text{ satisfies WBV and } P_{\text{top}}(T,\phi) < \infty\}.$$

Then we can easily see that the pressure function $P_{\text{top}}(T, .) : \mathcal{W}_0(T) \to \mathbf{R}$ satisfies continuity for bounded functions and convexity.

3 Weak Gibbs measures associated to potentials of WBV

Definition ([7],[11],[13],[15-19]) A Borel probability measure ν is called a *weak Gibbs measure* for a function ϕ with a constant P if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that ν-a.e.x,

$$K_n^{-1} \le \frac{\nu(X_{i_1\ldots i_n}(x))}{\exp(\sum_{i=0}^{n-1}\phi T^i(x) + nP)} \le K_n,$$

where $X_{i_1\ldots i_n}(x)$ denotes the cylinder containing x.

Definition A Borel probability measure ν on X is called a *f-conformal measure if* $\frac{d(\nu T)|X_i}{d\nu|X_i} = f|_{X_i}(\forall i \in I)$.

Lemma 3.1 *([17]) Let (T, X, Q) be a piecewise C^0-invertible Markov system with FRS satisfying the transitivity and int$X \in \mathcal{U}$. Let $\phi \in \mathcal{W}_0(T)$ and ν be an* $\exp[P_{top}(T,\phi) - \phi]$*-conformal measure. Then ν is a weak Gibbs measure for ϕ with $-P_{top}(T,\phi)$.*

For $\phi : X \to \mathbf{R}$ we define the Ruelle-Perron-Frobenius operator $\mathcal{L}_\phi$ by

$$\mathcal{L}_\phi g(x) = \sum_{i\in I}\exp[\phi(v_i(x))]g(v_i(x)) \ (\forall g \in C(X), \forall x \in X).$$

Lemma 3.2 *([11],[13]) If there exist $p > 0$ and a Borel probability measure ν on X satisfying $\mathcal{L}_\phi^*\nu = p\nu$, then ν is an* $\exp[\log p - \phi]$*-conformal measure and* $p = \exp[P_{top}(T,\phi)]$.

4 Indifferent periodic points associated to potentials of WBV

Lemma 4.1 $P_{top}(T,\phi) \ge \frac{1}{q}\sum_{h=0}^{q-1}\phi T^h(x_0)(\forall x_0 \in X, T^q x_0 = x_0)$.

Definition x_0 is called an *indifferent periodic point* with period q with respect to ϕ if $P_{\text{top}}(T,\phi) = \frac{1}{q}\sum_{h=0}^{q-1}\phi T^h(x_0)$. If there exists an $\exp[P_{\text{top}}(T,\phi) - \phi]$ -conformal measure ν, then x_0 satisfies

$$\frac{d(\nu T^q)}{d\nu}|_{X_{i_1\ldots i_q}(x_0)}(x_0) = \exp[qP_{\text{top}}(T,\phi) - \sum_{h=0}^{q-1}\phi T^h(x_0)] = 1.$$

If x_0 is not indifferent, then we call x_0 a *repelling periodic point.*

Proposition 4.1 *([16-17]) Let x_0 be an indifferent periodic point with period q with respect to $\phi \in \mathcal{W}_0(T)$. Let ν be an* $\exp[P_{top}(T,\phi) - \phi]$ *-conformal measure. Then*

(i) $\forall s \ge 1, P_{top}(T, s\phi) = sP_{top}(T,\phi)$ *and* $\forall s < 1, P_{top}(T, s\phi) \ge sP_{top}(T,\phi)$.

(ii) $\nu(X_{i_1\ldots i_n}(x_0))$ *decays subexponentially fast.*

5 Jump transformations

Let J be a subset of the index set I and let $B_1 = \bigcup_{i\in J} X_i$. Define $\mathcal{B}_1 := \{X_i \in Q : X_i \subset B_1\}$ and for each $n > 1$ $\mathcal{B}_n := \{X_{i_1\dots i_n} \in \bigvee_{i=0}^{n-1} T^{-i}Q : X_{i_k} \subset B_1^c(k = 1,\dots,n-1), X_{i_n} \subset B_1\}$. Define a function $R : X \to \mathbf{N}\cup\{\infty\}$ by $R(x) = \inf\{n \geq 0 : T^n x \in B_1\} + 1$. Then we see that $B_n := \{x \in X | R(x) = n\} = \bigcup_{X_{i_1\dots i_n}\in\mathcal{B}_n} X_{i_1\dots i_n}$ and $D_n := \{x \in X|R(x) > n\} = \bigcap_{i=0}^{n} T^{-i}B_1^c$. Now we define the jump transformation $T^* : \bigcup_{n=1}^{\infty} B_n \to X$ by $T^*x = T^{R(x)}x$. We denote $X^* := X\backslash(\bigcup_{i=0}^{\infty} T^{*-i}(\bigcap_{n\geq 0} D_n))$ and $I^* := \bigcup_{n\geq 1}\{(i_1\dots i_n) \in I^n : X_{i_1\dots i_n} \subseteq B_n\}$. Then it is easy to see that $(T^*, X^*, Q^* = \{X_{\underline{i}}\}_{\underline{i}\in I^*})$ is a piecewise C^0-invertible Markov system with FRS and the property (1) :$B_{n+1} = D_n \cap T^{-n}B_1$ is valid for $n \geq 1$. Let $\phi : X \to \mathbf{R}$ be a potential of WBV with $P_{\text{top}}(T,\phi) < \infty$. We assume further the next condition :

(Local Bounded Distortion) $\exists\theta > 0$ and $\forall X_{i_1\dots i_n} \in \mathcal{B}_n, \exists 0 < L_\phi(i_1\dots i_n) < \infty$ such that

$$|\phi v_{i_1\dots i_n}(x) - \phi v_{i_1\dots i_n}(y)| \leq L_\phi(i_1\dots i_n)d(x,y)^\theta$$

and

$$\sup_{n\geq 1}\sup_{X_{i_1\dots i_n}\in\mathcal{B}_n}\sum_{j=0}^{n-1} L_\phi(i_{j+1}\dots i_n) < \infty.$$

Define $\phi^* : \bigcup_{n=1}^{\infty} B_n \to \mathbf{R}$ by $\phi^*(x) = \sum_{i=0}^{R(x)-1} \phi T^i(x)$ and denote the local inverses to $T^*|_{X_{\underline{i}}}(\underline{i} \in I^*)$ by $v_{\underline{i}}$. Then $\{\phi^* v_{\underline{i}}\}$ is a family of equi-Hölder continuous functions and if T^* satisfies the next property then ϕ^* satisfies summability of variation.

(Exponential Instability) $\sigma^*(n) := \sup_{\underline{i}\in I^*:|\underline{i}|=n} diam X_{\underline{i}}$ decays exponentially fast as $n \to \infty$.

The summable variation allows one to show the existence of an unique equilibrium Gibbs state μ^* for ϕ^* under the existence of an $\exp[P_{\text{top}}(T,\phi) - \phi]$ -conformal measure ν on X with $\nu(\bigcap_{n\geq 0} D_n) = 0$ and $\mu^* \sim \nu|_{X^*}$. The following formula gives a T-invariant σ-finite measure $\mu \sim \nu$.

$$(2) : \mu(E) = \sum_{n=0}^{\infty} \mu^*(D_n \cap T^{-n}E).$$

If $\sum_{n=0}^{\infty} \nu(D_n) < \infty$, then μ is finite. In particular, $\mu(B_1) = \mu^*(X^*) > 0$, since $\nu(X^*) = 1$. If the reference measure ν is ergodic, then both μ, μ^* are ergodic, too.

Theorem 5.1 (A construction of conformal measures) *([17]) Let (T, X, Q) be a piecewise C^0-invertible Markov system with FRS satisfying transitivity. Let T^* be the jump transformation associated to a union of full cylinders of rank 1 which satisfies exponential instability. Let $\phi : X \to \mathbf{R}$ be a potential of WBV satisfying (LBD), $P_{top}(T,\phi) < \infty$ and*

$||\mathcal{L}_{\phi^*}1|| < \infty$. *Suppose either* $P_{top}(T^*, \phi^*) \geq 0$ *or* $||\mathcal{L}_{(\phi - P_{top}(T^*,\phi^*))^*}1|| < \infty$. *Then there exists a Borel probability measure* ν *on* X *supported on* X^* *satisfying*

$$\frac{d\nu T}{d\nu}|_{X_i} = \exp[P_{top}(T, \phi) - \phi](\forall i \in I)$$

and $\nu(\bigcup_{i \in I} \partial X_i) = 0$.

We can associate the indifferent periodic points x_0 with respect to ϕ to the Marginal sets $\bigcap_{n \geq 0} \mathcal{D}_n$.

Proposition 5.1 *([17])*

(i) *(Failure of bounded distortion)*

$$C_{nq}(x_0) := \sup_{x,y \in X_{i_1 \dots i_{nq}(x_0)}} \frac{\exp[\sum_{i=0}^{nq-1} \phi T^i(x)]}{\exp[\sum_{i=0}^{nq-1} \phi T^i(y)]} \to \infty$$

monotonically as $n \to \infty$.

(ii) *(Singularity of the invariant density)* $x_0 \in \bigcap_{n \geq 0} D_n$ *and* $\frac{d\mu}{d\nu}(x_0) = \infty$.

For a T-invariant probability measure m on $(X, \mathcal{F})$, I_m denotes the conditional information of Q with respect to $T^{-1}\mathcal{F}$.

Theorem 5.2 (Variational principle) *([17]) Let* ν *be the* $\exp[P_{top}(T, \phi) - \phi]$ *-conformal measure obtained under assumptions in Theorem 5.1. We assume further that* $\Gamma := \bigcap_{n \geq 0} D_n$ *consists of periodic points. If* $\int_{X^*} R d\nu < \infty$ *and* $H_\nu(Q^*) < \infty$, *then there exists a* T*-invariant ergodic probability measure* μ *equivalent to* ν *which satisfies the following variational principle.*

$$P_{top}(T, \phi) = h_\mu(T) + \int_X \phi d\mu \geq h_m(T) + \int_X \phi dm$$

for all T*-invariant ergodic probability measure* m *on* X *with* $I_m + \phi \in L^1(m)$ *satisfying* $h_m(T) < \infty$ *or* $\int_X \phi dm > -\infty$.

Corollary 5.1 (Phase transition) *We assume all conditions in Theorem 5.2. If* Γ *consists of indifferent periodic points with respect to* ϕ, *then the set of equilibrium states for* ϕ *is the convex hull of* μ *and the set of invariant Borel probability measures supported on* Γ.

6 Slow decay of correlations

We denote $v'_{i_1\dots i_n}(x) = \frac{d(\mu v_{i_1\dots i_n})}{d\mu}(x)$ and let $P_\mu : L^1(\mu) \to L^1(\mu)$ be the normalized transfer operator with respect to μ, i.e.,

$$P_\mu f(x) = \sum_{i\in I} v'_i(x) f(\psi_i(x)) 1_{TX_i}(x) (\forall f \in L^1(\mu)).$$

In this section, we shall establish bounds on the L^1-convergence of iterated transfer operators $\{P_\mu^n\}_{n\geq 1}$ and bounds on the decay of correlations relative to bounded functions f satisfying a weak Lipschitz-type condition defined by :

(6-1) $\exists 0 < L_f < \infty$ such that

$$\sup_{X_{i(m)}\subset D_m^c} \sup_{x,y\in X_{i(m)}} |f(x) - f(y)| \leq L_f \sigma(m) \ (\forall m > 0)$$

under the following conditions.

(6-2) $\Delta_1(k) := \sup_{n\geq 1} \sup_{i(n)\in\mathcal{A}_n} \sup_{X_{j(k)}\subset D_k^c} \sup_{x,y\in X_{j(k)}} |1 - \frac{\psi'_{i(n)}(x)}{\psi'_{i(n)}(y)}| \to 0$ as $k \to \infty$.

(6-3) $\Delta_2(k) := \sup_{X_{j(k)}\subset D_k^c} \sup_{x,y\in X_{j(k)}} |1 - \frac{(d\mu/d\nu)(x)}{(d\mu/d\nu)(y)}| \to 0$ as $k \to \infty$.

Here $\sigma(m) := \sup_{i(m)\in\mathcal{A}_m} diam X_{i(m)}$, $i(m)$ denotes a sequence $i_1 \dots i_m$ of length m and D_m^c denotes $X\backslash D_m$.

Remark (1) If $d\mu^*/d\nu$ is Hölder continuous (with exponent θ), $\Delta_2(m)$ can be bounded from above by $O(\Delta_1(m)) + O(\sigma(m)^\theta)$. For all examples, we can easily estimate both $\Delta_1(m)$ and $\sigma(m)$.

Remark (2) The condition (6-1) is milder than the usual Lipschitz condition. For example, for $S_\beta(x) = x + x^{1+\beta}(\text{mod}1)$ $f(x) = x^{-\delta}$ for any $0 < \delta < \beta$ is a non-Lipschitz unbounded function satisfying (6-1).

We denote $\Delta(k) := \max_{i=1,2} \Delta_i(k)$.

Theorem 6.1 (Polynomial bounds) *Let (T, X, Q) be a piecewise C^0-invertible Bernoulli system and let ν and μ be the probability measures obtained in Theorems 5.1 and 5.2 respectively. Suppose that (6-2) and (6-3) are satisfied. Assume further that all $\mu(D_n), \Delta(n)$ and $\sigma(n)$ decay polynomially fast. Then $\forall f \in L^\infty(\mu)$ satisfying (6-1) we have the following results.*

1.(Rates of L^1-convergence of $\{P_\mu^n f\}_{n\geq 1}$) $\forall n \geq 1$ *and* $\forall 0 < \epsilon < 1$

$$||P_\mu^n f - \int_X f d\mu||_1 \leq \max\{O(\mu(D_{[n^\epsilon]})), O(\Delta([n^\epsilon])), O(\sigma(2[n^\epsilon]))\}.$$

2.(Decay of correlations) $\forall g \in L^\infty(\mu)$ *and* $\forall 0 < \epsilon < 1$

$$|\int_X f(gT^n)d\mu - \int_X f d\mu \int_X g d\mu| \leq \max\{O(\mu(D_{[n^\epsilon]})), O(\Delta([n^\epsilon])), O(\sigma(2[n^\epsilon]))\}.$$

The next result gives sufficient conditions for (6-2).

Lemma 6.1 *Suppose that* $\{\phi\psi_i\}_{i\in I}$, $\{\phi^*\psi_{\underline{i}}^*\}_{\underline{i}\in I^*}$ *are equi-Hölder continuous with exponents* θ_1, θ_2 *respectively. Then* $\forall X_{i_1\ldots i_m} \subset D_m^c$ *and* $\forall (j_1 \ldots j_n) \in \mathcal{A}_n$ *such that* $X_{j_k} \subset B_1$ *and* $X_{j_{k+1}\ldots j_n} \subset D_{n-k}$ *and* $\forall x, y \in X$ *we have*

$$|1 - \frac{\psi'_{j_1\ldots j_n}(\psi_{i_1\ldots i_m}x)}{\psi'_{j_1\ldots j_n}(\psi_{i_1\ldots i_m}y)}|$$

$$\leq \max\{O(\sigma(m+n-k)^{\theta_2}), O(\sum_{i=[m/2]}^{\infty} \sup_{X_{l_1\ldots l_i}\subset B_i} \{diam X_{l_1\ldots l_i}\}^{\theta_1}), O(\sigma([\frac{m}{2}])^{\theta_2})\}.$$

References

[1] J.Aaronson, M.Denker & M.Urbanski. Ergodic Theory for Markov fibred systems and parabolic rational maps. *Trans.AMS* , **337** (1993), 495 - 548.

[2] M.Denker & M.Yuri. A note on the construction of nonsingular Gibbs measures. *Colloquium Mathematicum* **84/85** (2000), 377 - 383.

[3] L.-S. Young. Recurrence times and rates of mixing. *Israel J.Math*, **110** (1999), 153-188.

[4] C.Liverani, B.Saussol & S.Vaienti. A probabilistic approach to intermittency. *Ergodic Theory and Dynamical Systems*, **19** (1999), 671 - 685.

[5] C.Maes, F.Redig, F.Takens, A.V.Moffaert, & E.Verbitski. Intermittency and weak Gibbs states. *Nonlinearity*, **13** (2000), 1681-1698.

[6] M.Pollicott & M.Yuri. Statistical properties of maps with indifferent periodic points. *Commun.Math.Phys.* **217** (2001), 503 - 520.

[7] M.Yuri. On a Bernoulli property for multi-dimensional maps with finite range structure. *Tokyo J.Math.* **9** (1986) 457 - 485.

[8] M.Yuri. Invariant measures for certain multi-dimensional maps. *Nonlinearity* **7** (3) (1994) 1093 - 1124.

[9] M.Yuri. Multi-dimensional maps with infinite invariant measures and countable state sofic shifts. *Indagationes Mathematicae* **6** (1995) 355 - 383.

[10] M.Yuri. On the convergence to equilibrium states for certain nonhyperbolic systems. *Ergodic Theory and Dynamical Systems* **17** (1997) 977-1000.

[11] M.Yuri. Zeta functions for certain nonhyperbolic systems and topological Markov approximations. *Ergodic Theory and Dynamical Systems* **18** (1998) 1589 - 1612.

[12] M.Yuri. Decay of correlations for certain multi-dimensional maps. *Nonlinearity* **9** (1996) 1439-1461.

[13] M.Yuri. Thermodynamic Formalism for certain nonhyperbolic maps. *Ergodic Theory and Dynamical Systems* **19** (1999) 1365-1378.

[14] M.Yuri. Statistical properties for nonhyperbolic maps with finite range structure. *Trans.AMS* , **352** (2000), 2369-2388.

[15] M.Yuri. Weak Gibbs measures for certain nonhyperbolic systems. *Ergodic Theory and Dynamical Systems* **20** (2000) 1495-1518, .

[16] M.Yuri. On the speed of convergence to equilibrium states for multi-dimensional maps with indifferent periodic points. Preprint.

[17] M.Yuri. Thermodynamic Formalism for weak Gibbs measures associated to potentials of weak bounded variation. Preprint.

[18] M.Yuri. Weak Gibbs measures and the local product structure. Preprint.

[19] M.Yuri. Multifractal analysis of weak Gibbs measures for intermittent systems. Preprint.