Geometric approach to certain systems of conservation laws *

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Abstract

Our problem: Global theory for nonlinear hyperbolic equations and systems.
Difficulty: Appearance of singularities in finite time.
Our aim: Extension of solutions beyond the singularities.
Our method: Resolution of singularities, i.e., analysis in cotangent space.
Conclusion: Some questions on the mathematical theory to fluid mechanics.

1 Our program.

In this talk we will consider the Cauchy problem for certain systems of hyperbolic type. It is well known that the Cauchy problem with smooth data has a smooth solution in a neighbourhood of the initial curve, and that singularities appear generally in finite time. But, even if singularities may appear in solutions, physical phenomena can exist with the singularities. Moreover it seems to us that the singularities might cause various kinds of interesting phenomena. We are interested in the global theory for the above Cauchy problem. Therefore we would like to extend the solutions beyond their singularities. Then what happens ? This is the subject of this talk.

Let us consider the following equation:

$$F(x,y,z,p,q,r,s,t) = Ar + Bs + Ct + D(rt - s^2) - E = 0 \qquad (1.1)$$

where $p = \partial z/\partial x, q = \partial z/\partial y, r = \partial^2 z/\partial x^2, s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$. Here we assume that A, B, C, D and E are real smooth functions of (x,y,z,p,q).

Before stating our program, we will give some historical comments on the method of integration of (1.1). It had been first investigated by the french school, especially by G. Monge, G. Darboux [2] and E. Goursat [4, 5]. One of their principal methods is how to reduce the solvability of (1.1) to the integration of first order partial differential equations. For this approach they assumed strong conditions on equations. The first paper which succeeded to remove their assumptions is H. Lewy [10]. A little

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later, J. Hadamard [6] gave another proof. But their assumptions on the coefficients $\{A, B, C, D, E\}$ were a little strong. The best result on the existence and uniqueness theorem is given by P. Hartman and A. Wintner [7].

Now we review our method. Equation (1.1) is regarded as a smooth surface defined in eight dimensional space $\mathbf{R}^8 = \{(x, y, z, p, q, r, s, t)\}$. As p and q are first order derivatives of $z = z(x, y)$, we put the relation $dz = pdx + qdy$. Moreover, as r, s and t are second order derivatives of $z = z(x, y)$, we introduce the relations $dp = rdx + sdy$ and $dq = sdx + tdy$. Let us call $\{ dz = pdx + qdy, dp = rdx + sdy, dq = sdx + tdy \}$ the "contact structure of second order". A solution of (1.1) is regarded as a maximal integral sub-manifold of the contact structure of second order in the surface $\{(x, y, z, p, q, r, s, t) \in \mathbf{R}^8; F(x, y, z, p, q, r, s, t) = 0\}$. For lifting the solution into the cotangent space, we define equation (1.1) in the cotangent space. To do so, we decompose (1.1) as a product of one forms. Let λ_1 and λ_2 be the solutions of $\lambda^2 + B\lambda + (AC + DE) = 0$. Then, in the case where $D \neq 0$, equation (1.1) is written as follows:

$$\omega_1 \wedge \omega_2 = \varpi_1 \wedge \varpi_2 = D\left\{Ar + Bs + Ct + D(rt - s^2) - E\right\} dx \wedge dy \qquad (1.2)$$

where $\omega_1 = Ddp + Cdx + \lambda_1 dy, \omega_2 = Ddq + \lambda_2 dx + Ady, \varpi_1 = Ddp + Cdx + \lambda_2 dy$, and $\varpi_2 = Ddq + \lambda_1 dx + Ady$.

Definition 1.1 *A regular geometric solution of* (1.1) *is a submanifold of dimension* 2 *defined in* $\mathbf{R}^5 = \{(x, y, z, p, q)\}$ *on which it holds* $dz = pdx + qdy$ *and* $\omega_1 \wedge \omega_2 = 0$.

Therefore we must solve a Pfaff problem for getting a geometric solution. The Pfaffian problem was also well studied by many people, especially by E. Cartan in analytic space. A difference between their fundamental study and ours is that we consider it in C^∞-space. Therefore we need some condition which is corresponding to "hyperbolicity". Next we project the geometric solution to the base space and construct a reasonable weak solution which satisfies some physical conditions. This is our program for the global theory to nonlinear hyperbolic equations. In fact, we could establish the global theory for single first order partial differential equations ([15]) from this point of view and fill, by the classical characteristic method, the gap between the theory of viscosity solutions and the singularities of solutions.

This note is the brief outline of our recent researches. The detailed paper will be published elsewhere.

2 System of conservation laws.

Putting $u(x, y) = {}^t(p, q)$, $H(u) = {}^t(f(q), p)$ and $u_0(y) = {}^t(p_0(y), q_0(y))$, we consider the following Cauchy problem

$$\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}H(u) = 0 \qquad \text{in} \quad \{x > 0, \ y \in \mathbf{R}^1\}, \qquad (2.1)$$

$$u(0, y) = u_0(y) \qquad \text{on} \quad \{x = 0, \ y \in \mathbf{R}^1\} \quad . \qquad (2.2)$$

Suppose that $f(q)$ is in $C^2(\mathbf{R}^1)$. As we consider the hyperbolic case, we assume that $f'(q) > 0$. This is a system of conservation laws called as "p-system". It is well known that, even if the initial data are sufficiently smooth, singularities generally appear in the solution of (2.1)-(2.2). Therefore we will construct a "solution with singularities" called "weak solution". Let us recall here the definition of weak solutions of (2.1)-(2.2) introduced by P. D. Lax [8].

Definition 2.1 *A bounded and measurable 2-vector function* $u = u(x,y)$ *is a weak solution of* (2.1)-(2.2) *if it satisfies* (2.1)-(2.2) *in the weak sense, i.e.,*

$$\int_{\mathbf{R}^2_+} \{u(x,y)\frac{\partial \Phi}{\partial x}(x,y) - H(u)\frac{\partial \Phi}{\partial y}(x,y)\}dxdy + \int_{\mathbf{R}^1} u_0(y)\Phi(0,y)dy = 0 \qquad (2.3)$$

for any 2-vector function $\Phi(x,y) \in C_0^\infty(\mathbf{R}^2)$.

If there exists a weak solution $u = u(x,y)$ in the sense of the above definition, the second equation of (2.1) means that there exists a continuous function $z = z(x,y)$ satisfying $dz = pdx + qdy$ (see [14]). We substitute $p = \partial z/\partial x$ and $q = \partial z/\partial y$ into the first equation of (2.1), we get

$$F(q,r,t) = \frac{\partial^2}{\partial x^2}z - \frac{\partial}{\partial y}f(\frac{\partial z}{\partial y}) = r - f'(q)t = 0. \qquad (2.4)$$

This is a second order hyperbolic equation. Here we construct a geometric solution of (2.4) by using the result stated in §1. See [14] for the detailed proof.

Remark. We can directly introduce the notion of a "geometric solution" to (2.1)-(2.2) without going back to (1.1). See [14].

We can prove that the Cauchy problem (2.1)-(2.2) admits a regular geometric solution globally. Here we have used the terminology "globally" in the sense that the projection of the geometric solution to the base space $R^2 = \{(x,y)\}$ coincides with the whole space R^2. In this case the geometric solution is written down by

$$p + \Lambda(q) = \psi_1(\beta) \quad \text{and} \quad p - \Lambda(q) = \psi_2(\alpha)$$

where $\Lambda(q) = \sqrt{f'(q)}$, $\Lambda'(q) \equiv \lambda(q)$, $\psi_1(\beta) = p_0(\beta) + \Lambda(q_0(\beta))$, and $\psi_2(\alpha) = p_0(\alpha) - \Lambda(q_0(\alpha))$. Here $x = x(\alpha, \beta)$ and $y = y(\alpha, \beta)$ are the solutions of the following Cauchy problem:

$$\lambda(q)\frac{\partial x}{\partial \alpha} + \frac{\partial y}{\partial \alpha} = 0, \quad -\lambda(q)\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \beta} = 0, \qquad (2.5)$$

$$x(\alpha, \alpha) = 0, \quad y(\alpha, \alpha) = \alpha. \qquad (2.6)$$

As system of equations (2.5) is essentially a system of linear wave equations concerning $x = x(\alpha, \beta)$ and $y = y(\alpha, \beta)$, the Cauchy problem (2.5)-(2.6) has globally a classical solution. We have believed that a weak solution of (2.1)-(2.2) would be constructed by the projection of the geometric solution to the base space. In fact, we could do

so for first order partial differential equations. See [15]. We think that R. Thom [11] developed the theory of catastrophe to understand the natural phenomena from this point of view. But we have proved the following

Theorem 2.2 ([14]) *Assume that $f'(q) > 0$ and $f''(q) \neq 0$. Then we can not generally construct a single-valued weak solution of the Cauchy problem (2.1)-(2.2) by cutting off some part of the above geometric solution.*

After getting this result, we have come back to the starting point and have reconsidered the fundamental notions used in the "mathematical theory on fluid mechanics". In the following section we will present several questions on this subject.

3 Rarefaction waves.

Many people have considered the Cauchy problem (2.1)-(2.2) in various kinds of functional spaces, for example $L^1(\Omega)$, $L^1_{loc}(\Omega)$, etc, etc. The most well-known method would be the differnce method. For using this method, they have reduced (2.1)-(2.2) to Riemann problem. For solving it, they have introduced "rarefaction wave".

Example. Let $u = u(t,x)$ be a scalar function and consider the Cauchy problem as follows:

$$\frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = 0, \tag{3.1}$$

$$u(0,y) = \begin{cases} u_+ & (y > 0) \\ u_- & (y < 0) \end{cases} \tag{3.2}$$

where u_+ and u_- are constants. In the case where $u_+ < u_-$, the weak solution is given by

$$u(x,y) = \begin{cases} u_+ & \text{in } \{(x,y); y > \gamma x\} \\ u_- & \text{in } \{(x,y); \gamma x > y\} \end{cases} \tag{3.3}$$

where $\gamma = (u_+ + u_-)/2$. The jump discontinuity along the line $y = \gamma x$ is called a "shock wave", or simply "shock". Concerning this point, we do not have any question. If $u_+ > u_-$, then the solution accepted in the mathematical theory is given by

$$u(x,y) = \begin{cases} u_+ & \text{in } \{(x,y); y > u_+ x\} \\ y/x & \text{in } \{(x,y); u_+ x > y > u_- x\} \\ u_- & \text{in } \{(x,y); u_- x > y\} \end{cases} \tag{3.4}$$

This is called as "centred rarefaction wave" or simply "rarefaction wave". This solution has been well accepted in the mathemathecal theory for first order partial equations. One of the reasons comes from the viscosity theory. Let us add the viscosity term to (3.1) as follows:

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = \mu\frac{\partial^2 u}{\partial x^2} \tag{3.5}$$

where μ is a positive constant. Write the solution of (3.5)-(3.2) by $u_\mu(x,y)$. When μ tends to $+0$, $u_\mu(x,y)$ converges to the function given by (3.4). This means that, though the initial data has a jump discontinuity, it disappears immediately. It seems to us that this phenomenanon would be strange in the mechanics of continuum. In this sense, equation (3.5) also might be inappropriate for the description of continuum mechanics. Then, how should we understand this ?

In the mathematical model of continuum mechanics, we think that the jump discontinuity would change continuously, that is to say that it would gradually become small and finally disappear. It is possible to construct such a solution. Assume $u_+ > u_-$. Let T be any positive constant. We define a function $\varphi_T(y)$ by

$$\varphi_T(y) = \begin{cases} u_+\,, & y > u_+T \\ \\ u_-\,, & y < u_-T \end{cases} \tag{3.6}$$

where $\varphi_T(y)$ is in $C^2(R^1)$ and $\varphi_T'(y) > 0$ for $y \in (u_-T, u_+T)$. We put the initial condition at a time $x = T$ by $u(T,y) = \varphi_T(y)$. We consider the Cauchy problem for (3.1) under this initial condition. Then, though the solution of this Cauchy problem has the jump discontinuity in a neighbourhood of $x = 0$, the singularity disappears in finite time x_0 and the solution becomes smooth for all $x > x_0$. But this solution has several inappropriate properties. The first one is that it does not satisfy the entropy condition introduced by Lax [8]. Therefore the uniqueness of weak solution does not follow. If we may accept the above solution, we must introduce a new "entropy condition".

The principal reason why we insist the above thing is that, if the equation might describe the mechanics of continuum, the jump discontinuity should change continuously.

Remark. In the case where $u_+ > u_-$, we define a function $\varphi_T(y)$ by

$$\varphi_T(y) = \begin{cases} u_+\,, & y > u_+T \\ \\ y/T\,, & u_+T > y > u_-T \\ \\ u_-\,, & y < u_-T \end{cases} \qquad . \tag{3.7}$$

Then the solution of the Cauchy problem for (3.1) with $u(T,y) = \varphi_T(y)$ is equal to the rarefaction wave (3.3). Therefore the rarefaction wave is obtained if we combine, in the initial data $\varphi_T(y)$, two points $(x,y,u) = (T, u_+T, u_+)$ and $(x,y,u) = (T, u_-T, u_-)$ by a straight line. If we may combine these two point smoothly and monotonely, we get a weak solution whose jump discontinuity changes continuously.

4 Piecewise smooth solutions.

We suppose that the initial data are sufficiently smooth. What kinds of solutions should we look for ? Many people believe that the weak solution would become piecewise smooth. Recently we have found a paper [9] on this subject. First we write his result. We consider the Cauchy problem (2.1)-(2.2). For the diagonalization of (2.1), we introduce the Riemann invariants, that is to say, we do the change of dependent variables as follows:

$$w = (p - \Lambda(q))/2, \quad v = (p + \Lambda(q))/2. \tag{4.1}$$

Then w and v satisfy

$$\frac{\partial w}{\partial x} + h(v - w)\frac{\partial w}{\partial y} = 0, \quad \frac{\partial v}{\partial x} - h(v - w)\frac{\partial v}{\partial y} = 0. \tag{4.2}$$

where $h(v - w) = \lambda(q)$. Assume that the initial data satisfy

$$w(0, y) = c, \quad v(0, y) = v_0(y) \tag{4.3}$$

where c is a constant. Then M. -P. Lebaud [9] proved the following

Theorem 4.1 ([2]) *Une solution faible entropique de* (4.2)-(4.3) *présente un choc au point* (x_0, y_0) *le long de la courbe* $y = \gamma(x)$. *La courbe de choc* $y = \gamma(x)$ *est de classe* C^1 *et les fonctions z et w sont continues à droite et à gauche de celle-ci.*

In Theorem 4.1, (x_0, y_0) is a starting point of shock which is determined by the initial data. The proof of this theorem is very long and complicate. In fact, the author spent almost all the pages of [9] to prove Theorem 4.1. But, if we might consider this problem from our point of view, we can understand the result easily. As $w = w(x, y)$ is one of Riemann invariants, $w(x, y) = c$. Therefore $v = v(x, y)$ satisfies the single consevation law as follows:

$$\frac{\partial v}{\partial x} - h(v - c)\frac{\partial v}{\partial y} = 0. \tag{4.4}$$

For (4.4), we can easily construct a weak solution which satisfies the entropy condition introduced by Lax. See [15]. Hence we get "une solution faible entropique de (4.2)-(4.3)".

After the statement of this theorem, the author added the explanations as follows: La solution de (2.1) construite vérifie les conditions de Lax [1]. This is wrong. This says, as it is well known, that the notion of "weak solution" is not conserved by the change of variables. We will briefly explain the meaning of this problem. As the preparation for this, we will give several lemmas concerning the Cauchy problem (2.1)-(2.2).

Lemma 4.2 *Assume that* $f'(q) > 0$ *and* $f''(q) \neq 0$, *and that the initial data* $p_0(y)$ *and* $q_0(y)$ *are in* $C^1(R^1)$. *Then the Cauchy problem* (2.1)-(2.2) *has uniquely a* C^1-*solution in a neighbourhood of* $x = 0$.

Proof. First we get $p = p(\alpha, \beta)$ and $q = q(\alpha, \beta)$ by solving (2.5) with respect to p and q. Next we get $x = x(\alpha, \beta)$ and $y = y(\alpha, \beta)$ by solving the Cauchy problrm

(2.5)-(2.6). Then the Jacobian $D(x,y)/D(\alpha,\beta)$ does not vanish in a neighbourhood of $\{(\alpha,\alpha); \alpha \in R^1\}$, i.e., in a neighbourhood of $x = 0$. In fact it holds from (2.6)

$$\frac{\partial x}{\partial \alpha}(\alpha,\alpha) + \frac{\partial x}{\partial \beta}(\alpha,\alpha) = 0, \quad \frac{\partial y}{\partial \alpha}(\alpha,\alpha) + \frac{\partial y}{\partial \beta}(\alpha,\alpha) = 1. \tag{4.5}$$

From (2.5) and (4.5) it follows

$$\frac{\partial x}{\partial \alpha}(\alpha,\alpha) = -\frac{1}{2\lambda(q)}, \quad \frac{\partial x}{\partial \beta}(\alpha,\alpha) = \frac{1}{2\lambda(q)}, \quad \frac{\partial y}{\partial \alpha}(\alpha,\alpha) = \frac{\partial y}{\partial \beta}(\alpha,\alpha) = \frac{1}{2}.$$

Hence we get $(D(x,y)/D(\alpha,\beta))(\alpha,\alpha) = -\{2\lambda(q)\}^{-1} \neq 0$.

Therefore we can solve the system of equations $\{x = x(\alpha,\beta), y = y(\alpha,\beta)\}$ with respect to α and β. Substituting $\alpha = \alpha(x,y)$ and $\beta = \beta(x,y)$ into $p = p(\alpha,\beta)$ and $q = q(\alpha,\beta)$, we obtain the solutions of (2.1)-(2.2). For the proof of the uniqueness of solution, we write (2.1) in a diagonal form as (4.2) by using (4.1). Concerning the system written in the form as (4.2), T. Wazewski [16] already proved the local uniqueness of C^1-solution.

Lemma 4.3 *Equation* (2.1) *has the property that the propagation speed is finite.*

Proof. $p = p(\alpha,\beta)$ and $q = q(\alpha,\beta)$ is obtained by (2.5). As (2.5) are linear wave equations, they have the property that the propagation speed is finite.

Remark. Similar properties as Lemma 4.2 and Lemma 4.3 can be proved for general quasi-linear systems. See P. Hartman and A. Wintner [7]. Though it seems to the author that this paper has been forgotten almost completely, it contains the fundamental results concerning nonlinear hyperbolic equations and systems of two variables.

Theorem 4.4 *Assume that* $f'(q) > 0$ *and* $f''(q) \neq 0$, *and that the initial data* $p_0(y)$ *and* $q_0(y)$ *are in* $C^1(R^1)$. *Let* $u = u(x,y)$ *be a piecewise smooth function. If the number of curve on which* $u(x,y)$ *has a jump discontinuity is only one,* $u = u(x,y)$ *can not generally become a weak solution of* (2.1)-(2.2).

Idea of the proof. As we have informed in §2, the Cauchy problem (2.1)-(2.2) has a regular geometric solution globally. We project the geometric solution to the base space. In a neighbourhood of a point where the projection is regular, the projected solution becomes a classical solution of (2.1). Moreover Lemma 4.2 assures the uniqueness of the classical solution. Therefore, the solution which satisfies the condition stated in the theorem must be obtained by the projection of the geometric solution. But we have already proved in [14] that a weak solution of (2.1) can not be generally obtained by cutting off some part of the geometric solution. In fact we can construct such an example. Hence we get the conclusion of this theorem.

What we would like to propose: The most typical mathematical method for solving the problem of singularity would be the "resolution of singularity". We think that, even in physical problems, the phenomena of singularities would be understood from the point of view of "resolution of singularity".

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