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Graphic of 3-manifolds and its applications

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1. Introduction

It is a classical result of Reidemeister and Singer ([R], [S]) that any two Heegaard splittings of a 3-manifold are stably equivalent. However, it is still unknown that whether there actually exist two Heegaard splittings of the same genus such that more than one stabilizations are required to make them equivalent. In 2001, D. Backman [Ba] introduced a property, called critical, for separating surfaces in 3-manifolds, and give some fundamental results on critical Heegaard surfaces. The author has a feeling that "critical Heegaard surface" could be a break through to the above Stabilization Problem. The purpose of this article is to give a rough sketch of Backman's idea of the proof of one of such results. The author afraids that there are many mistakes caused by misunderstandings of the author's in this arti

Of course the responsibility for such mistakes is due to him Unfortunately this article contains no new results at all. The author is glad if this article could be a good introduction for Backman's paper.

2. Critical surface

Let F be a separating surface in a 3-manifold M. Let us call the regions that are separated by F "red" and "blue".

Definition 2.1 Let D, D' be compressing disks for F. We say that D is equivalent to D' (denoted by $D \sim D'$) if there exists an ambient isotopy φ_t of M such that $\varphi_1(F) = F$, and $\varphi_1(D) = D'$
(Here φ_t $(0 \le t \le 1)$ may not preserve F, and it is possible that φ_1 exchanges red and blue regions.)

The isotopy invariant disk complex for F, denoted by $\Gamma(F)$, is the 1-complex such that:

(1) The vertices of $\Gamma(F)$ correspond to the equivalence classes of compressing disks of F via "$\sim$", and

(2) Two vertices, say v_1, v_2, of $\Gamma(F)$ are connected by an edge if there are compressing disks D_1, D_2 for F such that:

1. D_i represents v_i (i.e., $[D_i] = v_i$),
2. D_1 and D_2 are contained in different sides of F,

and

3. $|D_1 \cap D_2| \leq 1$.

(Suppose that F is a Heegaard surface. Then the above conditions 2, 3 are equivalent to:
$\{D_1, D_2\}$ is either weakly reducing pair of disks or stabilizing pair of disks.)

We say that a vertex of $\Gamma(F)$ is <u>isolated</u> if it is not an endpoint of any edge.

<u>Definition 2.2.</u> We say that F is <u>critical</u> if

$$\Gamma(F) - \{\text{isolated verticies of } \Gamma(F)\}$$

is not connected.

(Roughly speaking, F is critical if there exists a pair of compressing disks D_1, D_2 for F such that D_1 is not related to D_2 via pairs of disks giving weak reducibility or stabilization up to the equivalence "$\sim$".)

Then, in [Ba], Backman proved the following:

<u>Theorem 1</u>. ([Ba. Theorem 5.1])

Let M be a 3-manifold with a critical Heegaard surface F. Suppose that there is an incompressible surface S in M. Then there exists an incompressible surface S' which is homeomorphic to S such that $F \cap S'$ consists of simple closed curves which are essential in both F and S'.

<u>Theorem 2</u> ([Ba, Theorem 6.1])

Let F, F' be mutually non-isotopic strongly irreducible Heegaard surfaces of M. Suppose that minimal genus common stabilization of F and F' is not critical. Then M contains an incompressible surface.

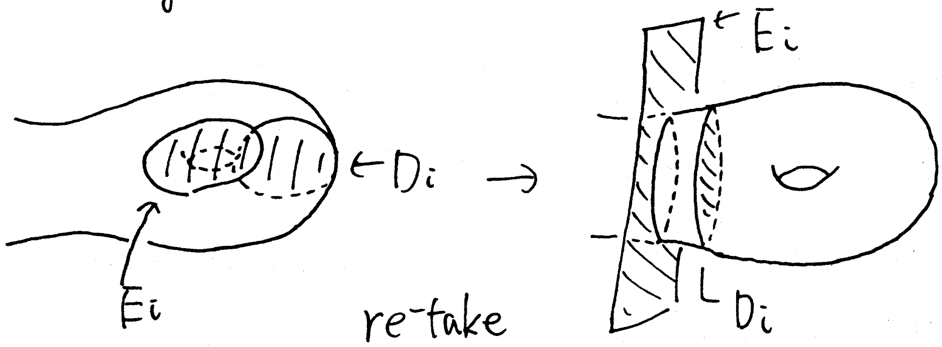
3. Sketch of the proof of Theorem 1.

The proof of Theorem 1 is based on the same stuff as Rubinstein-Scharlemann's paper [R-S].

<u>Convention</u>: We denote by $D_{\cdots}$ ($E_{\cdots}$ resp.) compressing disks of F contained in red region (blue region resp.).

<u>Stage 1</u>. Constructing the map, $\Phi : S \times D^2 \to M$

Since F is critical, there exist edges $[D_0]-[E_0]$, $[D_1]-[E_1]$ of $\Gamma(F)$ which belong different components. By re-taking D_i, E_i, if necessary, we may suppose that $D_0 \cap E_0 = \phi$, and $D_1 \cap E_1 = \phi$ (see the figure below).



By using outermost disk argument, it is easy to show that there is a sequence of compressing disks for F, $\{D_{\frac{i}{n}}\}_{i=0,1,\cdots,n}$, such that:

$$D_{\frac{0}{n}} = D_0,\ D_{\frac{n}{n}} = D_1, \text{ and } D_{\frac{i}{n}} \cap D_{\frac{i+1}{n}} = \phi \quad (i=0,1,\cdots,n-1)$$

and a sequence of compressing disks for F, $\{E_{\frac{i}{m}}\}_{i=0,1,\cdots,m}$, such that:

$$E_{\frac{0}{m}} = E_0,\ E_{\frac{m}{m}} = E_1, \text{ and } E_{\frac{i}{m}} \cap E_{\frac{i+1}{m}} = \phi \quad (i=0,1,\cdots,m-1)$$

Then let U_i $(i=0,1,\cdots,n)$ be a neighborhood of $D_{\frac{i}{n}}$ in M, and V_i $(i=0,1,\cdots,m)$ be a neighborhood of $E_{\frac{i}{m}}$ with the following properties.

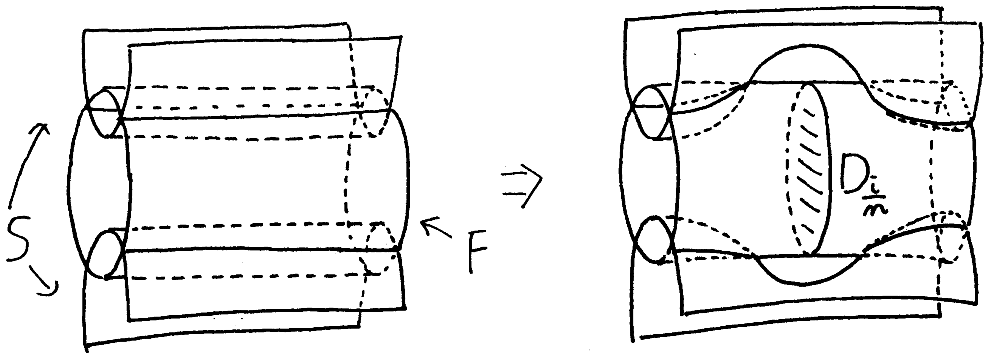
- $U_i \cap U_{i+1} = \phi \quad (i=0,1,\cdots,n-1)$
- $V_i \cap V_{i+1} = \phi \quad (i=0,1,\cdots,m-1)$
- $U_0 \cap V_0 = \phi$, $U_n \cap V_m = \phi$.

Let

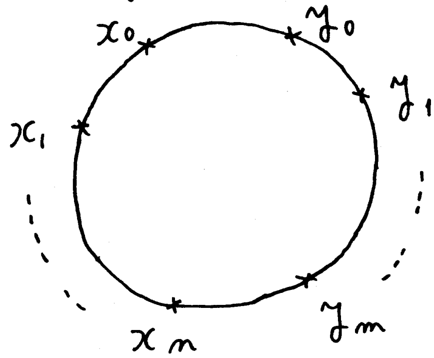
$$r_t^i(x) : M \times I \to M \quad (i=0,1,\cdots,n,\ x \in M,\ t \in I = [0,1])$$

be an isotopy with support in $\operatorname{Int} U_i$ which pushes $S \cap D_{\frac{i}{n}}$ out of $D_{\frac{i}{n}}$. See the next figure.

Similarly let δ_t^i be an isotopy with support in $\operatorname{Int} V_i$ which pushes $S \cap E_{\frac{i}{m}}$ out of $E_{\frac{i}{m}}$.

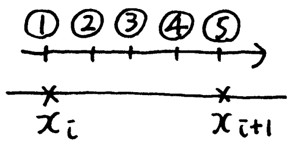


Then we distribute points $x_0, x_1, \cdots, x_m, y_m, \cdots, y_1, y_0$ on S^1 as in the following.

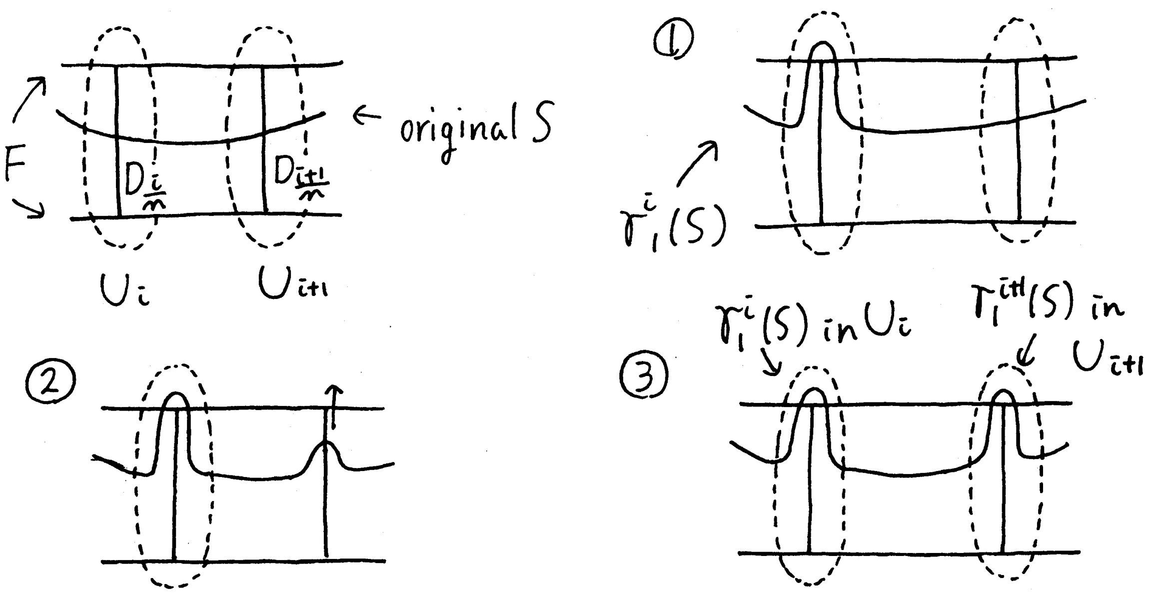


Then we regard S^1 as a parameter space of embeddings of S in M as follows:

If we move S^1 as



, then



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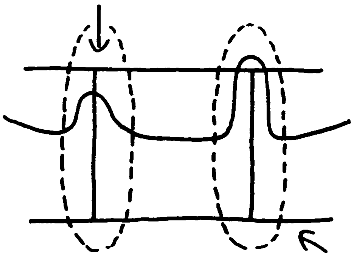

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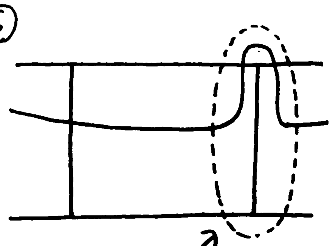
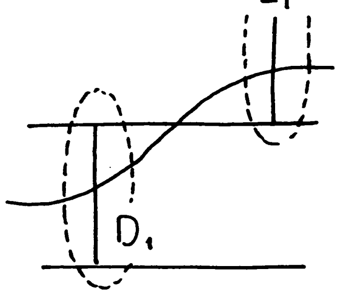


Similarly for $\overrightarrow{y_i \quad y_{i+1}}$

If we move S' as $\overrightarrow{x_m \quad y_m}$ (① ② ③ ④ ⑤), then S changes as



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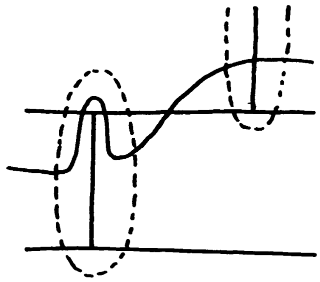

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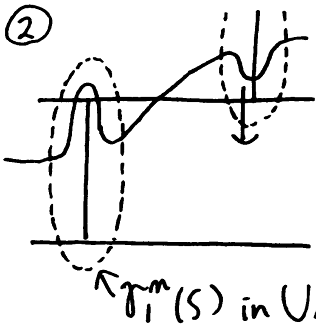

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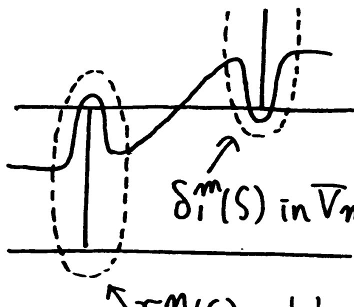

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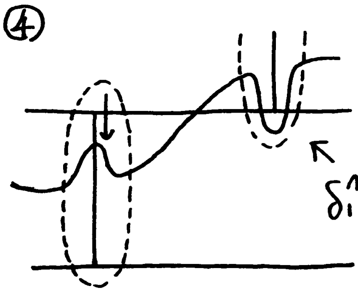

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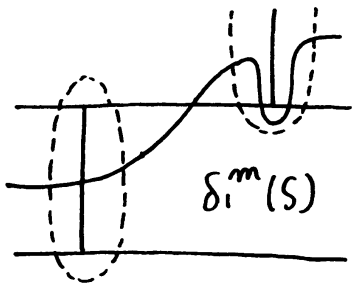


Similar for $\overrightarrow{y_0 \quad x_0}$

Then we extend this parametrization from $S^1 (= \partial D^2)$ to D^2 via linear extension, i.e., the center of D^2 is the original S, and along radial segment, it is deformed linearly to the position corresponding the endpoint in $S^1 (= \partial D^2)$.

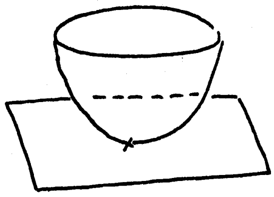
Hence we obtain:

$$\Phi : S \times D^2 \to M .$$

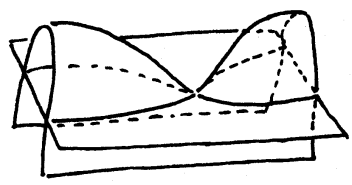
By Cerf theory [Ce], we can perturb Φ slightly so that D^2 is stratified into four parts below.

1. <u>Region</u>: Region is a component of the subset of D^2 consisting of points x such that S_x $(= \Phi(S \times \{x\}))$ and F intersect transversely. Region is an open set in D^2.

2. <u>Edge</u>: Edge is a component of the subset consisting of points x such that S_x and F intersect transversely except for one non-degenerate tangent point. (The tangent point is either a "center" or a "saddle".) Edge is a 1-dimensional subset of D^2.



center

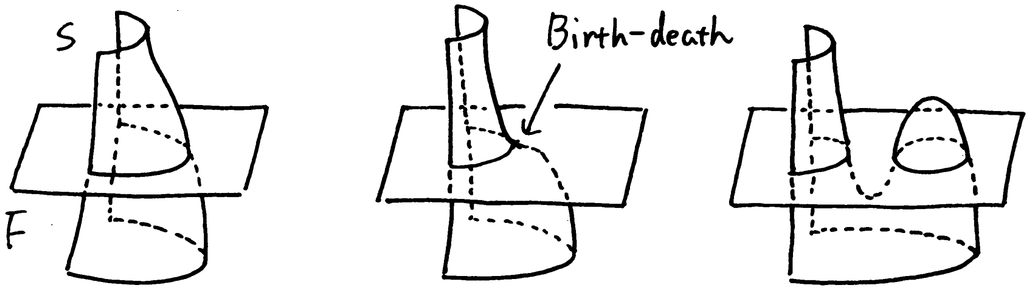


saddle

3. <u>Crossing vertex</u>: Crossing vertex is a component of the subset of D^2 consisting of points x such that S_x and F intersect transversely except for two non-degenerate tangent point. Crossing vertex is a 0-dimencional set (i.e., consists of a point.) and in a neighborhood of a crossing vertex, four edges are coming in.

4. <u>Birth-death vertex</u>: Birth-death vertex is a point x such that S_x and F intersect transversely except for a single degenerate tangent point. In particular, there is a parametrization $x = (\lambda, \mu)$ of a neighborhood of the point such that:

$$F = \{(x, y, z) \mid z = 0\} \text{ and } S_x = \{(x, y, z) \mid z = x^2 + \lambda + \mu y + y^3\}$$



Birth-death vertex is a 0-dimensional set and in a neighborhood of a birth-death vertex, two edges (one from center tangencies, and the other from saddle tangencies.)

Let Σ be the union of edges and vertices above. Σ is called a <u>graphic</u>.

Stage 2 Labelling regions

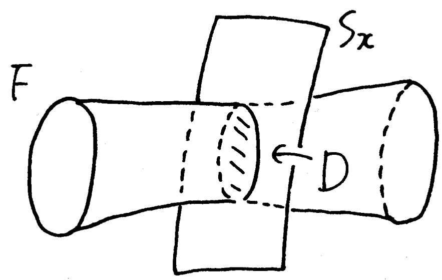
Let $\mathcal{C}_0$, $\mathcal{C}_1$ be sets of comp. of $\Gamma(F)$ such that:

1. Any isolated vertex of $\Gamma(F)$ is contained in both $\mathcal{C}_0$ and $\mathcal{C}_1$,
2. Any component of $\Gamma(F)$ which is not an isolated vertex is an element of exactly one of $\mathcal{C}_0$, $\mathcal{C}_1$,
3. The component of $\Gamma(F)$ containing the edge $[D_0]-[E_0]$ ($[D_1]-[E_1]$ resp.) is an element of $\mathcal{C}_0$ ($\mathcal{C}_1$ resp.).

(For example, let Γ_0 be the component of $\Gamma(F)$, which contains the edge $[D_0]$—$[E_0]$. Then we may take

$$\mathcal{C}_0 = \Gamma_0 \cup \{\text{isolated vertices of } \Gamma(F)\}, \text{ and}$$
$$\mathcal{C}_1 = \Gamma(F) \setminus \Gamma_0.\ )$$

Let R be a region. Then we label this region with "D_i" ($i=0$ or 1) if there exists a red compressing disk D for F such that $[D]$ is a vertex of an element of $\mathcal{C}_i$, and $D \subset S_x$ ($x \in R$). See figure below.

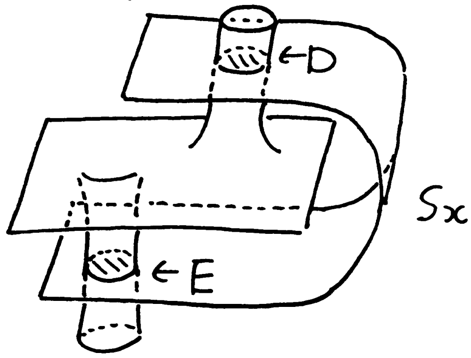


Similarly we label the region with "E_i" if there exists a blue compressing disk E for F such that $[E]$ is a vertex of an element of $\mathcal{C}_i$, and $E \subset S_x$.

<u>Remark</u> By this definition, we see that if there is an unlabelled region R, then a disk swap of S_x $(x \in R)$ gives the conclusion of Theorem 1. Hence in the remainder of this article, we show that there exists an unlabelled region.

<u>Claim 2.</u> No region can have both of the labels "D_i" and "E_{1-i}" ($i=0$ or 1).

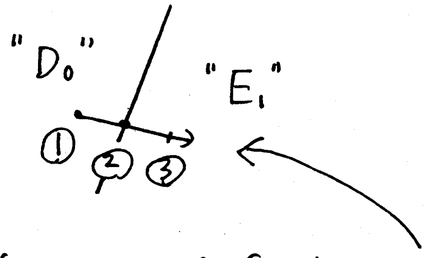
Proof. Suppose that there exists a region R with labels "D_0" and "E_1". Hence there is a red compressing disk $D\,(\subset S_x)$, and a blue compressing disk $E\,(\subset S_x)$, where $x \in R$ and $[D]$ ($[E]$ resp.) is a vertex of an element of $\mathcal{C}_0$ ($\mathcal{C}_1$ resp.).



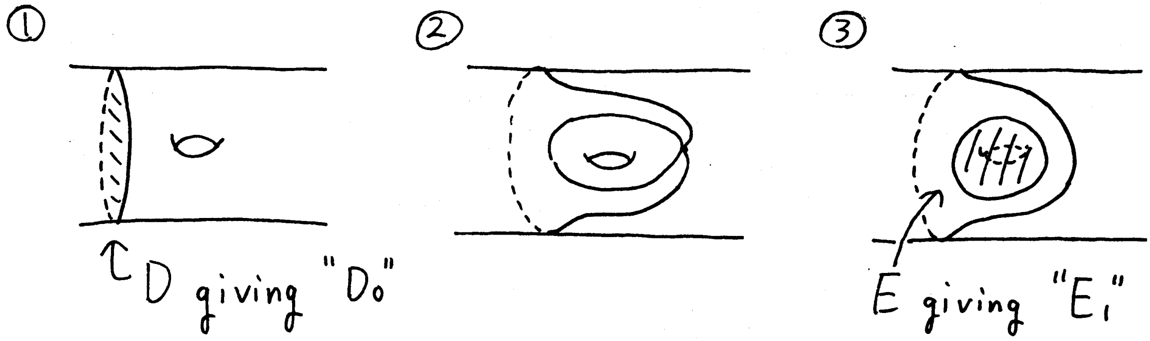
Since $D \cap E = \emptyset$, there is an edge $[D] - [E]$. However this contradicts the fact that $[D]$ and $[E]$ are contained in different components of $\Gamma(F)$. //

<u>Claim 3.</u> If a region is labelled by "D_i", then no adjacent region can be labelled by "E_{1-i}".

Proof. Suppose this is not true. Hence



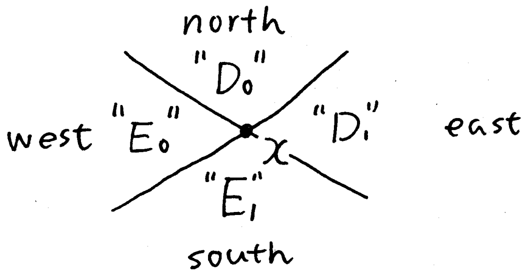
Consider the deformation of S along this path. Then $S_x \cap F$ changes as follows.



By isotopy, we can make $\partial D \cap \partial E = \phi$. Hence there is an edge $[D]$ — $[E]$, a contradiction. //

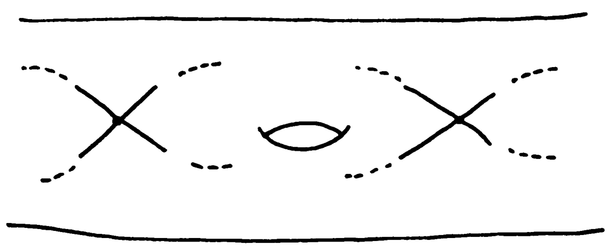
<u>Claim 4</u> All four labels cannot occur around a vertex of Σ.

Proof. Suppose that this is not true. Call the regions north, east, south, and west (see figure below).

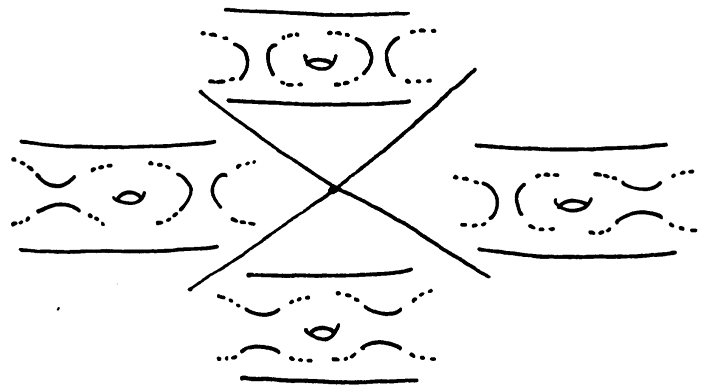


By Claims 2, 3 we see that the labelling must be as above

Then $S_x \cap F$ contains two saddle tangencies (and we schematic draw the intersection as follows).



Then in each region $S_x \cap F$ will look as follows.



It is elementary to see that either

1. "north intersection" and "south intersection" can be made disjoint on F by an isotopy, or

2. "east intersection" and "west intersection" can be made disjoint on F by an isotopy.

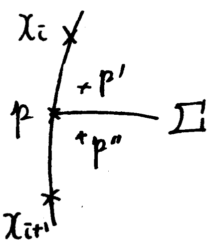
In either case, we have a contradiction as in Claim 2. //

Claim 5. Let $p (\in \partial D^2)$ be a point contained in a region, say R. Suppose that p lies between x_i and x_{i+1} on ∂D_2. Then R cannot have both of the labels "E_0" and "E_1" (Corresponding statement holds for points between y_i and y_{i+1}.

Proof. By the definition of Φ, we may suppose without loss of generality that $S_p \cap U_i = \gamma_1^i(S) \cap U_i$ (hence, $S_p \cap D_{\frac{i}{m}} = \phi \cdots ①$). Suppose that R has both labels "E_0" and "E_1", and $E'(\subset S_p)$, $E''(\subset S_p)$ be disks giving the labels "E_0", "E_1" respectively. Since $[E']$ and $[E'']$ are contained in different components of $\Gamma(F)$, we may suppose $[D_{\frac{i}{m}}]$ and $[E']$ are contained in different components of $\Gamma(F)$. However the above ① shows that there is an edge $[E']$ —— $[D_{\frac{i}{m}}]$, a contradiction. //

<u>Claim 6</u>. Let p be a point of $\Sigma \cap \partial D^2$. Suppose that p lies between x_i and x_{i+1} on ∂D^2. Let R_0 and R_1 be regions adjacent to p. Then {labels of R_0} $\cup$ {labels of R_1} does not contain {"E_0", "E_1"}.
(Corresponding statement holds for points between y_i and y_{i+1}.)

Proof. Suppose that this is not the case. By Claim 5, we may suppose that R_0 is labelled by "E_0", and R_1 is labelled by "E_1", and let $E'(\subset S_{p'})$, $E''(\subset S_{p''})$ be disks giving the labels "E_0", "E_1" respectively, where $p' \in R_0$, $p'' \in R_1$ are points close to p. As in the proof of Claim 5, we may suppose that $S_p \cap D_{\frac{i}{m}} = \phi$. Hence we have



$$[E'] —— [D_{\frac{i}{m}}] —— [E'']$$

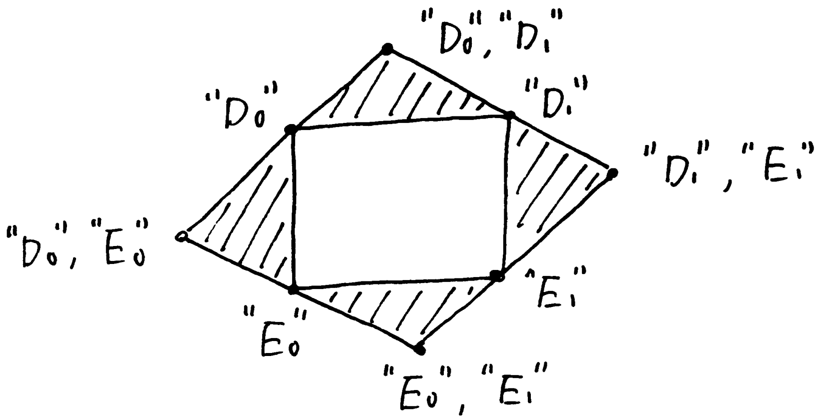
Hence $[E']$ and $[E'']$ are contained in the same component of $\Gamma(F)$, a contradiction. //

Claim 7. Let θ_0 ($\in \partial D^2$) be a point between x_0 and y_0 such that $S_{\theta_0} \cap U_0 = \tau_1^0(S) \cap U_0$, and $S_{\theta_0} \cap V_0 = \delta_1^0(S) \cap V_0$, and let R_0 be the region containing θ_0. Then R_0 cannot be labelled either by "D_1" or "E_1".
(Corresponding statement holds for point θ_1 between x_m and y_m.)

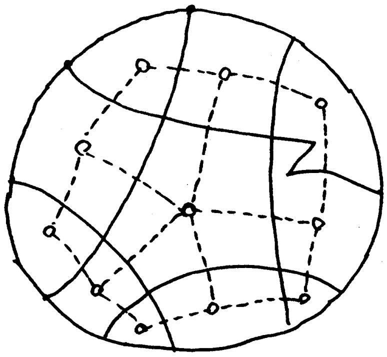
Proof. Suppose that R_0 is labelled by "D_1" with corresponding disk D ($\subset S_{\theta_0}$). Since $S_{\theta_0} \cap E_0 = \phi$, we see that there is an edge $[D] - [E_0]$. However this contradicts the fact that $[D_1] \in \mathcal{C}_1$ and $[E_0] \in \mathcal{C}_0$. We obtain a contradiction in the same manner for label "E_1". //

Stage 3. Finding an unlabelled region.

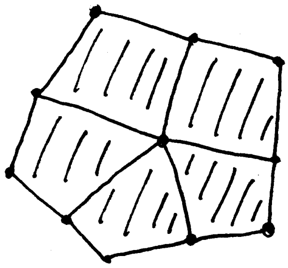
Suppose that each region is labelled. Let Π be the 2-complex with each vertex labelled as follows.



Let Σ' be the dual 2-complex of Σ.



Σ



Σ'

By Claim2, we see that there is a map:

vertices (Σ) $\longrightarrow$ vertices (Π),

where the labellings are exactly those of the corresponding regions.

By Claim3, we see that this map extends to:

edges (Σ) $\longrightarrow$ edges (Π).

By Claim4, we see that this map extends to a simplicial map: $\Sigma' \to \Pi$.

By Claims 5, 6, 7, we see that the restriction of the map to $\partial\Sigma'$ is a degree 1 map. However this contradicts the fact that Σ' is simply connected.

References

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