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Approximate resolutions and fractal geometry

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Abstract

A Lipschitz function between metric spaces is an important notion in fractal geometry as it is well-known to have a close connection to fractal dimension. In this note, we describe a new method of using the theory of approximate resolutions to study Lipschitz maps.

The purpose of this note is to present our recent work on approximate resolutions and applications to fractal geometry [MiW_2].

Recall that a function $f : X \to Y$ between metric spaces X and Y is a *Lipschitz map* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}(f(x), f(x')) \leq \alpha\, \mathrm{d}(x, x') \text{ for } x, x' \in X.$$

Being a Lipschitz map is an important property in fractal geometry, especially, in fractal dimensions since one of the required conditions for a fractal dimension is the Lipschitz subinvariance (see [F, p. 37]), i.e., if a map $f : X \to Y$ is a Lipschitz function, then the fractal dimension of $f(X)$ is at most that of X. In this note, we introduce a new method of using the theory of approximate resolutions to study Lipschitz maps.

Mardešić and Watanabe [MW] introduced the notion of approximate resolutions, which generalizes all compact limits, approximate limits of Mardešić and Rubin [MR] and resolutions of Mardešić [Ma]. This notion has proved to be useful in many problems in topology especially for nonmetric or noncompact spaces [W_2, MiW_1]. However, even for compact metric spaces, approximate resolutions are essential [Mio, W_1, W_2]. In fact, when we are given a map $f : X \to Y$ between compact metric spaces and limits $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, p_{ii+1}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, q_{jj+1}\}$, there may not exist a map of systems $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$, i.e., a function $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, and maps $f_j : X_{f(j)} \to Y_j$, $j \in \mathbb{N}$, with the property that for any $j < j'$, there is $i > f(j), f(j')$ such that

(M) $f_j p_{f(j)i} = q_{jj'} f_{j'} p_{f(j')i}$; and

(LM) $f_j p_{f(j)} = q_j f$, $j \in \mathbb{N}$.

In the theory of approximate resolutions, we replace those commutativity conditions by approximate commutativity conditions so that a map of systems $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ exists.

Throughout this note, a space means a compact metric space, and a map means a continuous map unless otherwise stated.

For any space X, let $\mathrm{Cov}(X)$ denote the set of all normal open coverings of X. For any subset A of X and $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $\mathcal{U}|A = \{U \cap A : U \in \mathcal{U}\}$. If $A = \{x\}$, we write $\mathrm{st}(x,\mathcal{U})$ for $\mathrm{st}(\{x\},\mathcal{U})$. For each $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}\,\mathcal{U} = \{\mathrm{st}(U,\mathcal{U}) : U \in \mathcal{U}\}$. Let $\mathrm{st}^{n+1}\mathcal{U} = \mathrm{st}(\mathrm{st}^n\mathcal{U})$ for each $n = 1, 2, \ldots$ and $\mathrm{st}^1\mathcal{U} = \mathrm{st}\,\mathcal{U}$. For any metric space (X, d) and $r > 0$, let $\mathrm{U_d}(x,r) = \{y \in X : \mathrm{d}(x,y) < r\}$. For any $\mathcal{U} \in \mathrm{Cov}(X)$, two points $x, x' \in X$ are $\mathcal{U}$-*near*, denoted $(x,x') < \mathcal{U}$, provided $x, x' \in U$ for some $U \in \mathcal{U}$. For any $\mathcal{V} \in \mathrm{Cov}(Y)$, two maps $f, g : X \to Y$ between spaces are $\mathcal{V}$-*near*, denoted $(f,g) < \mathcal{V}$, provided $(f(x), g(x)) < \mathcal{V}$ for each $x \in X$. For each $\mathcal{U} \in \mathrm{Cov}(X)$ and $\mathcal{V} \in \mathrm{Cov}(Y)$, let $f\mathcal{U} = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}\mathcal{V} = \{f^{-1}(V) : V \in \mathcal{V}\}$.

Approximate resolutions. First, let us recall the definitions and properties of approximate resolutions. For more details, the reader is referred to [MW].

An *approximate inverse system* (*approximate system*, in short) $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ consists of

i) a sequence of spaces X_i, $i \in \mathbb{N}$;

ii) a sequence of $\mathcal{U}_i \in \mathrm{Cov}(X_i)$, $i \in \mathbb{N}$; and

iii) maps $p_{ii'} : X_{i'} \to X_i$ for $i < i'$ where $p_{ii} = 1_{X_i}$ the identity map on X_i.

It must satisfy the following three conditions:

(A1) $(p_{ii'}p_{i'i''}, p_{ii''}) < \mathcal{U}_i$ for $i < i' < i''$;

(A2) For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i' > i$ such that $(p_{ii_1}p_{i_1i_2}, p_{ii_2}) < \mathcal{U}$ for $i' < i_1 < i_2$; and

(A3) For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i' > i$ such that $\mathcal{U}_{i''} < p_{ii''}^{-1}\mathcal{U}$ for $i' < i''$.

An *approximate map* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ of a space X into an approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ consists of maps $p_i : X \to X_i$ for $i \in \mathbb{N}$ with the following property:

(AS) For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i' > i$ such that $(p_{ii''}p_{i''}, p_i) < \mathcal{U}$ for $i'' > i'$.

An *approximate resolution* of a space X is an approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ of X into an approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ which satisfies the following two conditions:

(R1) For each ANR P, $\mathcal{V} \in \mathrm{Cov}(P)$ and map $f : X \to P$, there exist $i \in \mathbb{N}$ and a map $g : X_i \to P$ such that $(gp_i, f) < \mathcal{V}$; and

(R2) For each ANR P and $\mathcal{V} \in \mathrm{Cov}(P)$, there exists $\mathcal{V}' \in \mathrm{Cov}(P)$ such that whenever $i \in \mathbb{N}$ and $g, g' : X_i \to P$ are maps with $(gp_i, g'p_i) < \mathcal{V}'$, then $(gp_{ii'}, g'p_{ii'}) < \mathcal{V}$ for some $i' > i$.

If $\mathcal{C}$ is a collection of spaces, and if all X_i belong to $\mathcal{C}$, then the approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ is called an *approximate* $\mathcal{C}$-*resolution*. Let $\mathcal{POL}$ denote the collection of polyhedra. We have the following characterization for approximate resolutions:

Theorem 1 *An approximate map* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ *is an approximate resolution of a space* X *if and only if it satisfies the following two conditions:*

(B1) *For each* $\mathcal{U} \in \mathrm{Cov}(X)$*, there exists* $i_0 \in \mathbb{N}$ *such that* $p_i^{-1}\mathcal{U}_i < \mathcal{U}$ *for* $i > i_0$*; and*

(B2) *For each* $i \in \mathbb{N}$ *and* $\mathcal{U} \in \mathrm{Cov}(X_i)$*, there exists* $i_0 > i$ *such that* $p_{ii'}(X_{i'}) \subseteq \mathrm{st}(p_i(X), \mathcal{U})$ *for* $i' > i_0$.

We have the following existence theorem for approximate resolutions:

Theorem 2 *1) ([W₂]) Every topological space* X *admits an approximate* $\mathcal{POL}$*-resolution* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ *such that all* X_i *are finite polyhedra.*

2) ([MS]) Every connected compact Hausdorff space X *admits an approximate* $\mathcal{POL}$*-resolution* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ *such that all* X_i *are connected finite polyhedra, and all* p_i *and* $p_{ii'}$ *are surjective.*

Let $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ be approximate systems of spaces. An *approximate map* $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ consists of an increasing function $f : \mathbb{N} \to \mathbb{N}$ and maps $f_j : X_{f(j)} \to Y_j, j \in \mathbb{N}$, with the following condition:

(AM) For any $j, j' \in \mathbb{N}$ with $j < j'$, there exists $i \in \mathbb{N}$ with $i > f(j')$ such that

$$(q_{jj'} f_{j'} p_{f(j')i'}, f_j p_{f(j)i'}) < \mathrm{st}\, \mathcal{V}_j \text{ for } i' > i.$$

A map $f : X \to Y$ is a *limit* of $\boldsymbol{f}$ provided the following condition is satisfied:

(LAM) For each $j \in \mathbb{N}$ and $\mathcal{V} \in \mathrm{Cov}(Y_j)$, there exists $j' > j$ such that

$$(q_{jj''} f_{j''} p_{f(j'')}, q_j f) < \mathcal{V} \text{ for } j'' > j'.$$

For each map $f : X \to Y$, an *approximate resolution* of f is a triple $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ consisting of approximate resolutions $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ of X and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ of Y and of an approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ with property (LAM).

Theorem 3 *Let* X *and* Y *be spaces. For any approximate* $\mathcal{POL}$*-resolutions* $\boldsymbol{p} : X \to \boldsymbol{X}$ *and* $\boldsymbol{q} : Y \to \boldsymbol{Y}$*, every map* $f : X \to Y$ *admits an approximate map* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *such that* $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ *is an approximate resolution of* f.

For each approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, let $\mathrm{st}\,\boldsymbol{X}$ denote the approximate system $\{X_i, \mathrm{st}\,\mathcal{U}_i, p_{ii'}\}$. Then there is a natural approximate map $\boldsymbol{i_X} = \{1_{X_i}\} : \boldsymbol{X} \to \mathrm{st}\,\boldsymbol{X}$, where $1_{X_i} : X_i \to X_i$ is the identity map. For each approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, the map $\mathrm{st}\,\boldsymbol{p} = \{p_i\} : X \to \mathrm{st}\,\boldsymbol{X} = \{X_i, \mathrm{st}\,\mathcal{U}_i, p_{ii'}\}$ also satisfies (AS) and hence is an approximate map. Moreover, if $\boldsymbol{p} : X \to \boldsymbol{X}$ is an approximate resolution, so is $\mathrm{st}\,\boldsymbol{p} : X \to \mathrm{st}\,\boldsymbol{X}$.

For any approximate systems $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ and for each approximate map $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$, the map $\mathrm{st}\,\boldsymbol{f} = \{f_j, f\} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y}$ is also an approximate map. Moreover, if $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is an approximate resolution of a map $f : X \to Y$, then $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y}$ also satisfies (LAM) and hence $(\mathrm{st}\,\boldsymbol{f}, \mathrm{st}\,\boldsymbol{p}, \mathrm{st}\,\boldsymbol{q})$ is an approximate resolution of f.

Throughout the rest of the note, an approximate resolution means an approximate $\mathcal{POL}$-resolution unless otherwise stated.

An approach by normal sequences. Having recalled the notion of approximate resolutions, we follow the approach of Alexandroff and Urysohn (see [AU] and [N, 2-16]) to obtain a metric $\mathrm{d}_{\mathbb{U}}$ on X for a given space X and normal sequence $\mathbb{U}$ on X.

A family $\mathbb{U} = \{\mathcal{U}_i : i \in \mathbb{N}\}$ of open coverings on a space X is said to be a *normal sequence* provided $\mathrm{st}\,\mathcal{U}_{i+1} < \mathcal{U}_i$ for each i. Let $\Sigma\mathbb{U}$ denote the normal sequence $\{\mathcal{V}_i : \mathcal{V}_i = \mathcal{U}_{i+1}, i \in \mathbb{N}\}$ and $\mathrm{st}\,\mathbb{U}$ the normal sequence $\{\mathrm{st}\,\mathcal{U}_i : i \in \mathbb{N}\}$. For any normal sequences $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{\mathcal{V}_i\}$, we write $\mathbb{U} < \mathbb{V}$ provided $\mathcal{U}_i < \mathcal{V}_i$ for each i. Let $\Sigma^0\mathbb{U} = \mathbb{U}$, and for each $n \in \mathbb{N}$, let $\Sigma^n\mathbb{U} = \Sigma(\Sigma^{n-1}\mathbb{U})$, and also let $\mathrm{st}^0\,\mathbb{U} = \mathbb{U}$ and $\mathrm{st}^n\,\mathbb{U} = \mathrm{st}(\mathrm{st}^{n-1}\,\mathbb{U})$. For each map $f : X \to Y$ and for each normal sequence $\mathbb{V} = \{\mathcal{V}_i\}$, let $f^{-1}\mathbb{V} = \{f^{-1}\mathcal{V}_i\}$. For each closed subset A of X and for each normal sequence $\mathbb{U} = \{\mathcal{U}_i\}$ on X, let $\mathbb{U}|A = \{\mathcal{U}_i|A\}$.

Given a normal sequence $\mathbb{U} = \{\mathcal{U}_i\}$ on X, we define the function $\mathcal{D}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_{\mathbb{U}}(x,x') = \begin{cases} 9 & \text{if } (x,x') \not< \mathcal{U}_1; \\ \frac{1}{3^{i-2}} & \text{if } (x,x') < \mathcal{U}_i \text{ but } (x,x') \not< \mathcal{U}_{i+1}\ ; \\ 0 & \text{if } (x,x') < \mathcal{U}_i \text{ for all } i \in \mathbb{N}\ , \end{cases}$$

and the function $\mathrm{d}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_{\mathbb{U}}(x,x') = \inf\{\mathcal{D}_{\mathbb{U}}(x,x_1) + \mathcal{D}_{\mathbb{U}}(x_1,x_2) + \cdots + \mathcal{D}_{\mathbb{U}}(x_n,x')\}$$

where the infimum is taken over all points $x_1, x_2, ..., x_n$ in X and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. Then the function $\mathrm{d}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ defines a pseudometric on X with the property that

$$\mathrm{st}(x,\mathcal{U}_{i+3}) \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{U}}}(x, \frac{1}{3^i}) \subseteq \mathrm{st}(x,\mathcal{U}_i) \text{ for each } x \in X \text{ and } i.$$

Moreover, if $\mathbb{U}$ has the following property:

(B) $\{\mathrm{st}(x,\mathcal{U}_i) : i \in \mathbb{N}\}$ is a base at x for each $x \in X$.

then $\mathrm{d}_{\mathbb{U}}$ defines a metric on X, which we call the *metric induced by the normal sequence* $\mathbb{U}$. In particular, if $\mathbb{U} = \{\mathcal{U}_i\}$ is the normal sequence such that $\mathcal{U}_i = \{\mathrm{U}_{\mathrm{d}}(x, \frac{1}{3^i}) : x \in X\}$, then the metric $\mathrm{d}_{\mathbb{U}}$ induced by the nomal sequence $\mathbb{U}$ induces the uniformity which is isomorphic to that induced by the metric d.

Proposition 4 *Let X be a space, and let $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{\mathcal{V}_i\}$ be normal sequences on X. Then we have the following properties:*

1) If A is a closed subset of X, then $\mathrm{d}_{\mathbb{U}|A}(x,x') \geq \mathrm{d}_{\mathbb{U}}(x,x')$ for all $x, x' \in A$.

2) If $\mathbb{U} < \mathbb{V}$, then $\mathrm{d}_{\mathbb{U}}(x,x') \geq \mathrm{d}_{\mathbb{V}}(x,x')$ for all $x, x' \in X$.

3) $\mathrm{d}_{\Sigma\mathbb{U}}(x,x') = 3\,\mathrm{d}_{\mathbb{U}}(x,x')$ for all $x, x' \in X$.

4) $\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x,x') \leq \mathrm{d}_{\mathbb{U}}(x,x') \leq 3\,\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x,x')$ for all $x, x' \in X$.

Let X and Y be spaces, and let $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{\mathcal{V}_i\}$ be normal sequences on X and Y, respectively. A map $f : X \to Y$ is said to be a $(\mathbb{U},\mathbb{V})$-*Lipschitz map* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \alpha\, \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

In particular, if we can choose α such that $0 < \alpha < 1$, the map $f : X \to Y$ is said to be a $(\mathbb{U},\mathbb{V})$-*contraction map.*

Lipschitz maps and contraction maps between spaces are characterized in terms of normal sequences as follows:

Theorem 5 *Let X and Y be spaces with normal sequences $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{\mathcal{V}_i\}$, respectively, and let $f : X \to Y$ be a map. Consider the following statements:*

(L)$_m$ $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq 3^m\, \mathrm{d}_{\mathbb{U}}(x, x')$ *for $x, x' \in X$;*

(M)$_{m,n}$ $\Sigma^m\mathbb{U} < f^{-1}\,\mathrm{st}^n\,\mathbb{V}$; *and*

(N)$_{m,n}$ $\Sigma^m\mathbb{U} < f^{-1}\Sigma^n\mathbb{V}$.

Then the following implications hold for any $m, n \geq 0$:

1) *$(M)_{m,n} \Rightarrow (L)_{m+n}$;*

2) *$(N)_{m,n} \Rightarrow (L)_{n-m}$;*

3) *$(L)_m \Rightarrow (M)_{m+4,0} = (N)_{m+4,0}$; and*

4) *$(L)_{-m} \Rightarrow (N)_{4,m}$.*

An approach by approximate resolutions. Next, given a space X and an approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ of X, we obtain a metric $\mathrm{d}_{\boldsymbol{p}}$ on X.

For each approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, consider the following three conditions:

(U) $\mathrm{st}^2\,\mathcal{U}_j < p_{ij}^{-1}\mathcal{U}_i$ for $i < j$;

(A) $(p_{ij}p_j, p_i) < \mathcal{U}_i$ for $i < j$; and

(NR) $p_j^{-1}\,\mathrm{st}\,\mathcal{U}_j < p_i^{-1}\mathcal{U}_i$ for $i < j$.

An approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is said to be *admissible* provided it pocesses properties (U), (A), (NR) and the family $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$ has property (B). For any approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we can always find an admissible approximate resolution $\boldsymbol{p}' = \{p_{k_i}\} : X \to \boldsymbol{X}' = \{X_{k_i}, \mathcal{U}_{k_i}, p_{k_ik_j}\}$ by taking a subsystem, and we have the following property:

Proposition 6 *1) The family $\mathbb{U}_k = \{p_i^{-1}\,\mathrm{st}^k\,\mathcal{U}_i\}$ forms a normal sequence on X for $k \geq 0$;*

2) *The approximate resolution $\mathrm{st}^k\boldsymbol{p} = \{p_i\} : X \to \mathrm{st}^k\boldsymbol{X} = \{X_i, \mathrm{st}^k\,\mathcal{U}_i, p_{ii'}\}$ is admissible for $k \geq 1$.*

Let $\boldsymbol{p} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be any admissible approximate resolution of a space X. Then for any $x, x' \in X$, we define the function $\mathcal{D}_{\boldsymbol{p}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_{\boldsymbol{p}}(x, x') = \begin{cases} 9 & \text{if } (p_i(x), p_i(x')) \not< \mathcal{U}_i \text{ for any } i; \\ \frac{1}{3^{i-2}} & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ but } (p_i(x), p_i(x')) \not< \mathcal{U}_{i+1}\ ; \\ 0 & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ for all } i\ , \end{cases}$$

and the function $\mathrm{d}_{\boldsymbol{p}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_{\boldsymbol{p}}(x, x') = \inf\{\mathcal{D}_{\boldsymbol{p}}(x, x_1) + \mathcal{D}_{\boldsymbol{p}}(x_1, x_2) + \cdots + \mathcal{D}_{\boldsymbol{p}}(x_n, x')\}$$

where the infimum is taken over all finitely many points $x_1, x_2, ..., x_n$ of X. Note that $\mathrm{d}_{\boldsymbol{p}}(x, x') = \mathrm{d}_{\mathbb{U}}(x, x')$ for any $x, x' \in X$, where $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$.

For each approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we define the approximate system $\Sigma\boldsymbol{X}$ as $\{Z_i, \mathcal{W}_i, r_{ii'}\}$ where $Z_i = X_{i+1}$, $\mathcal{W}_i = \mathcal{U}_{i+1}$, $r_{ii'} = p_{i+1 i'+1} : Z_{i'} \to Z_i$ and the approximate resolution $\Sigma\boldsymbol{p}$ as $\{r_i : i \in \mathbb{N}\} : X \to \Sigma\boldsymbol{X}$ where $r_i = p_{i+1} : X \to X_{i+1}$. Let $\Sigma^0\boldsymbol{X} = \boldsymbol{X}$ and $\Sigma^0\boldsymbol{p} = \boldsymbol{p}$, and for each $i \in \mathbb{N}$, let $\Sigma^n\boldsymbol{X} = \Sigma(\Sigma^{n-1}\boldsymbol{X})$ and $\Sigma^n\boldsymbol{p} = \Sigma(\Sigma^{n-1}\boldsymbol{p})$.

Proposition 7 *Let X be a space, and let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution of X. Then we have the following properties:*

1) $\mathrm{d}_{\Sigma^n\boldsymbol{p}}(x, x') = 3^n\, \mathrm{d}_{\boldsymbol{p}}(x, x')$ *for $x, x' \in X$ and for each $n \in \mathbb{N}$; and*

2) $\mathrm{d}_{\mathrm{st}\,\boldsymbol{p}}(x, x') \leq \mathrm{d}_{\boldsymbol{p}}(x, x') \leq 3\, \mathrm{d}_{\mathrm{st}\,\boldsymbol{p}}(x, x')$ *for $x, x' \in X$.*

Let X and Y be spaces, and let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be normal approximate resolutions of X and Y, respectively. A map $f : X \to Y$ is said to be a $(\boldsymbol{p}, \boldsymbol{q})$-*Lipschitz map* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \leq \alpha\, \mathrm{d}_{\boldsymbol{p}}(x, x') \text{ for } x, x' \in X.$$

In particular, if we can choose α such that $0 < \alpha < 1$, a map $f : X \to Y$ is said to be a $(\boldsymbol{p}, \boldsymbol{q})$-*contraction map*.

For each $m \in \mathbb{Z}$, consider the following condition:

(Lip)$_m$ $\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \leq 3^m\, \mathrm{d}_{\boldsymbol{p}}(x, x')$ for $x, x' \in X$,

and for each $m \geq 0$ and for each approximate map $\boldsymbol{f} = \{f_i, f\} : \boldsymbol{X} \to \boldsymbol{Y}$, consider the following condition:

(ALip)$_m$ For each i, there exists $j_0 > i$ with the property that each $j > j_0$ admits $i_0 > f(j), i + m$ such that for each $i' > i_0$,

$$p_{i+m,i'}^{-1}\mathcal{U}_{i+m} < p_{f(j)i'}^{-1} f_j^{-1} q_{ij}^{-1} \mathcal{V}_i.$$

$(\boldsymbol{p}, \boldsymbol{q})$-Lipschitz maps are characterized in terms of condition (ALip)$_m$ for approximate resolutions as follows:

Theorem 8 *Let $f : X \to Y$ be a map between spaces, and let $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ be an approximate map such that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is an approximate resolution of f where $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ are admissible approximate resolutions of X and Y, respectively. Then the following implications hold for $m \geq 0$:*

1) *(ALip)$_m$ for $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y} \Rightarrow$ (Lip)$_m$ for $\boldsymbol{p}$ and $\mathrm{st}^2\boldsymbol{q} \Rightarrow$ (Lip)$_{m+2}$ for $\boldsymbol{p}$ and $\boldsymbol{q}$.*

Moreover, if each p_i is surjective, the following implication also holds:

2) *(Lip)$_m$ for $\boldsymbol{p}$ and $\boldsymbol{q} \Rightarrow$ (ALip)$_{m+4}$ for $\boldsymbol{i}_{\mathrm{st}\boldsymbol{Y}}\boldsymbol{i}_{\boldsymbol{Y}}\boldsymbol{f} : \boldsymbol{X} \to \mathrm{st}^2\boldsymbol{Y}$.*

In a similar way $(\boldsymbol{p}, \boldsymbol{q})$-contraction maps are characterized in terms of the following condition for $m \geq 0$:

(ACon)$_m$ For each i there exists $j_0 > i$ with the property that each $j > j_0$ admits $i_0 > f(j), i$ such that for each $i' > i_0$

$$p_{ii'}^{-1}\mathcal{U}_i < p_{f(j)i'}^{-1} f_j^{-1} q_{i+m,j}^{-1} \mathcal{V}_{i+m}.$$

Theorem 9 *Under the same setting as in Theorem 8, the following implications hold for $m \geq 0$:*

1) *(ACon)$_m$ for $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y} \Rightarrow$ (Lip)$_{-m}$ for $\boldsymbol{p}$ and $\mathrm{st}^2\boldsymbol{q} \Rightarrow$ (Lip)$_{-m+2}$ for $\boldsymbol{p}$ and $\boldsymbol{q}$.*

Moreover, if each p_i is surjective, the following implication also holds:

2) *(Lip)$_{-m}$ for $\boldsymbol{p}$ and $\boldsymbol{q} \Rightarrow$ (ACon)$_{m-4}$ for $\boldsymbol{i}_{\mathrm{st}\boldsymbol{Y}}\boldsymbol{i}_{\boldsymbol{Y}}\boldsymbol{f} : \boldsymbol{X} \to \mathrm{st}^2\boldsymbol{Y}$.*

As an easy application, we have the following unique fixed point theorem:

Corollary 10 *A map $f : X \to X$ has a unique fixed point if there is an approximate resolution $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ of f for some approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : X \to \boldsymbol{X}'$ and approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{X}'$ so that (ACon)$_m$ holds for $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{X}$ and for some $m \geq 2$.*

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