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Scaling Limit of a Dirac Particle Interacting with the Quantum Radiation Field

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Abstract

A quantum system of a Dirac particle — a relativistic charged particle with spin 1/2 — interacting with the quantum radiation field is considered and an effective particle Hamiltonian is derived as a scaling limit of the total Hamiltonian of the system.

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1 Introduction

We consider a quantum system of a Dirac particle — a relativistic charged particle with spin 1/2 — interacting with the quantum radiation field with momentum cutoffs. The total Hamiltonian H of the system is of the form:

$$\boxed{\text{a Dirac operator}} + \boxed{\text{the free Hamiltonian of the quantum radiation field}}$$
$$+ \boxed{\text{a perturbation term}}$$

Here, as usual, the perturbation term is given by the minimal interaction of the Dirac particle with the quantum radiation field. This is a well known model in relativistic quantum electrodynamics (QED), although rigorous mathematical analyses of it have been only recently initiated [3, 4, 5].

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In this note we focus our attention on scaling limits of H and derive an effective particle Hamiltonian, which is a modified Dirac operator containing fluctuation effects due to the interaction of the Dirac particle with the quantum radiation field. Such an effective Hamiltonian may be used as an approximate quantum mechanical particle Hamiltonian of the total Hamiltonian.

We remark that scaling limits in nonrelativistic QED have been discussed in [1, 7, 8, 9]. The present work may be regarded as a first step towards extensions of those studies to relativistic QED.

2 Description of the Model

2.1 The Hamiltonian of the Dirac particle

We denote the mass and the charge of the Dirac particle by $m > 0$ and $q \in \mathbf{R} \setminus \{0\}$ respectively. We consider the situation where the Dirac particle is in a potential V which is a *Hermitian-matrix-valued Borel measurable function* on $\mathbf{R}^3$. Then the Hamiltonian of the Dirac particle is given by the Dirac operator

$$H_{\mathrm{D}}(V) := \boldsymbol{\alpha} \cdot \boldsymbol{p} + m\beta + V \tag{2.1}$$

acting in the Hilbert space

$$\mathcal{H}_{\mathrm{D}} := \oplus^4 L^2(\mathbf{R}^3) \tag{2.2}$$

with domain $D(H_{\mathrm{D}}(V)) := \oplus^4 H^1(\mathbf{R}^3) \cap D(V)$ ($H^1(\mathbf{R}^3)$ is the Sobolev space of order 1), where α_j ($j = 1, 2, 3$) and β are 4×4 Hermitian matrices satisfying the anticommutation relations

$$\{\alpha_j,\, \alpha_k\} = 2\delta_{jk}, \quad j, k = 1, 2, 3, \tag{2.3}$$

$$\{\alpha_j,\, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \tag{2.4}$$

$\{A, B\} := AB + BA$, δ_{jk} is the Kronecker delta,

$$\boldsymbol{p} := (p_1, p_2, p_3) := (-iD_1, -iD_2, -iD_3) \tag{2.5}$$

with D_j being the generalized partial differential operator in the variable x_j, the j-th component of $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, and $\boldsymbol{\alpha} \cdot \boldsymbol{p} := \sum_{j=1}^{3} \alpha_j p_j$.

2.2 The quantum radiaiton field

We use the Coulomb gauge for the quantum radiation field. The Hilbert space of one-photon states in momentum representation is given by

$$\mathcal{H}_{\mathrm{ph}} := L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3), \tag{2.6}$$

where $\mathbf{R}^3 := \{\boldsymbol{k} = (k_1, k_2, k_3) | k_j \in \mathbf{R},\ j = 1,2,3\}$ physically means the momentum space of photons. Then a Hilbert space for the quantum radiation field is given by

$$\mathcal{F}_{\mathrm{rad}} := \oplus_{n=0}^{\infty} (\otimes_s^n \mathcal{H}_{\mathrm{ph}}) \tag{2.7}$$

the Boson Fock space over over $\mathcal{H}_{\mathrm{ph}}$, where $\otimes_s^n$ denotes n-fold symmetric tensor product of $\mathcal{H}_{\mathrm{ph}}$ and $\otimes_s^0 \mathcal{H}_{\mathrm{ph}} := \mathbf{C}$.

We denote by $a(F)$ ($F \in \mathcal{H}_{\mathrm{ph}}$) the annihilation operator with test vector F on $\mathcal{F}_{\mathrm{rad}}$. By definition, $a(F)$ is a densely defined closed linear operator and antilinear in F. The Segal field operator

$$\Phi_{\mathrm{S}}(F) := \overline{\frac{a(F) + a(F)^*}{\sqrt{2}}} \tag{2.8}$$

is self-adjoint [11, §X.7], where, for a closable operator T, $\overline{T}$ denotes its closure. For each $f \in L^2(\mathbf{R}^3)$, we define

$$a^{(1)}(f) := a(f, 0), \quad a^{(2)}(f) := a(0, f). \tag{2.9}$$

The mapping : $f \to a^{(r)}(f^*)$ restricted to $\mathcal{S}(\mathbf{R}^3)$ (the space of rapidly decreasing C^∞-functions on $\mathbf{R}^3$) defines an operator-valued destribution (f^* denotes the complex conjugate of f). We denote its symbolical kernel by $a^{(r)}(\boldsymbol{k})$: $a^{(r)}(f) = \int a^{(r)}(\boldsymbol{k}) f(\boldsymbol{k})^* d\boldsymbol{k}$.

We take a nonnegative Borel measurable function ω on $\mathbf{R}^3$ to denote the one free photon energy. We assume that, for almost everywhere (a.e.) $\boldsymbol{k} \in \mathbf{R}^3$ with respect to the Lebesgue measure on $\mathbf{R}^3$, $0 < \omega(\boldsymbol{k}) < \infty$. Then the function ω defines uniquely a multiplication operator on $\mathcal{H}_{\mathrm{ph}}$ which is nonnegative, self-adjoint and injective. We denote it by the same symbol ω also. The free Hamiltonian of the quantum radiation field is then defined by

$$H_{\mathrm{rad}} := d\Gamma(\omega), \tag{2.10}$$

the second quantization of ω. The operator H_{rad} is a nonnegative self-adjoint operator. The symbolical expression of H_{rad} is $H_{\mathrm{rad}} = \sum_{r=1}^{2} \int \omega(\boldsymbol{k}) a^{(r)}(\boldsymbol{k})^* a^{(r)}(\boldsymbol{k}) d\boldsymbol{k}$.

Remark 2.1 Usually ω is taken to be of the form $\omega_{\mathrm{phys}}(\boldsymbol{k}) := |\boldsymbol{k}|$, $\boldsymbol{k} \in \mathbf{R}^3$, but, in this note, for mathematical generality, we do not restrict ourselves to this case.

There exist $\mathbf{R}^3$-valued continuous functions $\mathbf{e}^{(r)}$ ($r = 1,2$) on the non-simply connected space $\mathbf{M}_0 := \mathbf{R}^3 \setminus \{(0,0,k_3) | k_3 \in \mathbf{R}\}$ such that, for all $\boldsymbol{k} \in \mathbf{M}_0$,

$$\mathbf{e}^{(r)}(\boldsymbol{k}) \cdot \mathbf{e}^{(s)}(\boldsymbol{k}) = \delta_{rs}, \quad \mathbf{e}^{(r)}(\boldsymbol{k}) \cdot \boldsymbol{k} = 0, \quad r, s = 1,2. \tag{2.11}$$

These vector-valued functions $\mathbf{e}^{(r)}$ are called the polarization vectors of one photon.

The time-zero quantum radiation field is given by

$$A_j(\boldsymbol{x}) := \sum_{r=1}^{2} \int d\boldsymbol{k} \frac{e_j^{(r)}(\boldsymbol{k})}{\sqrt{2(2\pi)^3\omega(\boldsymbol{k})}} \left\{ a^{(r)}(\boldsymbol{k})^* e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + a^{(r)}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right\}, \quad j = 1,2,3, \tag{2.12}$$

in the sense of operator-valued distribution.

Let ϱ be a real tempered distribution on $\mathbf{R}^3$ such that

$$\frac{\hat{\varrho}}{\sqrt{\omega}}, \quad \frac{\hat{\varrho}}{\omega} \in L^2(\mathbf{R}^3), \tag{2.13}$$

where $\hat{\varrho}$ denotes the Fourier transform of ϱ. The quantum radiation field with momentum cutoff $\hat{\varrho}$ is defined by

$$A_j(\boldsymbol{x};\varrho) := \Phi_{\mathrm{S}}(G_j^{\varrho}(\boldsymbol{x})) \tag{2.14}$$

with $G_j^{\varrho} : \mathbf{R}^3 \to \mathcal{H}_{\mathrm{ph}}$ given by

$$G_j^{\varrho}(\boldsymbol{x})(\boldsymbol{k}) := \left(\frac{\hat{\varrho}(\boldsymbol{k})^* e_j^{(1)}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}}{\sqrt{\omega(\boldsymbol{k})}}, \frac{\hat{\varrho}(\boldsymbol{k})^* e_j^{(2)}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}}{\sqrt{\omega(\boldsymbol{k})}} \right).$$

Symbolically $A_j(\boldsymbol{x};\varrho) = \int A_j(\boldsymbol{x}-\boldsymbol{y})\varrho(\boldsymbol{y})d\boldsymbol{y}$.

2.3 The total Hamiltonian

The Hilbert space of state vectors for the coupled system of the Dirac particle and the quantum radiation field is taken to be

$$\mathcal{F} := \mathcal{H}_{\mathrm{D}} \otimes \mathcal{F}_{\mathrm{rad}}. \tag{2.15}$$

This Hilbert space can be identified as

$$\mathcal{F} = L^2(\mathbf{R}^3; \oplus^4 \mathcal{F}_{\mathrm{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\mathrm{rad}} d\boldsymbol{x} \tag{2.16}$$

the Hilbert space of $\oplus^4\mathcal{F}_{\mathrm{rad}}$-valued Lebesgue square integrable functions on $\mathbf{R}^3$ [the constant fibre direct integral with base space $(\mathbf{R}^3, d\boldsymbol{x})$ and fibre $\oplus^4\mathcal{F}_{\mathrm{rad}}$ [12, §XIII.6]. We freely use this identification. The total Hamiltonian of the coupled system is defined by

$$H(V,\varrho) := H_{\mathrm{D}}(V) + H_{\mathrm{rad}} - q\sum_{j=1}^{3} \alpha_j A_j(\,\cdot\,;\varrho). \tag{2.17}$$

This is called a *Dirac-Maxwell operator* [5]. The self-adjointness of $H(V,\varrho)$ is discussed in [4]. Here we present only a self-adjointness result in a restricted case.

We assume the following:

Hypothesis (A)

(A.1) V is essentially bounded on $\mathbf{R}^3$.

(A.2) For $s = -1, 1/2$, $\omega^s \hat{\varrho} \in L^2(\mathbf{R}^3)$ and $|\boldsymbol{k}|\hat{\varrho}/\omega$, $|\boldsymbol{k}|\hat{\varrho}/\sqrt{\omega} \in L^2(\mathbf{R}^3)$.

Theorem 2.1 [4, Theorem 1.4] *Let $\mathcal{D}$ be a core of ω and $\mathcal{F}_{\mathrm{rad}}^{\mathrm{fin}}(\mathcal{D})$ be the subspace algebraically spanned by vectors of the form $a(F_1)^* \cdots a(F_n)^*\Omega$, $n \geq 0, F_j \in \mathcal{D}, j = 1, \cdots, n$, where $\Omega := \{1, 0, 0, \cdots\} \in \mathcal{F}_{\mathrm{rad}}$ is the Fock vacuum of $\mathcal{F}_{\mathrm{rad}}$. Then, under Hypothesis (A), $H(V, \varrho)$ is essentially self-adjoint on $[\oplus^4 C_0^\infty(\mathbf{R}^3)] \otimes_{\mathrm{alg}} \mathcal{F}_{\mathrm{rad}}^{\mathrm{fin}}(\mathcal{D})$, where $\otimes_{\mathrm{alg}}$ means algebraic tensor product.*

We denote the closure of $H(V, \varrho)$ by the same symbol.
The problem we consider here is stated as follows:

Problem

Find a family $\{H_\kappa(V, \varrho)\}_{\kappa \geq 1}$ of self-adjoint operators on $\mathcal{F}$ which are obtained by scaling prameters contained in $H(V, \varrho)$ with $H_\kappa(V, \varrho)|_{\kappa=1} = H(V, \varrho)$, a family $\{E(\kappa)\}_{\kappa \geq 1}$ of self-adjoint operators on $\mathcal{F}$, a unitary operator U on $\mathcal{F}$, a symmetric operator V_{eff} on $\mathcal{H}_{\mathrm{D}}$ and an orthogonal projection P acting on $\mathcal{F}_{\mathrm{rad}}$ such that, for all $z \in \mathbf{C} \setminus \mathbf{R}$

$$\text{s-}\lim_{\kappa \to \infty} (H_\kappa(V, \varrho) - E(\kappa) - z)^{-1} = U[(H_{\mathrm{D}}(V_{\mathrm{eff}}) - z)^{-1} \otimes P]U^{-1}. \tag{2.18}$$

This kind of limit is called a *scaling limit*. The change of the potential $V \to V_{\mathrm{eff}}$ corresponds to taking out effects of the quantum radiaiton field on the Dirac particle on a quantum particle mechanics level. The operator $E(\kappa)$ is a renormalization of $H_\kappa(V, \varrho)$, which may be divergent as $\kappa \to \infty$ in the sense that there exists a common subset $\mathcal{D} \subset D(E(\kappa))$ for all sufficiently large κ such that, for all $\psi \in \mathcal{D}$, $\|E(\kappa)\psi\| \to \infty$ $(\kappa \to \infty)$. The operators V_{eff} and $H_{\mathrm{D}}(V_{\mathrm{eff}})$ are called an *effective potential* and an *effective Hamiltonian* respectively. One may expect that $H_{\mathrm{D}}(V_{\mathrm{eff}})$ describes interaction effects of the quantum radiation field on the Dirac particle.

Remark 2.2 It has been shown that, in nonrelativistic QED, scaling limits indeed give interaction effects of the quantum radiation field on non-relativistic charged particles confined in a potential [1, 7, 8, 9].

3 Decomposition of the α-matrices and the Zitterbewegung

Let

$$H_{\mathrm{D}} := H_{\mathrm{D}}(0) = \boldsymbol{\alpha} \cdot \boldsymbol{p} + m\beta.$$

It is well-known [13] that $H_{\rm D}$ is bijective with

$$H_{\rm D}^{-1} = H_{\rm D}(\boldsymbol{p}^2 + m^2)^{-1} = H_{\rm D}(-\Delta + m^2)^{-1},$$

where $\Delta := \sum_{j=1}^{3} D_j^2$ is the generalized 3-dimensional Laplacian. Hence we can define for $j = 1, 2, 3$

$$\begin{aligned} \bar{\alpha}_j &:= p_j H_{\rm D}^{-1}, & (3.1) \\ \tilde{\alpha}_j &:= \alpha_j - p_j H_{\rm D}^{-1}, & (3.2) \end{aligned}$$

so that

$$\alpha_j = \bar{\alpha}_j + \tilde{\alpha}_j, \tag{3.3}$$

which gives a decomposition of α_j. The importance of the decomposition (3.3) lies in the facts stated in the following proposition:

Proposition 3.1 *For $j = 1, 2, 3$, $\bar{\alpha}_j$ and $\tilde{\alpha}_j$ are bounded self-adjoint operators on $\mathcal{H}_{\rm D}$ with*

$$\|\bar{\alpha}_j\| = 1, \quad \|\tilde{\alpha}_j\| = 1,$$

where, for a bounded linear operator T, $\|T\|$ denotes the operator norm of T. Moreover the following hold:

$$\begin{aligned} &[\bar{\alpha}_j, \bar{\alpha}_l] = 0, \quad \{\tilde{\alpha}_j, \bar{\alpha}_l\} = 0, & (3.4) \\ &[\bar{\alpha}_j, H_{\rm D}] = 0, \quad \{\tilde{\alpha}_j, H_{\rm D}\} = 0 \quad \text{on } D(H_{\rm D}), & (3.5) \\ &\{\tilde{\alpha}_j, \tilde{\alpha}_l\} = 2\delta_{jl} - 2p_j p_l (\boldsymbol{p}^2 + m^2)^{-1}, & (3.6) \\ &\bar{\alpha}_j \bar{\alpha}_l = p_j p_l (\boldsymbol{p}^2 + m^2)^{-1}, & (3.7) \end{aligned}$$

As for self-adjoint operators, there exists a strong notion on commutativity and anticommutativity respectively:

Definition 3.2 Let A and B be self-adjoint operators on a Hilbert space.

(i) We say that A and B *strongly commute* if their spectral measures commute.

(ii) We say that A and B *strongly anticommute* if $Be^{itA} \subset e^{-itA}B$ for all $t \in \mathbf{R}$.

Property (3.5) holds in the strong form:

Proposition 3.3 *For each $j = 1, 2, 3$, $\bar{\alpha}_j$ and $H_{\rm D}$ strongly commute, and $\tilde{\alpha}_j$ and $H_{\rm D}$ strongly anticommute.*

We remark that strong commutativity and strong anticommutativity of self-adjoint operators allow one to develop rich functional calculi (see, e.g., [2] and references therein).

For a linear opeartor T on $\mathcal{H}_{\mathrm{D}}$ we define

$$T(t) := e^{itH_{\mathrm{D}}} T e^{-itH_{\mathrm{D}}}, \tag{3.8}$$

the Heisenberg operator of T with respect to the free Dirac operator H_{D}.

We have by Proposition 3.3

$$\bar{\alpha}_j(t) = \alpha_j, \quad \tilde{\alpha}_j(t) = e^{2itH_{\mathrm{D}}} \tilde{\alpha}_j = \tilde{\alpha}_j e^{-2itH_{\mathrm{D}}}. \tag{3.9}$$

Hence

$$\alpha_j(t) = \bar{\alpha}_j + \tilde{\alpha}_j e^{-2itH_{\mathrm{D}}}. \tag{3.10}$$

The second term on the right hand side corresponds to the so-called "Zitterbewegung" (e.g., [13, p.19]). One may call $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$ the macroscopic velocity of the free Dirac particle [10].

4 Results

As a first step to analyze the problem proposed in Section 2, we consider a simplified version of the total Hamiltonian $H(V, \varrho)$:

$$H := H_{\mathrm{D}}(V) + H_{\mathrm{rad}} - q \sum_{j=1}^{3} \alpha_j A_j(0; \varrho), \tag{4.1}$$

the Hamiltonain in the dipole approximation. Let

$$g_j := G_j^{\varrho}(0) = \left(\frac{\hat{\varrho}^* e_j^{(1)}}{\sqrt{\omega}}, \frac{\hat{\varrho}^* e_j^{(2)}}{\sqrt{\omega}} \right), \quad j = 1, 2, 3, \tag{4.2}$$

and

$$E_0 := -\frac{q^2}{2} \sum_{j,l=1}^{3} \bar{\alpha}_j \bar{\alpha}_l \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle = -\frac{q^2}{2} \sum_{j,l=1}^{3} p_j p_l (-\Delta + m^2)^{-1} \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle, \tag{4.3}$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product of $\mathcal{H}_{\mathrm{ph}}$.

For $\kappa \geq 1$, we define a scaled Hamiltonian $H(\kappa)$ by

$$H(\kappa) := H_{\mathrm{D}}(V) + \kappa H_{\mathrm{rad}} - q\kappa \sum_{j=1}^{3} \alpha_j A_j(0; \varrho). \tag{4.4}$$

Let

$$h_{jl} := \left\langle \frac{g_j}{\omega}, \frac{g_l}{\omega} \right\rangle = \int_{\mathbf{R}^3} \frac{|\hat{\varrho}(\boldsymbol{k})|^2}{\omega(\boldsymbol{k})^3} \left(\delta_{jl} - \frac{k_j k_l}{|\boldsymbol{k}|^2} \right) d\boldsymbol{k}, \tag{4.5}$$

provided that $\hat{\varrho}/\omega^{3/2} \in L^2(\mathbf{R}^3)$, and

$$Q := \sum_{j,l=1}^{3} h_{jl}\bar{\alpha}_j\bar{\alpha}_l \tag{4.6}$$

Then we can define a bounded self-adjoint operator

$$V_{\text{eff}} := \sum_{n=0}^{\infty} \frac{q^{2n}}{2^n n!} \sum_{i_1,\cdots,i_n,j_1,\cdots,j_n=1}^{3} h_{i_1j_1}\cdots h_{i_nj_n}\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_n}e^{q^2Q/4}Ve^{-q^2Q/4}\bar{\alpha}_{j_1}\cdots\bar{\alpha}_{j_n} \tag{4.7}$$

on $\mathcal{H}_{\mathrm{D}}$. Note that the right hand side is convergent in operator norm with

$$\|V_{\text{eff}}\| \leq \|V\| e^{q^2\left(\sum_{j=1}^{3}\|g_j/\omega\|\right)^2}.$$

Let

$$U := e^{-iq\sum_{j=1}^{3}\bar{\alpha}_j\Phi_{\mathrm{S}}\left(\frac{ig_j}{\omega}\right)} \tag{4.8}$$

and P_0 be the orthogonal projection from $\mathcal{F}_{\text{rad}}$ onto the one-dimensional subspace $\{z\Omega | z \in \mathbf{C}\}$ spanned by the Fock vacuum $\Omega \in \mathcal{F}_{\text{rad}}$.

Theorem 4.1 *Assume Hypothesis (A) and $\hat{\varrho}/\omega^{3/2} \in L^2(\mathbf{R}^3)$. Let $z \in \mathbf{C}\setminus\mathbf{R}$. Then*

$$\text{s-}\lim_{\kappa\to\infty}\left(H(\kappa) - \kappa E_0 - \kappa\sum_{j=1}^{3}\tilde{\alpha}_jA_j(0;\varrho) - z\right)^{-1} = U(H_{\mathrm{D}}(V_{\text{eff}}) - z)^{-1} \otimes P_0U^{-1}. \tag{4.9}$$

This scaling limit corresponds to taking out effects coming from the interaction of the macroscopic velocity of the Dirac particle and the quantum radiation field.

We can also consider another scaled Hamiltonian. Let E_{D} be the spectral measure of the free Dirac operator H_{D} and, for a constant $L > 0$, set

$$H_{\mathrm{D}}^{L}(V) := E_{\mathrm{D}}([-L,\infty))H_{\mathrm{D}}E_{\mathrm{D}}([-L,\infty)) + V. \tag{4.10}$$

For a constant $s > 0$, we define

$$H_L(\kappa) := H_{\mathrm{D}}^{L}(V) + \kappa H_{\text{rad}} - q\kappa\sum_{j=1}^{3}\bar{\alpha}_jA_j(0;\varrho) - \frac{q}{\kappa^s}\sum_{j=1}^{3}\tilde{\alpha}_jA_j(0;\varrho). \tag{4.11}$$

Theorem 4.2 *Assume Hypothesis (A) and $\hat{\varrho}/\omega^{3/2} \in L^2(\mathbf{R}^3)$. Let $z \in \mathbf{C}\setminus\mathbf{R}$. Then*

$$\text{s-}\lim_{\kappa\to\infty}(H_L(\kappa) - \kappa E_0 - z)^{-1} = U(H_{\mathrm{D}}^{L}(V_{\text{eff}}) - z)^{-1} \otimes P_0U^{-1}. \tag{4.12}$$

Theorem 4.3 *Assume Hypothesis (A) and $\hat{\varrho}/\omega^{3/2} \in L^2(\mathbf{R}^3)$. Let $z \in \mathbf{C}\setminus\mathbf{R}$. Then*

$$\text{s-}\lim_{L\to\infty}\lim_{\kappa\to\infty}(H_L(\kappa) - \kappa E_0 - z)^{-1} = U(H_{\mathrm{D}}(V_{\text{eff}}) - z)^{-1} \otimes P_0U^{-1}. \tag{4.13}$$

Proofs of Theorems 4.1–4.3 will be given elsewhere [6].

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