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AN APPLICATION OF THE MOND-PEČARIĆ METHOD TO OPERATOR CONVEX FUNCTIONS

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ABSTRACT. As a converse of the arithmetic-geometric mean inequality, Specht estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m, M]$ with $M \geq m > 0$,

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n} \leq M_h(1) \sqrt[n]{x_1 \cdots x_n},$$

where $h = \frac{M}{m}$ and $M_h(1)$ is the Specht ratio.

In this report, we show some order relations between the arithmetic mean $A \nabla_\alpha B$, the power mean $(A^r \nabla_\alpha B^r)^{\frac{1}{r}}$ and the chaotically geometric mean $A \Diamond_\alpha B$ of positive operators A and B, i.e., $A \Diamond_\alpha B = e^{(1-\alpha)\log A + \alpha \log B}$ for $\alpha \in [0, 1]$. Among others, we show an operator version of Specht's theorem: If $0 < m \leq A,\ B \leq M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$, then

$$M_h(1)^{-1} A \Diamond_\alpha B \leq A \nabla_\alpha B \leq M_h(1) A \Diamond_\alpha B$$

holds for all $\alpha \in [0, 1]$.

1. INTRODUCTION

This report is based on the paper [2].

In 1960, as a converse of the arithmetic-geometric mean inequality, W.Specht [11] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m, M]$ with $M \geq m > 0$,

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n} \leq M_h(1) \sqrt[n]{x_1 \cdots x_n}, \tag{1}$$

where $h = \frac{M}{m} (\geq 1)$ is a generalized condition number in the sense of Turing [13] and the Specht ratio $M_h(1)$ is defined for $h \geq 1$ as

$$M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h > 1) \quad \text{and} \quad M_1(1) = 1.$$

It yields a rich hervest in operator theory. J.I.Fujii, S.Izumino and Y.Seo [1] showed an operator version of Specht's theorem (1): Let A be a positive operator on a Hilbert space H satisfying $0 < m \leq A \leq M$ for some scalars $0 < m < M$. Then

$$(Ax, x) \leq M_h(1) \exp(\log A\, x, x) \tag{2}$$

holds for every unit vector x in H. As a matter of fact, if we put $A = \operatorname{diag}(x_1, x_2, \cdots, x_n)$ and $x = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)$, then we have (1).

Also, we recall the geometric mean in the sense of Kubo-Ando theory [6]: For two positive operators A and B on a Hilbert space H, the geometric mean and arithmetic mean of A and B are defined as follows:

$$A \,\sharp_\alpha\, B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{and} \quad A \,\nabla_\alpha\, B = (1-\alpha)A + \alpha B$$

for $\alpha \in [0, 1]$. Like the numerical case, the arithmetic-geometric mean inequality holds:

$$A \,\sharp_\alpha\, B \leq A \,\nabla_\alpha\, B \qquad \text{for all } \alpha \in [0, 1]. \tag{3}$$

Tominaga [12] showed the following inequality as a reverse inequality of the noncommutative arithmetic-geometric mean inequality which differs from (2): Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $0 < \alpha < 1$. Then

$$A \,\nabla_\alpha\, B \leq M_h(1) A \,\sharp_\alpha\, B, \tag{4}$$

where $h = \frac{M}{m}$. It is considered as another operator version of Specht's theorem (1).

On the other hand, M.Fujii and R.Nakamoto discussed the monotonicity of a family of power means in [3] recently. For fixed $A,\ B > 0$, we put

$$F(r) = (A^r \,\nabla_\alpha\, B^r)^{\frac{1}{r}} \ (r \neq 0), \ = e^{\log A \,\nabla_\alpha\, \log B} \ (r = 0).$$

Then the power mean $F(r)$ is monotone increasing on $\mathbb{R}$ under the chaotic order $X \gg Y$, i.e., $\log X \geq \log Y$ for $X,\ Y > 0$, [3, Lemma 2]. In particular, $A \,\Diamond_\alpha\, B = e^{\log A \,\nabla_\alpha\, \log B}$ is called the chaotically α-geometric mean. In general, it does not concide with $A \,\sharp_\alpha\, B$.

In this report, we want to consider an operator version of Specht's theorem (1) on the chaotically geometric mean. We show some order relations between the power mean, the chaotically geometric mean and the arithmetic mean, which are based on the Mond-Pečarić method ([8, 9, 10]). As a result, we obtain Specht's type theorem on the chaotically geometric mean. Finally, we state an order relation between the geometric mean and the chaotically geometric one.

Concluding this section, we have to mention that almost all results in this report are based on our previous result [8, Corollary 4] coming from the Mond-Pečarić method. Namely this note might be understood as an application of the Mond-Pečarić method.

2. Results

Firstly, we shall show an order relation between the chaotically geometric mean and the arithmetic one, which is considered as another operator version of Specht's theorem (1).

Theorem 1. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$, $h = \frac{M}{m}$ and $0 < \alpha < 1$. Then*

$$\frac{1}{M_h(1)} A \Diamond_\alpha B \leq A \nabla_\alpha B \leq M_h(1) A \Diamond_\alpha B.$$

Though the power mean $F(s)$ converges to $A \Diamond_\alpha B$ as $s \to 0$ in the strong operator topology, it is not generally monotone increasing on $(0, 1]$ under the usual order. However, we have the following result.

Theorem 2. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $0 < \alpha < 1$. Then*

$$\frac{1}{M_h(1) M_h(s)^{1/s}} F(s) \leq A \Diamond_\alpha B \leq M_h(1) F(s) \qquad \text{for } s > 0,$$

where $h = \frac{M}{m}$ and $M_h(s) = M_{h^s}(1)$.

The power mean $F(r)$ is not monotone increasing on $(0, 1]$. So, we shall show an order relation between the operator function $F(s)$ and the arithmetic one.

Theorem 3. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$ and $0 < \alpha < 1$. Then*

$$K_+(h^r, \frac{1}{r})^{-1} F(r) \leq A \nabla_\alpha B \leq K_+(h^r, \frac{1}{r}) F(r) \qquad \textit{for } 0 < r < 1, \tag{5}$$

where $h = \frac{M}{m}$ and the Ky Fan-Furuta constant $K_+(h, r)$ ([5, 7]) is defined as

$$K_+(h, r) = \frac{(r-1)^{r-1}}{r^r} \frac{(h^r - 1)^r}{(h-1)(h^r - h)^{r-1}} \qquad \textit{for } r > 1. \tag{6}$$

Next, we shall investigate an order relation between the geometric mean and the chaotically geometric one.

Theorem 4. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$, $h = \frac{M}{m}$ and $0 < \alpha < 1$. Then*

$$\frac{1}{M_h(1)} A \,\sharp_\alpha\, B \leq A \,\Diamond_\alpha\, B \leq M_h(1)^2 A \,\sharp_\alpha\, B.$$

3. Preliminaries for proofs

We need some preliminaries in order to prove our results.

Let A be a positive operator on a Hilbert space H satisfying $0 < m \leq A \leq M$ for some scalars $0 < m < M$, and let $f(t)$ be a real valued continuous convex function on $[m, M]$. Mond and Pečarić [9] proved that

$$f((Ax, x)) \leq (f(A)x, x) \leq \lambda(m, M, f) f((Ax, x)) \tag{7}$$

holds for every unit vector $x \in H$, where

(8)

$$\lambda(m, M, f) = \max\left\{ \frac{1}{f(t)} \left(\frac{f(M) - f(m)}{M - m}(t - m) + f(m) \right); t \in [m, M] \right\}.$$

In fact, by the convexity of $f(t)$, we have $f(t) \leq \frac{f(M)-f(m)}{M-m}(t - m) + f(m)$ for all $t \in [m, M]$. Therefore, by the definition of $\lambda(m, M, f)$, it follows that

$$(f(A)x, x) \leq \frac{f(M) - f(m)}{M - m}((Ax, x) - m) + f(m) \leq \lambda(m, M, f) f((Ax, x))$$

holds for every unit vector $x \in H$ and hence we have (7).

The following result is a generalization of (7) and based on the idea due to Furuta's work [4, 5]. We here cite it for convenience:

Theorem A ([8]). *Let A_j $(j = 1, 2, \cdots, k)$ be positive operators on a Hilbert space H satisfying $0 < m \leq A_j \leq M$ for some scalars $0 < m < M$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$, and let $x_1, \cdots, x_k$ be vectors in H with $\sum_{j=1}^{k} \| x_j \|^2 = 1$. If $f(t)$ satisfies either* (i) $f(t) > 0$ *or* (ii) $f(t) < 0$ *on $[m, M]$, then*

$$\sum_{j=1}^{k}(f(A_j)x_j, x_j) \leq \lambda(m, M, f) f(\sum_{j=1}^{k}(A_j x_j, x_j)) \tag{9}$$

holds for $\lambda > 1$ in case (i), *or $0 < \lambda < 1$ in case* (ii), *where $\lambda = \lambda(m, M, f)$ is defined as* (8).

We note that Theorem A is a reverse inequality of the following known inequality, eg. [10]: Notation as in Theorem A and let $f(t)$ be a real valued continuous convex function on $[m, M]$. Then

$$f(\sum_{j=1}^{k}(A_j x_j, x_j)) \leq \sum_{j=1}^{k}(f(A_j)x_j, x_j). \tag{10}$$

For the power function $f(t) = t^p$, we know the following fact by Furuta [5], which is a reverse inequality of the Hölder-McCarthy inequality:

Theorem B . *Let A be a positive operator on a Hilbert space H satisfying $0 < m \leq A \leq M$ for some scalars $0 < m < M$ and put $h = \frac{M}{m}$. For each $p > 1$*

$$(A^p x, x) \leq K_+(h, p)(Ax, x)^p \tag{11}$$

holds for every unit vector $x \in H$ where the Ky Fan-Furuta constant $K_+(h, p)$ is defined as (6).

We obtain a complement of Theorem B by itself.

Lemma 1. *Assume that the conditions of Theorem B hold. If $0 < p < 1$, then*

$$K_+(h^p, \frac{1}{p})^{-p}(Ax, x)^p \leq (A^p x, x) \leq (Ax, x)^p$$

holds for every unit vector $x \in H$.

Proof. Since $0 < p < 1$, we have $1 < \frac{1}{p}$ and so Theorem B implies that

$$(A^{1/p}x, x) \leq K_+(h, 1/p)(Ax, x)^{1/p}.$$

Replacing A by A^p, we have $(Ax,x) \leq K_+(h^p, 1/p)(A^p x, x)^{1/p}$ and by raising both sides to the power p we obtain the desired result. □

Moreover, by Theorem B, Furuta [5] showed the following Kantorovich type order preserving inequality.

Theorem C . *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$. If $0 < A \leq B$, then*

$$A^p \leq K_+(h,p)B^p \qquad \text{for all } p \geq 1,$$

where $h = \frac{M}{m}$.

4. Reverse inequality on operator convexity

In this section, by virtue of Theorem A, we shall estimate the bounds of the operator convexity for convex functions.

Lemma 2. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$. If $f(t)$ is a real valued continuous convex function on $[m, M]$ such that $f(t) > 0$ on $[m, M]$, then for each $0 < \alpha < 1$*

$$\frac{1}{\lambda(m,M,f)} f(A\nabla_\alpha B) \leq f(A)\nabla_\alpha f(B) \leq \lambda(m,M,f) f(A\nabla_\alpha B), \tag{12}$$

where $\lambda(m, M, f)$ is defined as (8).

Proof. For each $0 < \alpha < 1$ and unit vector $x \in H$, put $A_1 = A$, $A_2 = B$, $x_1 = \sqrt{1-\alpha}x$ and $x_2 = \sqrt{\alpha}x$ in Theorem A. Then we have

$$(1-\alpha)(f(A)x,x) + \alpha(f(B)x,x) \leq \lambda(m,M,f) f((1-\alpha)(Ax,x) + \alpha(Bx,x)).$$

Hence it follows that

$$\begin{aligned}(((1-\alpha)f(A) + \alpha f(B))x,x) &\leq \lambda(m,M,f) f(((1-\alpha)A + \alpha B)x,x)) \\ &\leq \lambda(m,M,f)\,(f((1-\alpha)A + \alpha B)x,x)\end{aligned}$$

and the last inequality holds by the convexity of $f(t)$. Therefore we have

$$f(A)\nabla_\alpha f(B) \leq \lambda(m,M,f) f(A \nabla_\alpha B).$$

Next, since $f(t)$ is convex, it follows from (10) that

$$(1-\alpha)(f(A)x,x)+\alpha(f(B)x,x)\geq f((1-\alpha)(Ax,x)+\alpha(Bx,x)).$$

Since $0<m\leq(1-\alpha)A+\alpha B\leq M$, it follows from (7) that

$$\begin{aligned} f((1-\alpha)(Ax,x)+\alpha(Bx,x)) &= f(((A\ \nabla_\alpha\ B)x,x)) \\ &\geq \frac{1}{\lambda(m,M,f)}(f(A\ \nabla_\alpha\ B)x,x) \end{aligned}$$

holds for every unit vector $x\in H$. Therefore we have

$$\frac{1}{\lambda(m,M,f)}\, f(A\nabla_\alpha B)\leq f(A)\nabla_\alpha f(B).$$

□

We have the following complementary result of Lemma 2 for concave functions.

Lemma 3. *Let A and B be positive operators on a Hilbert space H satisfying $0<m\leq A,B\leq M$ for some scalars $0<m<M$. If $f(t)$ is a real valued continuous concave function on $[m,M]$ such that $f(t)>0$ on $[m,M]$, then for each $0<\alpha<1$*

$$\frac{1}{\mu(m,M,f)}\, f(A\nabla_\alpha B)\geq f(A)\nabla_\alpha f(B)\geq \mu(m,M,f)f(A\nabla_\lambda B), \tag{13}$$

where

$$\mu(m,M,f)=\min\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)\right);t\in[m,M]\right\}.$$

Next, consider the functions $f(t)=t^r$ on $[0,\infty)$. Then $f(t)$ is operator concave if $0\leq r\leq 1$, operator convex if $1\leq r\leq 2$ and $f(t)$ is not operator convex though $f(t)$ is convex if $r\geq 2$. By Lemmas 2 and 3, we obtain the reverse inequalities on operator convexity and operator concavity for $f(t)=t^r$.

Lemma 4. *Let A and B be positive operators on a Hilbert space H satisfying $0<m\leq A,B\leq M$ for some scalars $0<m<M$ and $0<\alpha<1$.*

(i) *If $0<r\leq 1$, then*

$$(A\nabla_\alpha B)^r\geq A^r\nabla_\alpha B^r\geq K_+(h^r,\frac{1}{r})^{-r}(A\nabla_\alpha B)^r.$$

(ii) *If $1\leq r\leq 2$, then*

$$(A\nabla_\alpha B)^r\leq A^r\nabla_\alpha B^r\leq K_+(h,r)(A\nabla_\alpha B)^r.$$

(iii) *If* $r > 2$, *then*

$$\frac{1}{K_+(h,r)}(A\nabla_\alpha B)^r \le A^r\nabla_\alpha B^r \le K_+(h,r)(A\nabla_\alpha B)^r,$$

where $h = \frac{M}{m}$ *and* $K_+(h,r)$*is defined as* (6).

Proof. Put $f(t) = t^r$ for $r > 1$ in Lemma 2, then we obtain $\lambda(m, M, f) = K_+(h,r)$. Also, in the case of $0 < r \le 1$, we have $\mu(m, M, f) = K_+(h^r, 1/r)^{-r}$ in Lemma 3. □

Though the power mean $F(r)$ is monotone increasing under the chaotic order, $F(r)$ is not monotone increasing for $0 < r < 1$ under the usual order. By virtue of Lemma 4, we see that $F(r)$ is monotone increasing for $r > 0$ in the following sense:

Lemma 5. *Let* A *and* B *be positive operators on a Hilbert space* H *satisfying* $0 < m \le A, B \le M$ *for some scalars* $0 < m < M$. *Let* $0 < r \le s$ *and* $0 < \alpha < 1$.
(i) *If* $0 < r \le 1$, *then*

$$K_+(h^r, \frac{1}{r})^{-1} K_+(h^r, \frac{s}{r})^{-1/s} F(s) \le F(r) \le K_+(h^r, \frac{1}{r}) F(s).$$

(ii) *If* $r \ge 1$, *then*

$$K_+(h^r, \frac{s}{r})^{-1/s} F(s) \le F(r) \le F(s),$$

where $h = \frac{M}{m}$ *and* $K_+(h,r)$ *is defined as* (6).

Proof. Since $0 < \frac{r}{s} \le 1$ and $0 < m^s \le A^s, B^s \le M^s$, we apply Lemma 4 to obtain the following inequality

$$(A^s\nabla_\alpha B^s)^{\frac{r}{s}} \ge A^r\nabla_\alpha B^r \ge K_+(h^r, \frac{s}{r})^{-\frac{r}{s}}(A^s\nabla_\alpha B^s)^{\frac{r}{s}}. \tag{14}$$

If $r \ge 1$, then $1 \ge \frac{1}{r} > 0$ and by raising both sides of (14) to the power $\frac{1}{r}$ it follows from the Löwner-Heinz Theorem that

$$(A^s\nabla_\alpha B^s)^{\frac{1}{s}} \ge (A^r\nabla_\alpha B^r)^{1/r} \ge K_+(h^r, \frac{s}{r})^{-\frac{1}{s}}(A^s\nabla_\alpha B^s)^{\frac{1}{s}}.$$

Also if $0 < r < 1$, then $\frac{1}{r} > 1$ and by raising both sides of (14) to the power $\frac{1}{r}$ it follows from Theorem B that

$$K_+(h^r, \frac{1}{r})(A^s\nabla_\alpha B^s)^{\frac{1}{s}} \ge (A^r\nabla_\alpha B^r)^{1/r} \ge K_+(h^r, \frac{1}{r})^{-1}K_+(h^r, \frac{s}{r})^{-\frac{1}{s}}(A^s\nabla_\alpha B^s)^{\frac{1}{s}}.$$

5. Proof of the results

Finally, we give proofs of Theorems stated in section 2.

Proof of Theorem 2.

By (ii) of Lemma 5, if $0 < r \leq s$ and $0 < r < 1$, then we have

$$K_+(h^r, \frac{1}{r})^{-1} K_+(h^r, \frac{s}{r})^{-1/s} F(s) \leq F(r) \leq K_+(h^r, \frac{1}{r}) F(s). \tag{15}$$

Since $\lim_{r \to +0} K_+(h^r, \frac{s}{r}) = M_h(s)$ which is shown in [14], we have the desired result as $r \to +0$ in (15):

$$\frac{1}{M_h(1) M_h(s)^{1/s}} F(s) \leq A \Diamond_\alpha B \leq M_h(1) F(s) \qquad \text{for } s > 0.$$

□

Proof of Theorem 3.

If $0 < r < 1$, then $\frac{1}{r} > 1$ and by (iii) of Lemma 4 it follows that

$$\frac{1}{K_+(h, \frac{1}{r})} (A \nabla_\alpha B)^{\frac{1}{r}} \leq A^{\frac{1}{r}} \nabla_\alpha B^{\frac{1}{r}} \leq K_+(h, \frac{1}{r}) (A \nabla_\alpha B)^{\frac{1}{r}}.$$

By replacing A and B by A^r and B^r respectively, we have

$$\frac{1}{K_+(h, \frac{1}{r})} (A^r \nabla_\alpha B^r)^{\frac{1}{r}} \leq A \nabla_\alpha B \leq K_+(h, \frac{1}{r}) (A^r \nabla_\alpha B^r)^{\frac{1}{r}}. \tag{16}$$

Therefore, we have the desired result:

$$K_+(h^r, \frac{1}{r})^{-1} F(r) \leq A \nabla_\alpha B \leq K_+(h^r, \frac{1}{r}) F(r).$$

□

Proof of Theorem 1.

If we put $r \to 0$ in (5) of Theorem 3, then it follows that $K_+(h^r, \frac{1}{r}) \to M_h(1)$ and $(A^r \nabla_\alpha B^r)^{\frac{1}{r}} \to A \Diamond_\alpha B$. Therefore we have

$$\frac{1}{M_h(1)} A \nabla_\alpha B \leq A \Diamond_\alpha B \leq M_h(1) A \nabla_\alpha B,$$

and hence

$$\frac{1}{M_h(1)} A \Diamond_\alpha B \leq A \nabla_\alpha B \leq M_h(1) A \Diamond_\alpha B.$$

□

Proof of Theorem 4.

It follows from (3),(4) and Theorem 1. □

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