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Core and Equilibria in Non-convex Economies

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Abstract

In this paper, we consider large economies in which both consumption and production sets may be non-convex. We show that each economy is approximated by a sequence of economies having equilibria. We prove that the core equivalence is also a dense property. Neither desirability assumption nor free-disposal is needed for these results. If we allow perfectly indivisible commodities, some desirability conditions are needed for our purposes.

1 Introduction.

In this paper, we consider large economies in which both consumption and production sets may be non-convex. In particular, we do not impose any desirability assumptions on preferences or free-disposability on productions. Our purpose is to show that each economy can be approximated in an appropriate topology by a sequence of economies having equilibria. Moreover, it will be shown that the core equivalence is also a dense property.

In our model, it is assumed that each consumer is also an individual producer and there is no production sector independent of the consumption side.[1] It is well-known that in coalition production economies, the production process can be decomposed into individual consumers under some plausible assumptions. Therefore, our model is comparable with the model of Hildenbrand (1974). In Hildenbrand (1974), the individual production set correspondence is closed and convex-valued with measurable graph and satisfies free disposability. In this paper, we do not impose the convexity and free disposability of individual production sets, while a slightly stronger measurability condition is required on the production set correspondence. Hence, there is no logical relationship between these two models.

In our model, however, we have a lot of difficulties in establishing the existence of an equilibrium in contrast to Hildenbrand (1974). At first, without free disposability, the multi-dimensional Fatou's lemma is not applicable so that the standard method would be useless. Second, production sets may be non-convex, which may complicate the arguments. Third, the non-convexity of consumption sets is another source of difficulty. A non-convex consumption set may result a discontinuous excess demand. An excess demand may be discontinuous if the

[1]Similar models have been analyzed by many authors such as Rashid (1978), Greenberg, Shitovitz and Wieczorek (1979), Suzuki (1995) and others.

budget line contains points having no local cheaper point. In particular, if an isolation point of the consumption set is on the budget line, the demand may have a critical jump.

Yamazaki (1978a) shows that the set of incomes at which the corresponding budget line contains such critical points is at most countable. Then, if the distribution of endowments is dispersed and the set of agents having a particular income is negligible, one can obtain a continuous mean demand function in exchange economies. (See, also Mas-Colell (1977a).) In economies with production, however, the dispersed endowment distribution is not sufficient for the existence of equilibrium. The income of each agent comes from the sales of outputs as well as endowments. Then, even if the endowment distribution is dispersed, the income distribution may concentrate on a particular value and there may exist non-negligible agents having discontinuous demand, which may result non-existence of equilibrium. This possibility is recently pointed out by Suzuki (1995).

Despite these observations, we can show that the set of production economies having equilibria is dense in the α-topology used by Hildenbrand (1974) and Mas-Colell (1977b). In the first model, we consider economies in which endowments are implicit in order to focus on profit distribution. We prove that each production set is approximated by a compact set. An economy with compact production sets is approximated by a simple economy. Finally, we show that each simple economy having compact production sets is approximated by an economy having dispersed profit distribution. The equilibrium existence is obtained in this case.

In our arguments in the first model, we allow perturbations of production sets in all directions. This may exclude the existence of perfectly indivisible commodities. In the second model, we would like to consider the non-convexity resulting from indivisibilities. But, we need some additional desirability assumptions on preferences which restricts the structure of the consumption set. Because we do not impose free-disposability, this would be the minimum cost that we must pay to obtain positive results in large economies. We will show that the existence of equilibrium is also a dense property in this setting.

In the final section, it will also be shown that the core equivalence is dense in a fairly large class of non-convex economies.

The paper is constructed as follows. The next section summarizes some mathematics that we need in the subsequent sections. The third section presents our first model. The forth section gives the second model. The final section discusses the core equivalence.

2 Mathematical Preliminaries.

In this section, we collect mathematical results which will be used in the subsequent sections. We start from the following definition.

Definition 1. A function f from a measurable space $(A, \mathscr{A})$ into a set X is called *simple* if there exists a finite measurable partition $\{A_k\}_{k=1}^{K}$ of A such that f is constant over A_k for each $k = 1, \cdots, K$.

The first result is as follows.

Proposition 2.1. *Let f be a Borel measurable function from a probability space $(A, \mathscr{A}, \nu)$ into a separable metric space (X, ρ). Then, there exists a sequence of simple functions $\{f_n\}$ converging to f a.e.*

Proof. Since X is separable, the range $f(A)$ of f is also separable. Then, $f(A)$ has a countable dense subset,

$$D(f) = \{x_1, x_2, \cdots, x_k, \cdots\} \subset f(A).$$

For each positive integer k, the function

$$A \ni a \to \rho(f(a), x_k) \in \mathbb{R}_+$$

is a measurable real-valued function. Hence, for each positive integer n and k, the set

$$B_{kn} = \{a \in A \mid \rho(f(a), x_k) < \frac{1}{n}\}$$

is measurable in A. Because $\{x_k\}_{k=1}^{\infty}$ is dense in $f(A)$, $A = \cup_{k=1}^{\infty} B_{kn}$. For each positive integer n, we define a countable measurable partition $\{A_{kn}\}_{k=1}^{\infty}$ of A by,

$$A_{1n} = B_{1n} \text{ and}$$
$$A_{kn} = B_{kn} \setminus \bigcup_{j=1}^{k-1} B_{jn} \text{ for all } k \geqq 2.$$

For each positive integer n, let us define a function f'_n by $f'_n(a) = x_k$ for $a \in A_{kn}$. Then, for each $a \in A$ and for each n, we have,

$$\rho(f(a), f'_n(a)) = \rho(f(a), x_k) < \frac{1}{n},$$

which implies $f'_n(a) \to f(a)$ for each $a \in A$. Now, we construct a sequence of simple functions $\{f_n\}$ converging to f a.e. using the sequence $\{f'_n\}$ that we have obtained. At first, for each positive integer n, we can find a positive integer K_n satisfying,

$$\nu(\bigcup_{j=1}^{K_n} A_{jn}) \geqq 1 - \frac{1}{n} \quad \text{and} \quad \bigcup_{j=1}^{K_n} A_{jn} \subset \bigcup_{j=1}^{K_{n+1}} A_{jn+1}.$$

Define f_n by,

$$f_n(a) = \begin{cases} f'_n(a) & \text{if } a \in \bigcup_{j=1}^{K_n} A_{jn}, \\ x_{K_n} & \text{otherwise.} \end{cases}$$

Let $C_n = A \setminus \cup_{j=1}^{K_n} A_{jn}$ and $C = \cap_{n=1}^{\infty} C_n$. Since $\nu(C) < \nu(C_n) < \frac{1}{n}$ for all n, $\nu(C) = 0$. Since for each $a \in A \setminus C$, there exists $\bar{n}$ such that $a \in \cup_{j=1}^{K_{\bar{n}}} A_{j\bar{n}}$, $a \in \cup_{j=1}^{K_n} A_{jn}$ for all $n \geq \bar{n}$. Therefore, by the definition of f_n, for all $n \geq \bar{n}$,

$$\rho(f(a), f_n(a)) = \rho(f(a), f'_n(a)),$$

which implies $\rho(f(a), f_n(a)) \to 0$ as $n \to \infty$. Because each f_n is simple, we may conclude that $\{f_n\}$ is the desired sequence. □

Proposition 2.2. *Let $(A, \mathscr{A}, \nu)$ be an atomless probability space. Then, there exists a Borel measurable function $f : A \to [0,1]$ such that $\nu \circ f^{-1}(\{s\}) = 0$ for all $s \in [0,1]$.*

Proof. Because $(A, \mathscr{A}, \nu)$ is atomless, for each positive integer n, there exists a finite measurable partition $\{A_{kn} \mid k = 1, 2, \dots, 2^n\}$ of A such that,

$$\nu(A_{kn}) = \frac{1}{2^n} \text{ for each } k = 1, 2, \dots, 2^n,$$
$$A_{(2k-1)(n+1)} \cup A_{(2k)(n+1)} = A_{kn} \text{ for each } k = 1, 2, \dots, 2^{n-1}.$$

For each positive integer n, let us define $f^n : A \to [0,1]$ by

$$f^n(a) = \frac{k}{2^n} \text{ for } a \in A_{kn}.$$

If $a \in A_{kn}$, then $f^{n+1}(a) = \frac{2k-1}{2^{n+1}}$ or $f^{n+1}(a) = \frac{2k}{2^{n+1}}$ because $a \in A_{(2k-1)(n+1)} \cup A_{(2k)(n+1)}$. Hence for each $a \in A$ and for each n, $f^n(a) \geqq f^{n+1}(a) \geqq 0$. Therefore, for each $a \in A$, $f(a) = \lim_{n\to\infty} f^n(a)$ is well-defined. Because it is a limit of simple functions, the function f is measurable.

Let $s \in [0,1)$ be given. For each positive integer n, there exists $k_n \in \{1, 2, \dots, 2^n\}$ such that

$$\frac{k_n}{2^n} \leqq s < \frac{k_n + 1}{2^n}.$$

If $f(a) = s$, then $\frac{k_n}{2^n} \leq f^n(a) \leqq \frac{k_n+1}{2^n}$ for all n. Indeed, if $f^n(a) < \frac{k_n}{2^n}$ for some n, then $s = f(a) \leqq f^n(a) < \frac{k_n}{2^n}$, which is a contradiction. On the other hand, if $k_n + 1 = 2^n$, then it is obvious that $f^n(a) \leqq \frac{k_n+1}{2^n} = 1$. Suppose that $k_n + 1 < 2^n$ and $\frac{k_n+1}{2^n} < f^n(a)$ for some n. Then, because $\frac{k_n+2}{2^n} \leqq f^n(a)$, $\frac{k_n+1}{2^n} \leqq f^m(a)$ for all $m \geqq n$. Therefore, $\frac{k_n+1}{2^n} \leq f(a) = s$, which is also a contradiction. Hence, it has been shown that

$$\{a \in A \mid f(a) = s\} \subset \{a \in A \mid \frac{k_n}{2^n} \leqq f^n(a) \leqq \frac{k_n + 1}{2^n}\}$$

for all n. Thus, for all n,

$$\nu(\{a \in A \mid f(a) = s\}) \leqq \frac{1}{2^{n-1}},$$

which implies $\nu(\{a \in A \mid f(a) = s\}) = 0$.

Finally, let $f(a) = 1$. Then, for all positive integer n, $\frac{2^n-1}{2^n} \leqq f^n(a)$. Otherwise, for some n, $f^n(a) < \frac{2^n-1}{2^n}$, which implies $1 = f(a) < \frac{2^n-1}{2^n} < 1$, a contradiction. Therefore, for all n,

$$\nu(\{a \in A \mid f(a) = 1\}) \leqq \frac{1}{2^{n-1}},$$

which implies $\nu(\{a \in A \mid f(a) = 1\}) = 0$. This completes the proof. □

Corollary 1. *Let $(A, \mathscr{A}, \mu)$ be an atomless measure space such that $\mu(A) > 0$. Then, there exists a Borel measurable function $f : A \to [0,1]$ such that $\mu \circ f^{-1}(\{s\}) = 0$ for each $s \in [0,1]$.*

Proof. For each $B \in \mathscr{A}$, let $\nu(B) = \frac{\mu(B)}{\mu(A)}$. Then, $(A, \mathscr{A}, \nu)$ is an atomless probability space. Applying proposition 2.2 to this probability space, we obtain a measurable function $f : A \to [0,1]$ such that $\nu \circ f^{-1}(\{s\}) = 0$ for each $s \in [0,1]$. This is the desired one since for each $B \in \mathscr{A}$, $\mu(B) = 0$ if and only if $\nu(B) = 0$. □

3 The First Model

In our model, there are ℓ commodities and hence the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$ is the underlying commodity space. The non-negative orthant of $\mathbb{R}^\ell$ is written as $\mathbb{R}^\ell_+$. The set of all agents is denoted by an atomless probability space $(A, \mathscr{A}, \nu)$. The consumption set of agent $a \in A$ is a closed subset X_a of $\mathbb{R}^\ell$. The preference of agent $a \in A$ is given by an irreflexive and transitive binary relation $\succ_a$ on X_a which is open in $X_a \times X_a$. The set of all pairs of a consumption set X and a preference relation $\succ$ on X is denoted by $\mathscr{P}$. Endowed with the topology of closed convergence, $\mathscr{P}$ is a separable metric space.

Each agent $a \in A$ is also an individual producer, whose production set is given by Y_a. Each production set Y_a is a closed subset of $\mathbb{R}^\ell$. Let $\mathscr{F}(\mathbb{R}^\ell)$ be the set of all closed subsets of $\mathbb{R}^\ell$. When no confusion arises, we denote $\mathscr{F}(\mathbb{R}^\ell)$ by $\mathscr{F}$. An economy is a function,

$$\mathscr{E} : (A, \mathcal{A}, \nu) \to \mathscr{P} \times \mathscr{F},$$

where $\mathscr{E}(a) = (X_a, \succ_a, Y_a)$ for each $a \in A$. We assume that the consumption sector $A \ni a \to (X_a, \succ_a) \in \mathscr{P}$ is a Borel measurable function and the production set correspondence Y_a satisfies the measurability in the following sense, that is, for any closed subset F of $\mathbb{R}^\ell$, the set $\{a \in A \mid Y_a \cap F \neq \emptyset\}$ is $\mathscr{A}$-measurable. We also assume that the correspondence $A \ni a \to X_a \cap Y_a$ admits a bounded selection b, i.e., b is a bounded function on A such that $b(a) \in X_a \cap Y_a$ for each $a \in A$.

Now, we introduce the definitions of quasi-equilibrium and of market equilibrium.

Definition 2. A *quasi-equilibrium* is a list (f, g, p) of integrable functions f and g from A into $\mathbb{R}^\ell$ and a price vector $p \neq 0$ satisfying the following conditions.

(1) For almost every $a \in A$, $p \cdot f(a) \leqq \sup p \cdot Y_a$ and $x \succ_a f(a)$ implies that $p \cdot x \geqq \sup p \cdot Y_a$.

(2) For almost every $a \in A$, $g(a) \in Y_a$ and $p \cdot g(a) = \sup p \cdot Y_a$.

(3) $\int f = \int g$.

Definition 3. A quasi-equilibrium (f, g, p) is a *market equilibrium* if for almost every $a \in A$, $f(a)$ is $\succ_a$-maximal in the budget set $\{x \in X \mid p \cdot x \leqq \sup p \cdot Y_a\}$ at p.

At first, we show that the consumption set correspondence X_a has a measurable graph. Because the projection $(X, \succ) \to X$ is continuous, the consumption set correspondence X_a is a Borel measurable function. Now, consider the set

$\mathfrak{F}_B = \{F \in \mathfrak{F} \mid F \cap B \neq \emptyset\}$ for an open set of $\mathbb{R}^\ell$. Because $\mathfrak{F}_B$ is open in the topology of closed convergence and X_a is Borel measurable,

$$\{a \in A \mid X_a \in \mathfrak{F}_B\} = \{a \in A \mid X_a \cap B \neq \emptyset\} \in \mathscr{A}.$$

Since X_a is closed, by proposition 4 in page 61 of Hildenbrand (1974), X_a has a measurable graph.

In order to discuss the relationship between a quasi-equilibrium and a market equilibrium, we need the following definition.

Definition 4. *The profit distribution is dispersed* if for any $p \neq 0$ and for any $w \in \mathbb{R}$,

$$\nu(\{a \in A \mid \sup p \cdot Y_a = w\}) = 0.$$

By an analogous argument as in Yamazaki (1978b), we may obtain the following result.

Theorem 3.1. *If the profit distribution is dispersed, then a quasi-equilibrium is a market equilibrium.*

The next existence theorem of a quasi-equilibrium will be a fundamental step towards our main result.

Theorem 3.2. *If the correspondences X_a and Y_a are integrably bounded, then there exists a quasi-equilibrium.*

Proof. Because X_a and Y_a are integrably bounded and closed-valued, they are compact-valued. Hence, the integrals $\int X_a$ and $\int Y_a$ are well-defined, compact and convex. For each $a \in A$ and for each price vector p in the unit disk $D = \{p \in \mathbb{R}^\ell \mid \|p\| \leqq 1\}$, where $\|p\|$ denotes the Euclidean norm of p, let

$$\begin{aligned} s(p,a) &= \{y \in Y_a \mid p \cdot y = \sup p \cdot Y_a\}, \\ \hat{B}(p,a) &= \{x \in X_a \mid p \cdot x \leqq \sup p \cdot Y_a + (1 - \|p\|)\}, \\ d(p,a) &= \{x \in \hat{B}(p,a) \mid x' \succ_a x \text{ implies } p \cdot x' > \sup p \cdot Y_a + (1 - \|p\|)\}. \end{aligned}$$

For each $a \in A$, $s(p,a)$ is well-defined and upper hemi-continuous in p. Because $X_a \cap Y_a \neq \emptyset$ for every $a \in A$, $d(p,a)$ is well-defined and upper hemi-continuous in p. The measurability of these correspondences also follows from the standard arguments. Because they are integrably bounded, $\int s(p,a) \neq \emptyset$ and $\int d(p,a) \neq \emptyset$. Therefore, the mean excess demand correspondence $\eta(p) = \int d(p,a) - \int s(p,a)$ is well-defined, compact and convex-valued and upper hemi-continuous in p. Let us define a correspondence $\varphi(p,x)$ from $D \times (\int X_a - \int Y_a)$ into itself as follows. If $\|x\| \neq 0$, let $\varphi(p,x) = \{\frac{x}{\|x\|}\} \times \eta(p)$ and if $\|x\| = 0$, let $\varphi(p,x) = D \times \eta(p)$. It is easy to see that $\varphi(p,x)$ is compact and convex-valued and upper hemi-continuous. By Kakutani's theorem, there exits $(p,x) \in D \times (\int X_a - \int Y_a)$ such that $(p^*,x^*) \in \varphi(p^*,x^*)$. By the same arguments as in Bergstrom (1975), we may prove that $\|p^*\| = 1$ and $x^* = 0$. By construction, there exist integrable functions f and g such that $f(a) \in d(p^*,a)$, $g(a) \in s(p^*,a)$ for almost every $a \in A$ and $\int f = \int g$. Therefore, the list (f, g, p^*) is a quasi-equilibrium. $\square$

Because we have assumed that for each closed subset F of $\mathbb{R}^\ell$, the weak inverse of F by Y_a is $\mathscr{A}$-measurable, the remark in page 61 of Hildenbrand (1974) implies that for each open set B of $\mathbb{R}^\ell$, the set $\{a \in A \mid Y_a \cap B \neq \emptyset\}$ is $\mathscr{A}$-measurable. Then, for each finite collection $\mathscr{B}$ of open subsets of $\mathbb{R}^\ell$ and a compact subset $K \subset \mathbb{R}^\ell$,

$$
\begin{aligned}
&\{a \in A \mid Y_a \cap K = \emptyset, \ \ Y_a \cap B \neq \emptyset \text{ for each } B \in \mathscr{B}\} \\
&= \bigcap_{B \in \mathscr{B}} \{a \in A \mid Y_a \cap B \neq \emptyset\} \cap \{a \in A \mid Y_a \cap K \neq \emptyset\}^c \in \mathscr{A}.
\end{aligned}
$$

Therefore, Y_a is a Borel measurable function from A into $\mathscr{F}$, where $\mathscr{F}$ is endowed with the topology of closed convergence. Conversely, if Y_a is a Borel measurable function in this sense, the weak inverse of a closed subset by Y_a is measurable. Therefore, the economy

$$\mathscr{E} : (A, \mathscr{A}, \nu) \to \mathscr{P} \times \mathscr{F}$$

is a Borel measurable function from A into a separable metric space. For a positive integer K, let

$$C^K = \{x \in \mathbb{R}^\ell \mid \|x\| \leqq K\}.$$

Let an economy $\mathscr{E} = \{(X_a, \succ_a, Y_a)\}_{a \in A}$ be given. We define a measurable function

$$\mathscr{E}^K : (A, \mathscr{A}, \nu) \to \mathscr{P} \times \mathscr{F}$$

by $\mathscr{E}^K(a) = (X_a^K, \succ_a^K, Y_a^K)$ where $X_a^K = X_a \cap C^K$, $\succ_a^K$ is the restriction of $\succ_a$ to X_a^K and $Y_a^K = Y_a \cap C^K$. For sufficiently large K, $\mathscr{E}^K$ is an economy and $\mathscr{E}^K(a) \to \mathscr{E}(a)$ for each $a \in A$ as $K \to \infty$. By proposition 2.1, each $\mathscr{E}^K$ can be approximated by an economy with finitely many values in almost everywhere convergence. Therefore, any given economy $\mathscr{E}$ can be approximated by a simple economy in which consumption sets and production sets are compact.

Proposition 3.1. *Let $Y : (A, \mathscr{A}, \nu) \to \mathscr{F}$ be a compact-valued simple function. Then, Y can be approximated in the sense of almost convergence by Y' whose profit distribution is dispersed.*

Proof. Since Y is a simple function, there exists a finite measurable partition $\{A_i \mid i = 1, \ldots, n\}$ of A such that $\nu(A_i) > 0$ and $Y_i = Y_a$ for each $a \in A_i$ and for each $i = 1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$ and for each $\varepsilon > 0$, let

$$Y_i^\varepsilon = \{y \in \mathbb{R}^\ell \mid d(y, Y_i) \leqq \varepsilon\}$$

where $d(y, Y_i)$ is the distance from y to the set Y_i. Then, define Y^ε by the following way.

$$Y_a^\varepsilon = Y_i^{\varepsilon f_i(a)} \text{ if } a \in A_i,$$

where $f_i : A_i \to [0, 1]$ is the function given by corollary 2.1. Then, the profit distribution determined by Y^ε is dispersed. Indeed, for each $p \in \mathbb{R}^\ell$ with $p \neq 0$ and for each $w \in \mathbb{R}$,

$$\nu(\{a \in A_i \mid \sup p \cdot Y_a^\varepsilon = w\}) = 0$$

for each $i = 1, 2, \ldots, n$ by the construction of f_i. It is obvious that $Y^\varepsilon \to Y$ everywhere as $\varepsilon \to 0$. This completes the proof. $\square$

Hence, by theorem 3.2, we may conclude that,

Theorem 3.3. *Each economy $\mathscr{E}$ can be approximated by an economy having a market equilibrium in the sense of almost everywhere convergence.*

4 The Second Model

In the previous model, we allow to perturb production sets in any directions. This may not be justified if there exists perfectly indivisible commodities. In this section, we would like to consider the existence of perfectly indivisible commodities. But, this requires us to introduce additional assumption on preferences which may restrict the non-convexity of the consumption set. Because we do not impose free disposability at all, this would be the minimum cost we must pay in the setting of large economies.

In the following, each consumer has a common consumption set X. Suppose that the ℓ-th commodity is perfectly divisible and the consumption set X always contains the ℓ-th commodity. Namely, the consumption set X is a closed subset of $\mathbb{R}^\ell$ bounded from below satisfying the following condition.

For any $x = (x_1, x_2, \cdots, x_{\ell-1}, x_\ell)$, if $x'_\ell \geq x_\ell$, then $(x_1, x_2, \cdots, x_{\ell-1}, x'_\ell) \in X$.

The preference relation $\succ \subset X \times X$ of each consumer satisfies the following additional assumptions.

(1) (Local Nonsatiation): For any $x \in X$ and for any neighborhood U of x, there exists $y \in U \cap X$ such that $y \succ x$.

(2) (Weak Desirability): For any $x \in X$ and any $i = 1, \cdots, \ell$, there exists $y \in X$ such that $y_i > x_i$, $y_j \leq x_j$ for all $j \neq i$ and $y \succ x$.

(3) (Overriding Desirability of the Divisible Commodity): For any $x, y \in X$, there exists $z \in X$ such that $z \succ y$ and $z_\ell > x_\ell$, $z_j \leq x_j$ for all $j \neq \ell$.

Let $\mathscr{P}^*$ be the set of all preference relations on X satisfying the above conditions. We also explicitly introduce the initial endowment of each agent. On the other hand, each individual production set Y is a closed subset of $\mathbb{R}^\ell$ such that $Y \cap \mathbb{R}^\ell_+ = \{0\}$. Let

$$\mathscr{F}^* = \{Y \in \mathscr{F} \mid Y \cap \mathcal{R}^\ell_+ = \{0\}\}.$$

Then, an economy $\mathscr{E}^*$ is a function,

$$\mathscr{E}^* : (A, \mathscr{A}, \nu) \to \mathscr{P}^* \times \mathbb{R}^\ell \times \mathscr{F}^*$$

which is Borel measurable. For each $a \in A$, $\mathscr{E}^*(a) = (\succ_a, e_a, Y_a)$ and assume that $e_a \in X$ for all $a \in A$, $\int e < +\infty$ and that there exist $\bar{x} \in \int X$ and $\bar{y} \in \int Y$ such that $\bar{x} \ll \int e + \bar{y}$. Note that $\int Y \neq \emptyset$ because $0 \in \int Y$ and $\int X = \mathrm{co}X$, where $\mathrm{co}X$ is the convex hull of X.

Definition 5. A *market equilibrium* for an economy $\mathscr{E}^*$ is a list (p, f, g) of a price vector $p \in S = \{p \in \mathbb{R}^\ell_+ \mid \sum_{i=1}^{\ell} p_i = 1\}$ and a pair of integrable functions (f, g) from A into $\mathbb{R}^\ell$ such that,

(i) $p \cdot f(a) \leq p \cdot e(a) + \sup p \cdot Y_a$ and $y \succ_a f(a)$ implies $p \cdot y > p \cdot e(a) + \sup p \cdot Y_a$ for a.e. $a \in A$.

(ii) $g(a) \in Y_a$ and $p \cdot g(a) = \sup p \cdot Y_a$ for a.e. $a \in A$.

(iii) $\int f = \int e + \int g$.

Definition 6. Given an economy $\mathscr{E}^*$, the endowment distribution is *dispersed* if for any price $p \in S$ and for any $w \in \mathbb{R}$,

$$\nu(\{a \in A \mid p \cdot e(a) = w\}) = 0.$$

Then, we have the following theorem.

Theorem 4.1. *Suppose that for an economy $\mathscr{E}^*$, the endowment distribution is dispersed and the production set correspondence Y is simple and compact valued. Then, $\mathscr{E}^*$ has a market equilibrium.*

Proof. For each $p \in S$ and $a \in A$, the budget set $B(p, a)$ is defined in the usual way. We define the weak demand set $d_w(p, a)$ by,

$$d_w(p, a) = \{x \in B(p, a) \mid y \succ_a x \text{ implies } p \cdot y \geq p \cdot e(a) + \sup p \cdot Y_a\}$$

For any given positive integer n, let $S_{\frac{1}{n}} = \{p \in S \mid p_i \geqq \frac{1}{n} \text{ for each } i = 1, \ldots, \ell.\}$. For each $p \in S_{\frac{1}{n}}$, $d_w(p, a)$ is non-empty and compact. It is not difficult to show that the individual weak demand correspondence,

$$S_{\frac{1}{n}} \ni p \longrightarrow d_w(p, a) \subset \mathbb{R}^\ell$$

is upper hemi-continuous. Let $\alpha = \max\{\sup p \cdot Y_a \mid p \in S_{\frac{1}{n}}, a \in A\} < +\infty$ and $\beta = \max\{\ |b_i| \mid i = 1, \cdots, \ell\}$, where $b = (b_1, \ldots, b_\ell)$ is the lower bound of the consumption set X. Define $\hat{e}(a) = \max\{|e_i(a)| \mid i = 1, \ldots, \ell\}$ for $a \in A$. $\hat{e}$ is integrable because e is integrable. For each $x \in d_w(p, a)$ and $i = 1, \ldots, \ell$, $x_i - b_i \geqq 0$. Hence,

$$\frac{1}{n}(x_i - b_i) \leqq p_i(x_i - b_i) \leqq \sum_{i=1}^{\ell}(p_i x_i - p_i b_i) \leqq \hat{e}(a) + \alpha + \beta.$$

Then,

$$-\beta \leqq x_i \leqq n(\hat{e}(a) + \alpha + \beta) + \beta$$

and thus $|x_i| \leqq h(a)$ for all $i = 1, \ldots, \ell$, where $h(a) \equiv \max\{n(\hat{e}(a) + \alpha + \beta) + \beta, \beta\}$, which is integrable. Therefore, for each $p \in S_{\frac{1}{n}}$, the correspondence,

$$A \ni a \longrightarrow d_w(p, a) \subset \mathbb{R}^\ell$$

is integrably bounded. Then, the mean weak demand $\int d_w(p, a)$ is non-empty. Therefore, the mean weak demand correspondence,

$$S_{\frac{1}{n}} \ni p \longrightarrow \int d_w(p, a) \subset \mathbb{R}^\ell$$

is upper hemi-continuous, non-empty, compact and convex valued.

On the other hand, it is relatively easy to show that the mean supply correspondence

$$S_{\frac{1}{n}} \ni p \longrightarrow \int s(p,a) \subset \mathbb{R}^\ell$$

is upper hemi-continuous, non-empty, compact and convex valued.

For each $p \in S_{\frac{1}{n}}$, define,

$$\eta^n(p) = \int d_w(p,a) - \int s(p,a) - \int e.$$

η^n is upper hemi-continuous, compact and convex valued and has a compact range. For any $z^n \in \eta^n(p)$, $p \cdot z^n = 0$ by the local non-satiation. Then, by the fixed point theorem of Gale and Nikaido, we have a sequence $\{p^n\}$ of prices and sequences $\{f^n\}$ and $\{g^n\}$ of selections from the individual weak demand $d_w(p^n,a)$ and the individual supply $s(p^n,a)$ respectively satisfying,

$$\int f^n - \int g^n - \int e \in S^\circ_{\frac{1}{n}}$$

for all n, where $S^\circ_{\frac{1}{n}} = \{x \in \mathbb{R}^\ell \mid p \cdot x \leqq 0 \text{ for all } p \in S_{\frac{1}{n}}\}$. Then, it is not difficult to show that there exist a price vector $p^* \in S$ and integrable functions f and g such that $p^n \to p^*$ and

(1) $f(a) \in d_w(p^*,a)$ a.e. $a \in A$,

(2) $g(a) \in s(p^*,a)$ a.e. $a \in A$,

(3) $\int f - \int g - \int e \leq 0$ and $p^* \cdot (\int f - \int g - \int e) = 0$.

Since we assume that $\bar{x} \ll \int e + \bar{y}$ for some $\bar{x} \in \int X$ and $\bar{y} \in \int Y$, $p^* \cdot \bar{x} < \int p^* \cdot e + p^* \cdot \bar{y} \leq \int p^* \cdot e + \int \sup p^* \cdot Y$. Therefore, there exists a subset $\bar{A}$ with $\nu(\bar{A}) > 0$ such that for each $a \in \bar{A}$, there exists $\tilde{x} \in X$ satisfying $p^* \cdot \tilde{x} < p^* \cdot e(a) + p^* \cdot Y_a$. By way of contradiction, let us suppose that $p^*_\ell = 0$. By overriding desirability, for each $a \in \bar{A}$, there exists $\tilde{z} \in X$ such that $\tilde{z} \succ_a f(a)$, $\tilde{z}_\ell > \tilde{x}_\ell$ and $\tilde{z}_j \leq \tilde{x}_j$ for all $j \neq \ell$. Since $p^*_\ell = 0$, $p^* \cdot \tilde{z} \leq p^* \cdot \tilde{x} < p^* \cdot e(a) + \sup p^* \cdot Y_a$, this contradicts the fact that $f(a) \in d_w(p^*,a)$ a.e. $a \in A$. Thus, it has been shown that $p^*_\ell > 0$.

Because the endowment distribution is dispersed, by the analogous way as in Yamazaki (1978a), $f(a)$ is $\succ_a$-maximal in the budget $B(p^*,a)$ for almost every $a \in A$. Finally, let us suppose that $p^*_j = 0$ for some $j \neq \ell$. By the weak desirability, for all $a \in A$, there exists $z \in X$ such that $z \succ_a f(a)$, $z_j > f_j(a)$ and $z_i \leq f_i(a)$ for all $i \neq j$. Since $p^* \cdot z \leq p^* \cdot f(a) = p^* \cdot e(a) + \sup p^* \cdot Y_a$, this is a contradiction. Therefore, $p^* \gg 0$ and it follows from (3) that $\int f = \int g + \int e$. This completes the proof. □

Theorem 4.2. *For each economy $\mathscr{E}^*$, there exists a sequence $\{\mathscr{E}^n\}$ of economies having an equilibrium which converges to $\mathscr{E}^*$ almost everywhere.*

Proof. Let $\mathscr{E}^*(a) = (\succ_a, e(a), Y_a)$ for $a \in A$. At first, there exists a sequence $\{\tilde{e}^n\}$ of simple functions converging to e a.e. Since $\tilde{e}^n(A) \subset e(A)$, $\tilde{e}^n(a) \in X$ for all $a \in A$. For each n, define,

$$e^n(a) = \tilde{e}^n(a) + (0, \cdots, 0, \frac{1}{n} f(a)),$$

where f is the function given in Proposition 2.2. By the assumption on X, $e^n(a) \in X$ for all $a \in A$. Let $p \in S$ with $p_\ell > 0$. For a finite partition $\{A_k\}$ of A, $\tilde{e}_n(a)$ is constant over each A_k. Because $p \cdot e_n(a) = p \cdot \tilde{e}_n(a) + (p_\ell/n)f(a)$, for any $w \in \mathbb{R}$ and for any k,

$$\nu(\{a \in A_k \mid p \cdot e_n(a) = w\}) = \nu(\{a \in A_k \mid f(a) = (n/p_\ell)(w - \tilde{w})\}) = 0$$

where $\tilde{w} \equiv p \cdot \tilde{e}(a)$ for all $a \in A_k$. Therefore, for any $p \in S$ with $p_\ell > 0$ and for any $w \in \mathcal{R}$,

$$\nu(\{a \in A \mid p \cdot e_n(a) = w\}) = 0.$$

It is easy to see that $e_n \to e$ a.e. and hence $\int e_n \to \int e$. On the other hand, by assumption, there exist $\bar{x} \in \int X$ and $\bar{y} \in \int Y$ such that $x \ll \int e + y$. Then, there exists an integrable selection y from Y such that $\bar{y} = \int y$. Since $y(a) \in Y_a$ a.e., there exists a set A_0 of ν-measure zero such that $y(a) \in Y_a$ for all $a \in A \setminus A_0$. Define a function,

$$Z : A \to \mathscr{F}^* \times \mathbb{R}^\ell$$

by $Z(a) = (Y_a, y(a))$ for all $a \in A \setminus A_0$ and $Z(a) = (Y_a, 0)$ for $a \in A_0$. Here, we remark that $y(a) \in Y_a$ for all $a \in A$. Then, Z is measurable and hence by Proposition 2.1, there exists a sequence $\{Z^n\}$ of simple functions converging to Z a.e. For each n, there is a partition $\{A_{kn}\}$ of A such that Z^n is constant over each A_{kn}. That is, $Z^n(a) \equiv (Y^k, y^k)$ for all $a \in A_{kn}$. Because $Z^n(A) \in Z(A)$, $y^k \in Y^k$ for all k. Define simple functions $\tilde{Y}^n$ and y^n by $\tilde{Y}^n_a = Y^k$ for $a \in A_{kn}$ and $y^n(a) = y^k$ for $a \in A_{kn}$. Then, it is easy to see that $\tilde{Y}^n \to Y$ and $y^n \to y$ a.e. and $y^n(a) \in \tilde{Y}^n_a$ a.e. For each n, there exists a positive integer m_n such that $y^n(A) \subset D_{m_n}$, where D_{m_n} is the disk with radius m_n. Let $Y^n_a = \tilde{Y}_a \cap D_{m_n}$. Then, because we may choose $\{m_n\}$ to satisfy $D_{m_n} \to \mathbb{R}^\ell$ as $n \to \infty$, $Y^n \to Y$ a.e. It is obvious that $y^n(a) \in Y^n_a$ a.e. Therefore, since $\int e_n \to \int e$ and $\int y_n \to \int y$, for sufficiently large n, we have $\bar{x} \ll \int e^n + \int y^n$. Now, let us define $\mathscr{E}^n(a) = (\succ_a, e^n(a), Y^n_a)$ for all $a \in A$. Then, $\mathscr{E}^n$ satisfies the conditions in previous theorem for sufficiently large n and has an equilibrium. Hence, we may assume that each $\mathscr{E}^*_n$ has a market equilibrium. By construction, $\mathscr{E}^n \to \mathscr{E}^*$ almost everywhere. This completes the proof. □

5 The Core Equivalence.

In this section, we discuss about the equivalence between the core and the set of competitive equilibrium allocations. In large economies with convex consumption set, the equivalence has been established by Aumann (1964). Hildenbrand (1968) shows that the non-convexity of each individual production set Y_a is not an obstacle for the equivalence. Under some additional assumptions, the equivalence theorem is extended to the case of non-convex consumption set by Yamazaki (1978b). Since our model can be viewed as a coalition production economy with a Radon-Nykodym derivative and non-convex consumption set, these observations suggest that the equivalence is very likely also in our setting. Indeed, while the general equivalence fails due to the non-convexity of the consumption set, we can prove that the core equivalence is a dense property.

Let X be the consumption set, a non-empty closed subset of $\mathbb{R}^\ell$ bounded from below and $\hat{\mathscr{P}}$ the set of irreflexive, transitive and continuous preference relations on X satisfying local non-satiation. An economy is a measurable function,

$$\hat{\mathscr{E}} : (A, \mathscr{A}, \nu) \to \hat{\mathscr{P}} \times \mathbb{R}^\ell \times \mathscr{F}^*.$$

In this section, we define an allocation (f, g) feasible in an economy $\hat{\mathscr{E}}$ if $f(a) \in X$, $g(a) \in Y_a$ for a.e. $a \in A$ and $\int f = \int e + \int g$.

Definition 7. A feasible allocation (f, g) is *blocked* by a coalition $S \in \mathscr{A}$ if $\nu(S) > 0$ and if there exist integrable functions

$$f' : S \to \mathbb{R}^\ell \text{ and } g' : S \to \mathbb{R}^\ell$$

satisfying the following conditions.

(1) $f'(a) \succ_a f(a)$ for a.e. $a \in S$.

(2) $g'(a) \in Y_a$ for a.e. $a \in S$.

(3) $\int_S f' = \int_S e + \int_S g'$.

The core is the set of feasible allocations that have no blocking coalition.

The following two lemmas are easy consequences of Theorems 1 and 2 in Hildenbrand (1968).

Lemma 5.1. Let $\hat{\mathscr{E}}$ be an economy whose production set correspondence is simple. Then, the set of market equilibrium allocations is contained in the core of the economy.

Lemma 5.2. Let $\hat{\mathscr{E}}$ be an economy whose production set correspondence is simple. Then, the core of the economy is contained in the set of quasi-equilibrium allocations.

The next result can be proved in the same way as in the proof of Theorem 4.1.

Lemma 5.3. Let $\hat{\mathscr{E}}$ be an economy whose endowment distribution is dispersed and production set correspondence is simple. Then, a quasi-equilibrium allocation is a market equilibrium allocation.

Then, the main result of this section is as follows.

Theorem 5.1. *For any economy $\hat{\mathscr{E}}$, there exists a sequence $\{\hat{\mathscr{E}}^n\}$ of economies converging to $\hat{\mathscr{E}}$ almost everywhere such that for each n, the core of the economy $\hat{\mathscr{E}}^n$ is equal to the set of market equilibrium allocations.*

Proof. By the same argument as in the proof of Theorem 4.2, there exists a sequence $\{\hat{\mathscr{E}}^n\}$ of economies converging to $\hat{\mathscr{E}}$ almost everywhere such that for each n, the endowment distribution is dispersed and the production set correspondence is simple. Then, by lemmas 5.1, 5.2 and 5.3, for each n, the core of $\hat{\mathscr{E}}^n$ is equal to the set of market equilibrium allocations. This completes the

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